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Garfunkel's Inequality

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Abstract. Let I be the incenter of triangle ABC and U, V, W the intersections of the segments IA, IB, IC with the incircle. If the centroid G is inside the incircle, and D, E, F the intersections of the segments GA, GB, GC with the incircle. Jack Garfunkel [1] asked for a proof that the perimeter of UVW is not greater than that of DEF. This problem is hitherto unsolved. We give a proof in this note.

Consider a triangle ABC with centroid G lying inside its incircle (I). Let the segments AG, BG, CG, AI, BI, CI intersect the incircle at D, E, F, U, V, W respectively. Garfunkel posed the inequality $\partial(UVW) \leq \partial(DEF)$ as Problem 648(b) of *Crux Mathematicorum* [1, 2]. [1] Here, $\partial(\cdot)$ denotes the perimeter of a triangle. The problem is hitherto unresolved. In this note we give a proof of this inequality. We adopt standard notations: a, b, c, are the sidelengths of triangle ABC, s the semiperimeter and r the inradius.

Lemma 1. *If the centroid G of the triangle ABC is inside the incircle (I), then*

$$a^2 < 4bc, \quad b^2 < 4ca, \quad c^2 < 4ab.$$

Proof. Because G is inside (I), we have $\overrightarrow{IG}^2 \leq r^2$, $(\overrightarrow{AG} - \overrightarrow{AI})^2 \leq r^2$, $\overrightarrow{AG}^2 + \overrightarrow{AI}^2 - 2\overrightarrow{AG} \cdot \overrightarrow{AI} \leq r^2$. This inequality is equivalent to the following

$$\overrightarrow{AG}^2 + (\overrightarrow{AI}^2 - r^2) - \frac{2}{3}(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{AI} \leq 0$$

$$\frac{2(b^2+c^2) - a^2}{9} + (s-a)^2 - \frac{2(b+c)(s-a)}{3} \leq 0$$

$$8(b^2+c^2) - 4a^2 + 9(b+c-a)^2 - 12(b+c)(b+c-a) \leq 0$$

$$3(b+c-a)^2 + 2(b-c)^2 \leq 2(4bc - a^2)$$

which implies $a^2 < 4bc$ and similarly $b^2 < 4ac$, $c^2 < 4ab$. □

Let the external bisectors of triangle UVW bound the triangle PQR, and intersect the incircle of ABC at U', V', W' respectively.

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[1] Problem 648(a) asked for a proof of $\partial(XYZ) \leq \partial(UVW)$, XYZ being the intouch triangle. See Figure 1. A proof by Garfunkel was given in [1].

Lemma 2. *If the centroid G of ABC is inside the incircle, then the points D, E, F are on the minor arcs UU', VV', WW' respectively.*

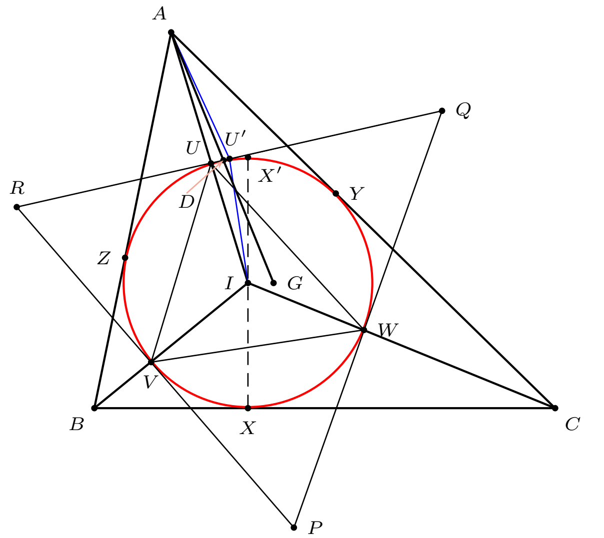

Figure 1

Proof. If $b = c$ then obviously U, D and U' are the same point.

Assume without loss of generality $b > c$. We set for brevity $\varphi = \frac{A}{2}$, $\theta = \frac{B-C}{4}$. Note that U' is the midpoint of the arc VUW. We have

$$\angle UIU' = \frac{1}{2}(\angle UIW - \angle UIV) = \frac{1}{2}\left(90^\circ + \frac{B}{2} - 90^\circ - \frac{C}{2}\right) = \theta.$$

Let X' be the antipode of the touch point X of the incircle with BC. Since $\angle UIV = \angle X'IW$, the point U' is the mid point of the arc UX'. We have

$$\begin{aligned}
\overrightarrow{AU'} =& \overrightarrow{AI} + \overrightarrow{IU'} = \overrightarrow{AI} + \frac{1}{2\cos\theta}\left(\overrightarrow{IU} + \overrightarrow{IX'}\right) \\
=& \overrightarrow{AI} + \frac{1}{2\cos\theta}\left(\sin\varphi\overrightarrow{IA} - \overrightarrow{IA} - \overrightarrow{AX}\right) \\
=& \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\overrightarrow{AI} - \frac{1}{2\cos\theta}\overrightarrow{AX} \\
=& \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\left(\frac{b}{2s}\overrightarrow{AB} + \frac{c}{2s}\overrightarrow{AC}\right) \\
& - \frac{1}{2\cos\theta}\left(\frac{s-c}{a}\overrightarrow{AB} + \frac{s-b}{a}\overrightarrow{AC}\right) \\
=& \left(\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{b}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-c}{a}\right)\overrightarrow{AB} \\
& + \left(\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{c}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-b}{a}\right)\overrightarrow{AC}.
\end{aligned}$$

Since $b > c$, the centroid G lies inside the angle $\angle IAC$. To prove that D lies on the minor arc UU' it is sufficient to prove that the coefficient of $\overrightarrow{AC}$ is greater than that of $\overrightarrow{AB}$ in the above expression of $\overrightarrow{AU'}$. We need, therefore, to prove the inequality

$$\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{c}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-b}{a} > \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{b}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-c}{a}.$$

Factoring and grouping common terms, the inequality is equivalent to

$$\frac{1}{2\cos\theta}\cdot\frac{b-c}{a} - \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{b-c}{2s} > 0$$

$$\frac{b-c}{4s\cos\theta}\left(\frac{b+c}{a} - 2\cos\theta + \sin\varphi\right) > 0$$

$$(b + c + a\sin\varphi)^2 > 4a^2\cos^2\theta. \tag{1}$$

Using the well-known identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, and $a\cos 2\theta = (b+c)\sin\varphi$ by the law of sines, inequality (1) can be written in the form

$$(b + c + a\sin\varphi)^2 > 2a^2 + 2a(b+c)\sin\varphi$$

$$(b+c)^2 - a^2 > a^2 - a^2\sin^2\varphi$$

$$2bc + 2bc\cos A > a^2\cos^2\varphi$$

$$4bc\cos^2(A/2) > a^2\cos^2\varphi$$

$$4bc > a^2.$$

This inequality holds by Lemma 1 since G is inside the incircle. This shows that D is on the minor arc UU'. The same reasoning also shows that E and F are on the minor arcs VV', WW' respectively. □

Theorem (Garfunkel's inequality)**.** *If the centroid G lies inside the incircle, then* $\partial(UVW) \le \partial(DEF)$.

Proof. By Lemma 2, the points D, E, F lie on the minor arcs UU', VV', WW' respectively. Let X'' be the intersection point of DE and QR, Y'' be the intersection point of EF and RP, and Z'' be the intersection point of FD and PQ. Note that X'', Y'', Z'' belong to the segments DE, EF, FD respectively. See Figure 2. It follows that

$$\begin{aligned}
\partial(DEF) &= DE + EF + FD \\
&= DX'' + X''E + EY'' + Y''F + FZ'' + Z''D \\
&= (EX'' + EY'') + (FY'' + FZ'') + (DZ'' + DX'') \\
&\ge X''Y'' + Y''Z'' + Z''X'' \\
&= \partial(X''Y''Z'').
\end{aligned}$$

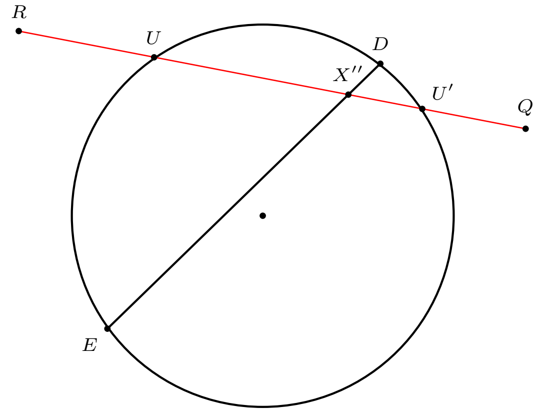

Figure 2

Therefore, $\partial(DEF) \geq \partial(X''Y''Z'')$. On the other hand, triangle PQR is acute and triangle UVW is its orthic triangle. See Figure 1. By Fagnano's theorem, we have $\partial(X''Y''Z'') \geq \partial(UVW)$. It follows that $\partial(DEF) \geq \partial(UVW)$. The equality holds if and only if triangle ABC is equilateral. $\square$

References

[1] J. Garfunkel, Problem 648, *Crux Math.*, 7 (1981) 178; solution, 8 (1982) 180–182.

[2] S. Rabinowitz, *Index to Mathematical Problems 1980-1984*, MathPro Press, Westford, Massachusetts USA 1992, p. 469.

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