Garfunkel’s Inequality

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Abstract. Let $I$ be the incenter of triangle $ABC$ and $U, V, W$ the intersections of the segments $IA, IB, IC$ with the incircle. If the centroid $G$ is inside the incircle, and $D, E, F$ the intersections of the segments $GA, GB, GC$ with the incircle. Jack Garfunkel [1] asked for a proof that the perimeter of $UVW$ is not greater than that of $DEF$. This problem is hitherto unsolved. We give a proof in this note.

Consider a triangle $ABC$ with centroid $G$ lying inside its incircle $(I)$. Let the segments $AG, BG, CG, AI, BI, CI$ intersect the incircle at $D, E, F, U, V, W$ respectively. Garfunkel posed the inequality $\partial(\triangle UVW) \leq \partial(\triangle DEF)$ as Problem 648(b) of *Crux Mathematicorum* [1, 2]. Here, $\partial(\cdot)$ denotes the perimeter of a triangle. The problem is hitherto unresolved. In this note we give a proof of this inequality. We adopt standard notations: $a, b, c$, are the sidelengths of triangle $ABC$, $s$ the semiperimeter and $r$ the inradius.

**Lemma 1.** If the centroid $G$ of the triangle $ABC$ is inside the incircle $(I)$, then

$$a^2 < 4bc, \quad b^2 < 4ca, \quad c^2 < 4ab.$$ 

**Proof.** Because $G$ is inside $(I)$, we have $IG^2 \leq r^2, (AG - AI)^2 \leq r^2, AG^2 + AI^2 - 2AG \cdot AI \leq r^2$. This inequality is equivalent to the following

$$\frac{2(b^2 + c^2) - a^2}{9} + (s - a)^2 - \frac{2(b + c)(s - a)}{3} \leq 0$$

$$8(b^2 + c^2) - 4a^2 + 9(b + c - a)^2 - 12(b + c)(b + c - a) \leq 0$$

$$3(b + c - a)^2 + 2(b - c)^2 \leq 2(4bc - a^2)$$

which implies $a^2 < 4bc$ and similarly $b^2 < 4ca, c^2 < 4ab$. \[\square\]

Let the external bisectors of triangle $UVW$ bound the triangle $PQR$, and intersect the incircle of $ABC$ at $U', V', W'$ respectively.
Lemma 2. If the centroid $G$ of $ABC$ is inside the incircle, then the points $D, E, F$ are on the minor arcs $UU', VV', WW'$ respectively.

![Figure 1]

Proof. If $b = c$ then obviously $U, D$ and $U'$ are the same point.

Assume without loss of generality $b > c$. We set for brevity $\phi = \frac{A}{2}, \theta = \frac{B - C}{4}$. Note that $U'$ is the midpoint of the arc $VUW$. We have

$$\angle UIU' = \frac{1}{2} (\angle UIW - \angle UIV) = \frac{1}{2} \left( 90^\circ + \frac{B}{2} - 90^\circ - \frac{C}{2} \right) = \theta.$$

Let $X'$ be the antipode of the touch point $X$ of the incircle with $BC$. Since $\angle UIV = \angle X'IW$, the point $U'$ is the mid point of the arc $UX'$. We have

$$\overrightarrow{AU'} = \overrightarrow{AI} + \overrightarrow{IU'} = \overrightarrow{AI} + \frac{1}{2 \cos \theta} \left( \overrightarrow{IU} + \overrightarrow{IX'} \right)$$

$$= \overrightarrow{AI} + \frac{1}{2 \cos \theta} \left( \sin \phi \overrightarrow{IA} - \overrightarrow{IA} - \overrightarrow{AX} \right)$$

$$= \left( 1 - \frac{\sin \phi - 1}{2 \cos \theta} \right) \overrightarrow{AI} - \frac{1}{2 \cos \theta} \overrightarrow{AX}$$

$$= \left( 1 - \frac{\sin \phi - 1}{2 \cos \theta} \right) \left( \frac{b}{2s} \overrightarrow{AB} + \frac{c}{2s} \overrightarrow{AC} \right)$$

$$- \frac{1}{2 \cos \theta} \left( \frac{s - c}{a} \overrightarrow{AB} + \frac{s - b}{a} \overrightarrow{AC} \right)$$

$$= \left( 1 - \frac{\sin \phi - 1}{2 \cos \theta} \right) \frac{b}{2s} \overrightarrow{AB} - \frac{1}{2 \cos \theta} \cdot \frac{s - c}{a} \overrightarrow{AB}$$

$$+ \left( 1 - \frac{\sin \phi - 1}{2 \cos \theta} \right) \frac{c}{2s} - \frac{1}{2 \cos \theta} \cdot \frac{s - b}{a} \overrightarrow{AC}.$$
Since \( b > c \), the centroid \( G \) lies inside the angle \( \angle IAC \). To prove that \( D \) lies on the minor arc \( UU' \) it is sufficient to prove that the coefficient of \( AC \) is greater than that of \( AB \) in the above expression of \( AU' \). We need, therefore, to prove the inequality

\[
\left( 1 - \frac{\sin \varphi - 1}{2 \cos \theta} \right) \frac{c}{2s} - \frac{1}{2 \cos \theta} \cdot \frac{s - b}{a} > \left( 1 - \frac{\sin \varphi - 1}{2 \cos \theta} \right) \frac{b}{2s} - \frac{1}{2 \cos \theta} \cdot \frac{s - c}{a}.
\]

Factoring and grouping common terms, the inequality is equivalent to

\[
\frac{1}{2 \cos \theta} \cdot \frac{b - c}{a} - \left( 1 - \frac{\sin \varphi - 1}{2 \cos \theta} \right) \frac{b - c}{2s} > 0
\]

\[
\frac{b - c}{4s \cos \theta} \left( \frac{b + c}{a} - 2 \cos \theta + \sin \varphi \right) > 0
\]

\[
(b + c + a \sin \varphi)^2 > 4a^2 \cos^2 \theta.
\]

Using the well-known identity \( \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \), and \( a \cos 2\theta = (b + c) \sin \varphi \) by the law of sines, inequality (1) can be written in the form

\[
(b + c + a \sin \varphi)^2 > 2a^2 + 2a(b + c) \sin \varphi
\]

\[
(b + c)^2 - a^2 > a^2 - a^2 \sin^2 \varphi
\]

\[
2bc + 2bc \cos A > a^2 \cos^2 \varphi
\]

\[
4bc \cos^2(A/2) > a^2 \cos^2 \varphi
\]

\[
4bc > a^2.
\]

This inequality holds by Lemma 1 since \( G \) is inside the incircle. This shows that \( D \) is on the minor arc \( UU' \). The same reasoning also shows that \( E \) and \( F \) are on the minor arcs \( VV' \), \( WW' \) respectively.

\[\square\]

**Theorem (Garfunkel’s inequality).** If the centroid \( G \) lies inside the incircle, then

\[\partial(UVW) \leq \partial(DEF).\]

**Proof.** By Lemma 2, the points \( D, E, F \) lie on the minor arcs \( UU' \), \( VV' \), \( WW' \) respectively. Let \( X'' \) be the intersection point of \( DE \) and \( QR \), \( Y'' \) be the intersection point of \( EF \) and \( RP \), and \( Z'' \) be the intersection point of \( FD \) and \( PQ \). Note that \( X'', Y'', Z'' \) belong to the segments \( DE, EF, FD \) respectively. See Figure 2. It follows that

\[
\partial(DEF) = DE + EF + FD
\]

\[
= DX'' + X''E + EY'' + Y''F + FZ'' + Z''D
\]

\[
= (EX'' + EY'') + (FY'' + FZ'') + (DZ'' + DX'')
\]

\[
\geq X''Y'' + Y''Z'' + Z''X''
\]

\[
= \partial(X''Y''Z'').
\]
Therefore, $\partial(DEF) \geq \partial(X''Y''Z'')$. On the other hand, triangle $PQR$ is acute and triangle $UVW$ is its orthic triangle. See Figure 1. By Fagnano’s theorem, we have $\partial(X''Y''Z'') \geq \partial(UVW)$. It follows that $\partial(DEF) \geq \partial(UVW)$. The equality holds if and only if triangle $ABC$ is equilateral. □

References


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