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On Some Actions of D_3 on a Triangle

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Abstract. The action of the dihedral group D_3 on the equilateral triangle is generalized to various actions on general triangles.

1. Introduction

The equilateral triangle admits in a natural way the action of the dihedral group D_3. The elements f of the group act as reflexions (order 2: $f^2 = 1$) or as rotations (order 3: $f^3 = 1$). If we relax the property of f from being isometry, we can define similar actions on an arbitrary triangle. In fact there are infinitely many actions of D_3 on an arbitrary triangle, described by the following setting.

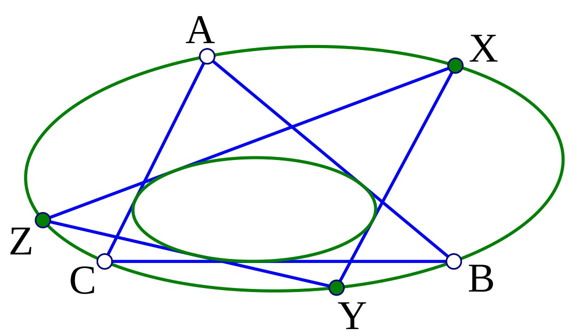

Figure 1. Projectivity preserving a conic

It is well known that given six points A, A', B, B', C, C' on a conic c, there is a unique projectivity preserving c and mapping A to A', B to B' and C to C'. Taking A', B', C' to be permutations of the set A, B, C we see that there is a group G of projectivities that permute the vertices of the triangle $t = (ABC)$ and preserve the conic c. It is not difficult to see that G is naturally isomorphic to the group of symmetries of the equilateral triangle. Thus from the algebraic point of view, the group action contains no significant information. But from the geometric point of view the situation is quite interesting. For example, fixing such a group, we can consider generalized rotations i.e. $f \in G$ of order three $f^3 = 1$, which applied to a point $X \in c$ generate an *orbital triangle* $X, Y = f(X), Z = f(f(X))$. All these orbital triangles envelope a second conic which is also invariant under the group G. For definitions, general facts on triangles, transformations and especially projectivities I refer to [1]. For special conics, circumscribed on a triangle, this setting unifies several dispersed properties and presents them under a new light.

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I shall illustrate this aspect by applying the above method to two special cases. Then I shall discuss an exceptional, similar setting, which results by replacing the circumconic with the circumcircle of the triangle and the projectivities by Moebius transformations. The first case will be that of the exterior Steiner ellipse of the triangle.

2. Steiner dihedral group of a triangle

We start with a triangle $t = (ABC)$ and its exterior Steiner ellipse. Then we consider the projectivities that preserve this conic and permute the vertices of the triangle. First I shall state the facts. The group, which I call the *Steiner dihedral group* of the triangle, comprises two kinds of maps: involutions, that resemble to reflections, and cyclic permutations of the vertices that resemble to rotations.

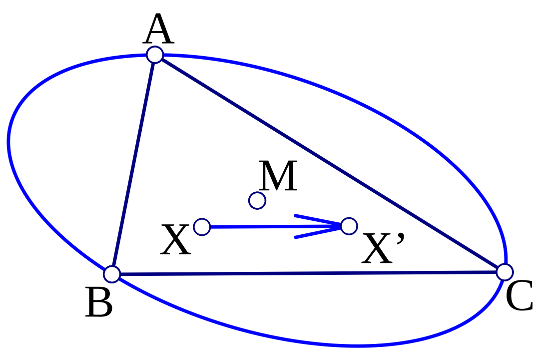

Figure 2. Isotomic conjugation

The involutions are related to the sides of the triangle and coincide with the isotomic conjugations with respect to the corresponding medians: Side a of the triangle defines an involution on the conic: $I_a(X) = Y$, where XY is parallel to the side a and bisected by the median to a. I_a has the median to a as its line of fixed points, which coincides with the conjugate diameter of a relative to the conic. The corresponding isolated fixed point (Fregier point of the involution) is the point at infinity of line a. Analogous definitions and properties have the involutions I_b, I_c.

More important seems to be the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, that preserves the conic and cycles the vertices of the triangle. I call it the *isotomic rotation*.

As is the case with every projectivity f, preserving a conic, for all points X on c, the lines $[X, f(X)]$ envelope another conic, which in this case is the inner Steiner ellipse. By the same argument all *orbital* triangles i.e. triangles of the form $t' = (X, f(X), f(f(X)))$, are circumscribed on the inner Steiner ellipse. More precisely the following statements are valid and easy to prove:

(1) The centroid G of the triangle is the fixed point of f.

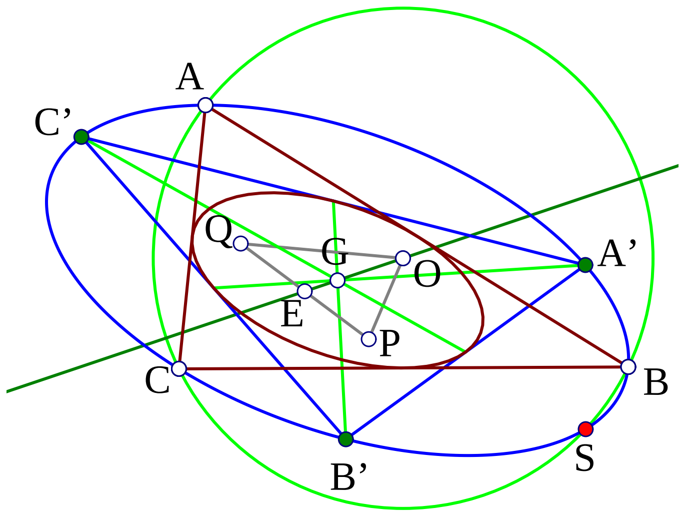

Figure 3. Isotomic rotation

(2) Every point X of the plane defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which has G for its centroid.

(3) The orbital triangles s, as above, which have X on the external Steiner ellipse, are all circumscribed to the inner Steiner ellipse. They are precisely the only triangles that have these two ellipses as their external/internal Steiner ellipses.

(4) The inner and outer Steiner ellipses generate a family of homothetic conics, with homothety center the centroid G of the triangle. For every point X of the plane the orbital triangle s generated by X has the corresponding conics-family-member c, passing through X, as its outer Steiner ellipse. Besides, for all points X on c, the corresponding orbital triangles circumscribe another conics-family-member c', which is the inner Steiner ellipse of all these triangles.

(5) For a fixed orbital triangle $t = (ABC)$, the orbit of its circumcenter O, defines a triangle $u = (OPQ)$, whose median through O is the Euler line of the initial triangle t. The middle E of PQ is the center of the Euler circle of t.

(6) The trilinear coordinates of points $P = f(O)$ and $Q = f(P)$ are respectively:

$$P = \left(\frac{\sin 2C}{\sin A}, \frac{\sin 2A}{\sin B}, \frac{\sin 2B}{\sin C}\right),$$

$$Q = \left(\frac{\sin 2B}{\sin A}, \frac{\sin 2C}{\sin B}, \frac{\sin 2A}{\sin C}\right).$$

Deferring for a later moment the proofs, I shall pass now to the analogous group, of projectivities, which results by replacing the external ellipse with the circumcircle of the triangle. For a reason that will be made evident shortly I call the corresponding group the *Lemoine* dihedral group of the triangle.

3. Lemoine dihedral group of a triangle

We start with a triangle $t = (ABC)$ and its circumcircle c. Then we consider the projectivities that preserve c and permute the vertices of the triangle. There are again two kinds of such maps. Involutions, and maps of order three.

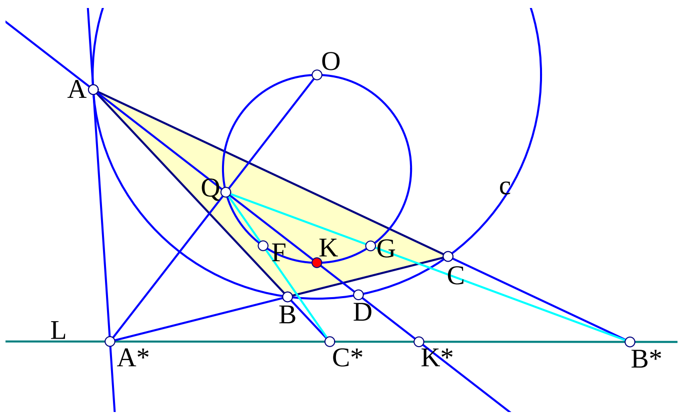

Figure 4. Lemoine reflexion

Side a of the triangle defines a projective involution $I_a(X) = X'$, by the properties $I_a(A) = A$ and $I_a(B) = C, I_a(C) = B$. Its line of fixed points, is the symmedian line AD. The corresponding isolated fixed point (Fregier point) is the pole A^* of the symmedian with respect to the circumcircle, which lies on the Lemoine axis L of the triangle. In the figure above, K is the symmedian point and Q is the projection of the circumcenter on the symmedian AD (is a vertex of the second Brocard triangle of t). From the invariance of cross-ratio and the fact that I_a maps L to itself, follows that $(C^*B^*K^*A^*) = 1$, hence the symmedian bisects the angle B^*QC^*. Joining Q with B^*, C^* we find the intersections F, G of these lines with the Brocard circle (with diameter OK). Below (in §6) we show that F, G coincide with the Brocard points of the triangle.

I_a could be called the *Lemoine reflexion* (on the symmedian through A). Analogous is the definition and the properties of the involutions I_b and I_c, corresponding to the other sides of the triangle.

More important seems to be the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, which preserves the circumcircle and cycles the vertices of the triangle. I call it the *Lemoine rotation*.

As before, for all points X on c, the lines $[X, f(X)]$ envelope another conic, which in this case is the Brocard ellipse c' of the triangle t. By the same argument all *orbital* triangles i.e. triangles of the form $t' = (X, Y = f(X), Z = f(f(X)))$,

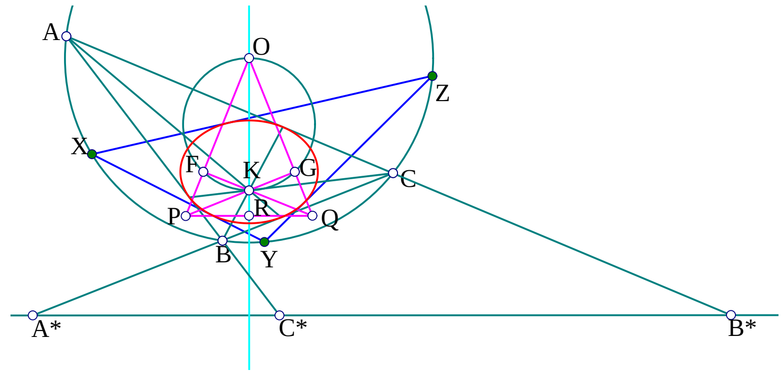

Figure 5. Lemoine rotation

are circumscribed on the Brocard ellipse. More precisely the following statements are valid and easy to prove:

(1) f leaves invariant each member of the family of conics generated by the circumcircle and the Brocard ellipse of t. In particular the Lemoine axis of t remains invariant under f, and permutes points A^*, B^*, C^*.
(2) The symmedian (or Lemoine) point K of the triangle is the fixed point of f.
(3) Every point X of the circle c defines an *orbital* triangle
$$s = (X, f(X), f(f(X))),$$
which has K as symmedian point.
(4) The orbital triangles s, as above, which have X on c, are all circumscribed to the Brocard ellipse c'. They are precisely the only triangles that have c and c' as circumcircle and Brocard ellipse, respectively.
(5) For a fixed orbital triangle $t = (ABC)$, the orbit of its circumcenter O, defines a triangle $u = (OPQ)$, whose median through O is the Brocard axis of the initial triangle t.
(6) The triangle u is isosceles and symmetric on the Brocard axis. The feet G, F of the altitudes of u from P and Q, respectively, coincide with the Brocard points of t.
(7) The triangles u, $u' = (PRF)$ and $u'' = (QRG)$ are similar. The similarity ratio of the two last to the first is equal to the sine of the Brocard angle.

Deferring once again the proofs at the end (§6), I shall pass to a third group, using now inversions instead of projectivities. For a reason that will be made evident shortly I call the corresponding group the *Brocard* dihedral group of the triangle.

4. Brocard dihedral group of a triangle

Once again we start with a triangle $t = (ABC)$ and its circumcircle c. Then we consider the Moebius transformations that permute the vertices of t. It is true that through such maps the sides are not mapped to sides. We do not have proper maps of the triangle's set of points onto itself, but we have a group that permutes its vertices, is isomorphic to D_3 and, as we will see, has intimate relations with the previous one and the geometry of the triangle.

Everything is based on the well known fact that a Moebius transformation is uniquely defined by prescribing three points and their images. Thus, fixing a vertex, A say, of the triangle and permuting the other two, we get a Moebius involution, I_a say. Analogously are defined the other two involutions I_b and I_c. I call them the *Brocard reflexions* of the triangle. Two of them generate the whole group. By the well known property of Moebius transformations, we know that all of them preserve the circumcircle c.

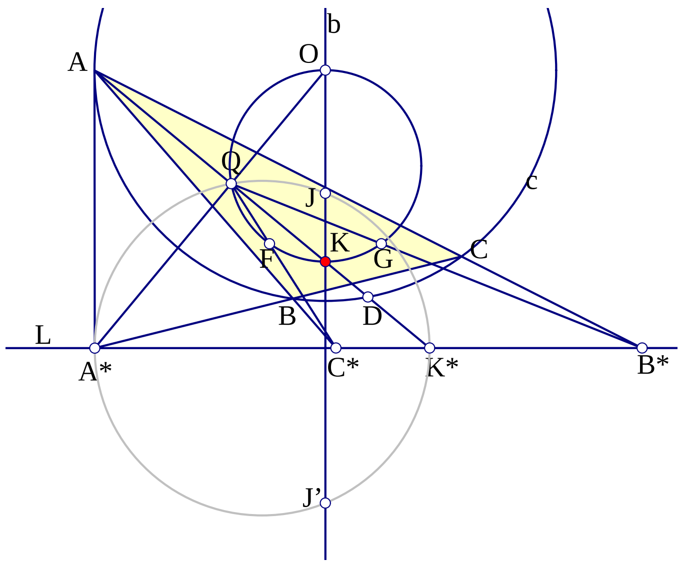

Figure 6. Brocard reflexion

I cite some properties of I_a that are easy to prove:

(1) On the points of the circumcircle the Brocard reflexion I_a coincides with the corresponding Lemoine reflexion.
(2) I_a leaves invariant each member of the bundle of circles through its fixed points A and D (D being the intersection of the symmedian from A with the circumcircle).
(3) I_a leaves invariant each member of the bundle of circles that is orthogonal to the previous one (i.e. the circles which are orthogonal to the symmedian AD and the circumcircle).
(4) In particular I_a leaves invariant the symmedian from A and maps the symmedian point K to the intersection K^* of the Lemoine axis with that symmedian.
(5) I_a permutes the circles of the bundle generated by the circumcircle and the Lemoine axis of t. The same happens with the orthogonal bundle to the previous one, which is the bundle generated by the Apollonian circles of t.

(6) I_a interchanges the circumcenter O with the pole A^* of the symmedian at A. It maps also the Brocard axis b onto the circle through the isodynamic points and A^*.
(7) All the circles through O, Q are mapped by I_a to lines through A^*. In particular the Brocard circle is mapped to the Lemoine axis.
(8) The line AB is mapped by I_a to the circle through A, C, tangent to this line at A.
(9) I_a maps the Brocard points F, G to the intersection points B^*, C^* of the sides AC and AB with the Lemoine axis respectively.

We pass now to the Moebius transformation that recycles the vertices of the triangle $t = (ABC)$. It is the product of two Brocard reflexions $f = I_b \circ I_a$. It is of order three: $f^3 = 1$ and I call it the *Brocard rotation*. The geometric properties of this transformation are related to the so called *characteristic parallelogram* of it. This is generally defined, for every Moebius transformation (may be degenarated), as the parallelogram whose vertices are the two fixed points and the poles of f and of its inverse f^{-1}. A short discussion of this parallelogram will be found in §8. Here are the main properties of our Brocard Rotation.

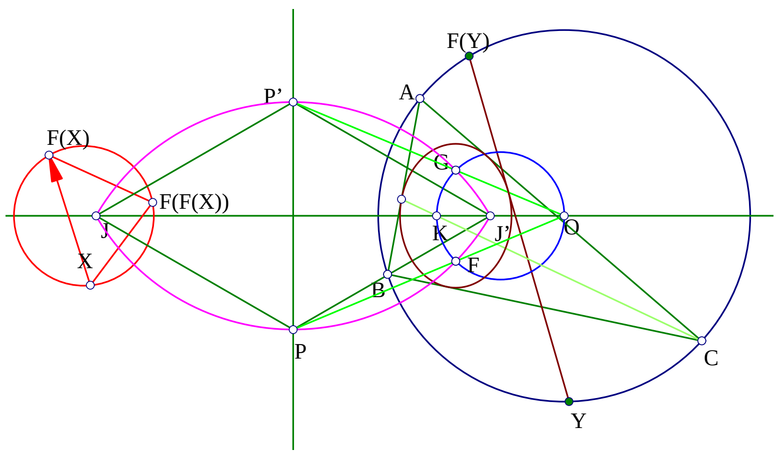

Figure 7. Brocard rotation

(10) On the points of the circumcircle c of t the Brocard Rotation coincides with the corresponding Lemoine rotation.
(11) The characteristic parallelogram of f is a rhombus with two angles of measure $\pi/3$. The vertices at these angles are the fixed points of f. They also coincide with the isodynamic points of the triangle. The other vertices of the parallelogram (angles $2\pi/3$) coincide with the inverses of the Brocard points with respect to the circumcircle.
(12) f leaves invariant every circle of the bundle of circles, generated by the circumcircle of t and its Brocard circle (circle throuch circumcenter and Brocard points).

(13) All circles of the bundle, which is orthogonal to the previous, pass through the isodynamic points J, J' of t. Each circle c of this bundle is mapped to a circle c' of the same bundle, which makes an angle of $\pi/6$ with c. In particular the Apollonian circles of the triangle are cyclically permuted by f.

(14) Every point X of the plane defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which shares with t the same isodynamic points J, J', hence Brocard and Lemoine axes. Conversely, every triangle whose isodynamic points are J and J' is an orbital triangle of f.

(15) The Brocard points of all the above orbital triangles s fill the two $\pi/3$-angled arcs JPJ' and $JP'J'$ on the two circles with centers at the poles P, P' of f, joining the isodynamic points J and J'.

(16) The orbital triangles s, as above, which have X on the circumcircle of t, are all circumscribed to the Brocard ellipse c' of t. They are precisely the only triangles that have c and c' as their circumcircle and Brocard ellipse, respectively.

(17) The other two points of the orbital triangle of the circumcenter O, are the two Brocard points of t.

(18) The second Brocard triangle $A_2B_2C_2$ is an orbital triangle of f.

5. Proofs on Steiner

A convenient method to define the two Steiner ellipses of a triangle, is to use a projectivity F, that maps the vertices of an equilateral triangle $t' = (A'B'C')$ onto the vertices of an arbitrary triangle $t = (ABC)$ and the center of t' onto the centroid of t. As is well known, prescribing four points and their images, uniquely determines a projectivity of the plane. Thus the previous conditions uniquely determine F (up to permutation of vertices). Let a', b' be the circumcircle and incircle, correspondingly of t'. Their images $a = F(a')$ and $b = F(b')$ are correspondingly the exterior and interior Steiner ellipses of t.

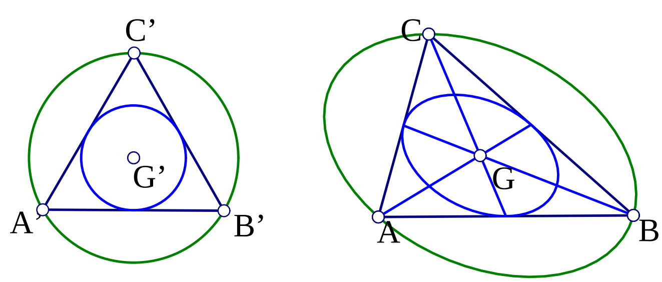

Figure 8. Creating the two Steiner ellipses of a triangle

From the general properties of projectivities result the main properties of Steiner's ellipses of the triangle t:

(1) From the invariance of cross-ratio, and the fact that F preserves the middles of the sides, follows that F preserves also the line at infinity. Thus, the images of circles are ellipses.
(2) The same reason implies, that the tangent to the outer ellipse at the vertex is parallel to the opposite side of the triangle.
(3) The same reason implies, that the centers of the two ellipses coincide with G and the ellipses are homothetic with ratio 2, with respect to that point.
(4) The invariance of cross-ratio implies also, that the *Steiner involution*, defined as the projectivity that fixes A and permutes B, C, coincides (on points of the conic) with the conjugation $X \mapsto Y$, where XY is parallel to a. It leaves the line at infinity fixed and coincides with the isotomic conjugation with respect to the median from A. The median being a conjugate direction to a with respect to the conic.
(5) The Fregier point of the involution I_a is the point at infinity of line $a = BC$ and the line of fixed points of I_a is the median from A.

The *isotomic rotation* is the projectivity $f = I_b \circ I_a$. One sees immediately that it has order three: $f^3 = 1$, that preserves the conic and cycles the vertices of the triangle. Besides it fixes the centroid G and cycles the middles of the sides. All the statements of §2, about orbital triangles, follow immediately from the previous facts and the property of f, to be conjugate, via F, to a rotation by $2\pi/3$ about G. For the statement on the particular orbital triangle of the circumcenter O of t, it suffices to do an easy calculation with trilinears. Actually the Euler line passes also through the symmetric O' of O with respect to G, which is one of the intersection points of the two conics of the figure below.

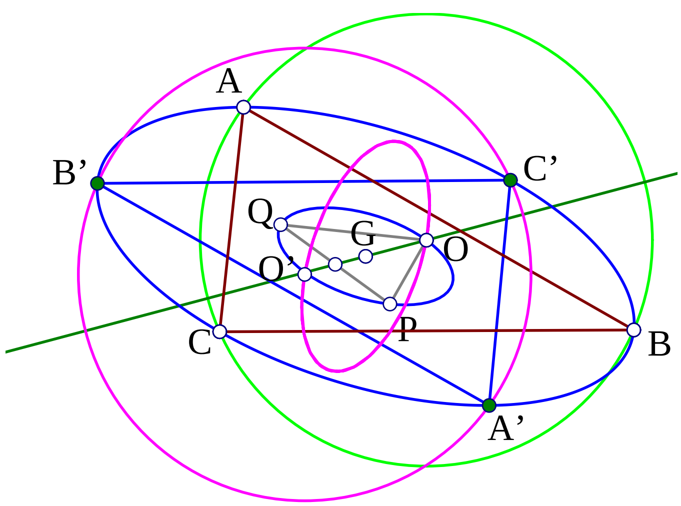

Figure 9. Circumcenters of orbital triangles

One of the conics is the member of the conics-family passing through O. The other ellipse has the same axes with the previous one and is the locus of the circumcenters of orbital triangles $u = (X, f(X), f^2(X))$, for X on the outer Steiner ellipse. O' is the circumcenter of the triangle $t' = (A'B'C')$ which is symmetric to t with respect to G.

6. Proofs on Lemoine

A convenient method to define the Brocard ellipse of a triangle, is to use a projectivity F, that maps the vertices of an equilateral triangle $t' = (A'B'C')$ onto the vertices of an arbitrary triangle $t = (ABC)$ and the center of t' onto the symmedian point of t. These conditions uniquely determine F (up to permutation of vertices).

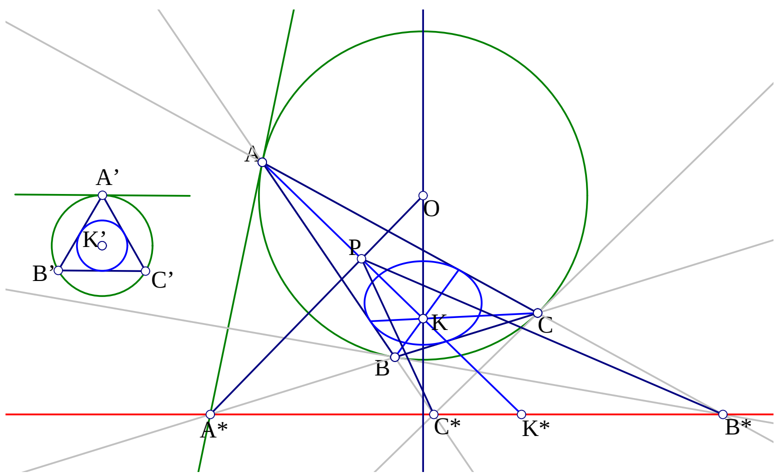

Figure 10. Creating the Brocard ellipse of a triangle

F maps the incircle of t' to the Brocard ellipse of t and the circumcircle of t' to the circumcircle of t. To see the later, notice that F preserves the cross ratio of a bundle of four lines through a point. Now the tangent of t' at A', the two sides $A'B'$, $A'C'$ and the median of t' from A' form a harmonic bundle. The same is true for the tangent of t at A the two sides AB, AC and the symmedian from A. Thus F maps the tangent of t' at A' to the tangent of t at A, and analogous properties hold for the other vertices. This forces the circumcircle of t to coincide with the image, under F, of the circumcircle of t'. The other statement, on the Brocard ellipse, follows from the fact, that this ellipse is characterized as the unique conic tangent to the sides of the triangle at the traces of the symmedians from the opposite vertices. The main properties of the *Lemoine reflexion* I_a result from the fact that it is conjugate, via F, to the reflexion of t' with respect to its median from A'. Thus the line of fixed points of I_a coincides with the symmedian from A. The intersection point A^* of the line BC with the tangent at A is the image, via F, of the point at infinity of the line $B'C'$. Analogous properties hold for the points

B^* and C^*. Since these points are known to be on the Lemoine axis, this implies that the line at infinity is mapped, via F, to the Lemoine axis of the triangle. All the lines through A^* remain invariant under I_a, hence this point coincides with the Fregier point of the involution.

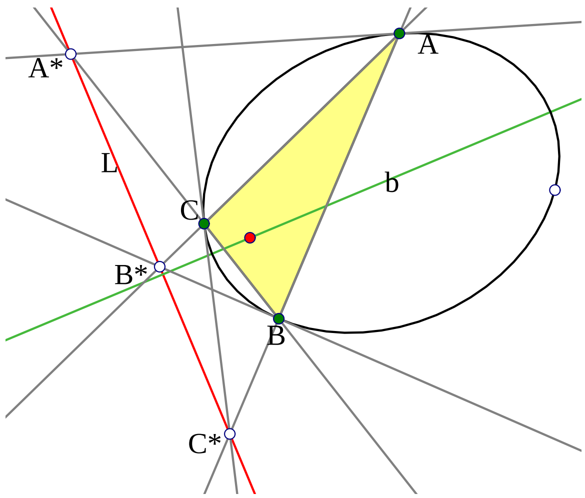

Figure 11. Orbital triangles

The *Lemoine rotation* is the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, that preserves the circumcircle and cycles the vertices of the triangle. Besides it fixes the symmedian point K of the triangle and cycles the symmedians. f is conjugate, via F, to a rotation by $2\pi/3$ about K'. f leaves invariant the family of conics generated by the circumcircle and the Brocard ellipse. This family is the image, under F, of the bundle of concentric circles about K'. In particular the line at infinity is mapped onto the Lemoine axis of t, which is also invariant under f. The conics of the family, left invariant by f, are all symmetric with respect to the Brocard diameter b. Besides all orbital triangles $s = (A = X, B = f(X), C = f(f(X)))$ of f have the property shown in the above figure.

In this figure the point A^* is the intersection point of BC and the tangent at A of the conic-family member passing through A. Analogously are defined , B^* and C^*. The three points lie on the Lemoine axis L of t and are cyclically permuted by f. The proof is a repetition of the argument on harmonic bundles at the beginning of the paragraph. This has though a nice consequence. First, if A is on the Brocard diameter b of t, which is the symmetry axis of all the conics of the invariant family, then the coresponding orbital triangle s is symmetric. Besides the lines AB and AC pass through two fixed points C^* and B^* of L respectively. In fact, in that case, the tangent at A meets L at its point at infinity. Consequently the corresponding BC is parallel to L and s is isosceles. In addition, since f cycles the corresponding points A^*, B^*, C^*, the two last points are the image of the point at infinity of L, under f and its image respectively. Thus they are independent of the position of A on b.

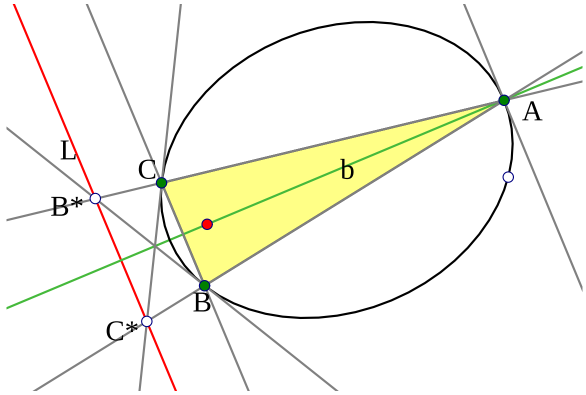

Figure 12. The orbit of the point at infinity of L

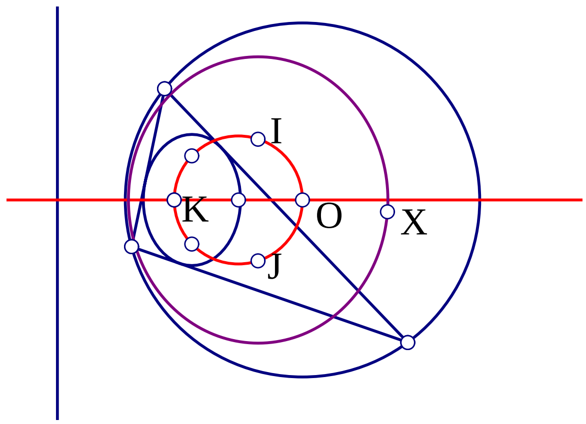

Figure 13. Focal points of the conics

Below B^*, C^* will be identified with the inverses of the Brocard points of t with respect to the circumcircle. Notice that the Brocard points of t are the focal points of the Brocard ellipse and they lie on the Brocard circle with diameter OK. It is well known, that in general the focal points of a family of conics lie on certain cubics. For a reference, see our paper with Apostolos Thoma [2], where we investigated such cubics from a geometric point of view. In the present case the family consists of conics that are symmetric with respect to the Brocard axis and the cubic must be reducible and equal to the product of a circle and a line. In fact a calculation shows that the cubic is the union of the Brocard cirlce and the Brocard axis. All points X inside the circumcircle of t define family members whose focal points are on the Brocard circle. All points X outside the circumcircle of t define family members whose focal points are on the Brocard axis. For X varying on b there are two positions, where the legs of the orbital isosceli contain the foci of the corresponding conic-member through X. One of these points is the center O of the circumcircle. Notice that the family of conics is generated also from the

Lemoine axis (squared) and the circumcircle. This representation makes simpler the computations of a proof of the last statements of §3, on the orbital triangle of the circumcenter. Another geometric proof of this fact may be derived from the arguments of the two next paragraphs.

7. Proofs on Brocard

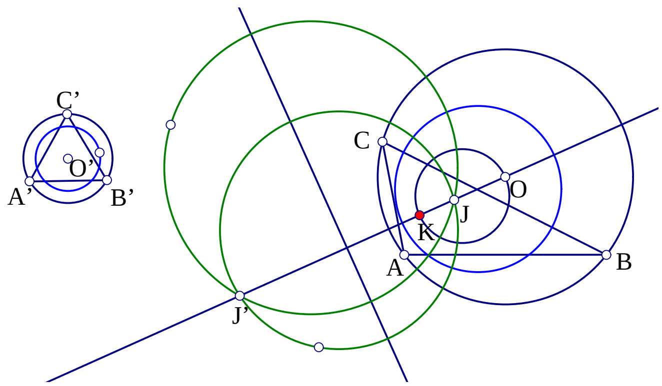

Figure 14. The isodynamic bundles of the triangle

In contrast to projectivities that need four, Moebius transformations are determined completely by three pairs of points. Imitating the procedures of the previous paragraphs, we define the Moebius transformation F that sends the vertices of an equilateral triangle $t' = (A'B'C')$ to the vertices of an arbitrary triangle $t = (ABC)$. Since Moebius transformations, preserve the set of circles and lines, the circumcircle of t' is mapped on the circumcircle of t. Moreover the bundle of concentric circles to the circumcircle of t' maps to the bundle Σ of circles generated by the circumcircle of t and its Lemoine axis. Below I call Σ the *Brocard bundle* of t. This is a hyperbolic bundle with focal (or limiting) points coinciding with the isodynamic points J, J' of t. Since F is conformal it maps the lines from O' to the circle bundle that is orthogonal to the previous one. All circles of this bundle pass through the isodynamic points. All these facts result immediately from the fact that the altitudes of t' map onto the corresponding Apollonian circles of t. This in turn follows from the invariance of the complex cross ratio, by considering the cross ratio of the vertices $(ABCD) = (A'B'C'D') = 1$. D on the circumcircle is uniquely determined by this condition and coincides with the trace of the symmedian from A. The conformality of Moebius transforms implies also that the Apollonian circles meet at J at angles equal to $\pi/3$. Below I call the bundle Σ' of circles through J, J' the *Apollonian bundle* of t. Now to the proofs of the statements in §4.

The first statement (1) is a general fact on Moebius transformations preserving a circle c. Given three pairs of points on c, there is a unique Moebius f and a unique projectivity f' preserving c and corresponding the points of the pairs. f and f' coincide on points $X \in c$. In fact, taking cross ratios $(ABCX)$ in complex or by projecting the points on a line, from a fixed point, $Z \in c$ say, gives the same result. The same is true for the images $(A'B'C'X')$ under both transformations, thus the images of X under f and f' coincide.

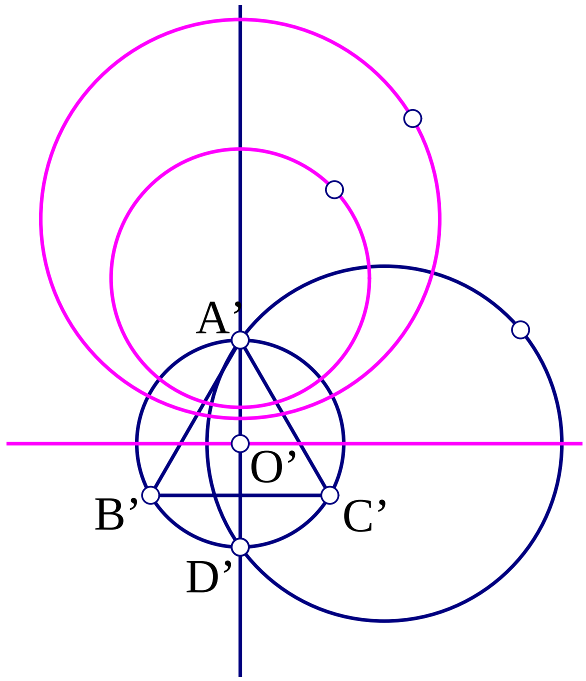

Figure 15. The I'_a invariant bundles

The next two statements (2,3) follow immediately from the fact that I_a is conjugate, via F, to the Moebius transformation I'_a fixing A', D' and mapping B' to C'. A short calculation shows that I'_a preserves the circles passing through A', D' and also preserves the circles of the orthogonal bundle to the previous one. These two I'_a-invariant bundles, map under F to the corresponding I_a-invariant bundles of the statements. The previous argument shows also that the bundle of concentric circles at O' is permuted by I'_a, consequently the same is true for the bundle of lines through O'. But these two bundles map under F to the main bundles of our configuration, the Brocard Σ and the Apollonian Σ' correspondingly. This proves also statement (4).

Next statement (5) follows from the invariance of cross ratio, along the I_a-invariant symmedian from A, and the fact that the Lemoine axis is the polar of the symmedian point with respect to the circumcircle. A consequence of this, taking into account that I_a permutes the Brocard bundle, is that the Brocard circle of t maps via I_a to the Lemoine axis.

From the previous considerations, on the Brocard and Apollonian bundles, follows that I_a does the following: (a) It interchanges O, P, (b) sends Q (the projection of the circumcenter on the symmedian) at the point at infinity, (c) maps the circles with center at Q to circles with the same property, (d) maps the lines e

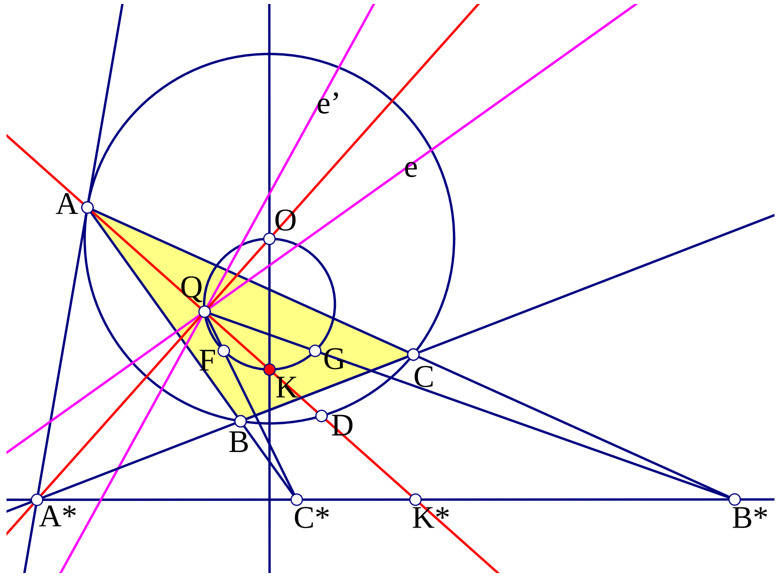

Figure 16. I_a on Brocard points

through Q to their symetrics e' with respect to PQ (or the symmedian at A). As a consequence I_a maps the line QB^* to the line QC^* and points G, F onto C^*, B^* correspondingly. Consider now the image of line AB via I_a. By the properties just described, points A, B, C^* are mapped onto A, C, G correspondingly. Also the point at infinity is mapped onto Q, thus the line maps to a circle r passing through the points (A, Q, C, G). It is trivial to show that the circle through the points (A, Q, C) is tangent to line AB at A. This identifies G with one of the two Brocard points of t. Statements (6-10) follow immediately from the previous remarks. Before to proceed to the proofs of the remaining statements of §4, let us review some facts about the characteristic parallelograms of Moebius transformations.

8. Characteristic parallelogram

For proofs of properties of Moebius transformations and their characteristic parallelogram I refer to Schwerdtfenger [3]. The characteristic parallelogram of a Moebius transformation f has one pair of opposite vertices coinciding with the fixed points of f, the other pair of vertices coinciding with the poles of f and f^{-1} respectively. The parallelogram can be degenerated or have infinite sides. It characterizes completely f, when we know which vertices are the fixed points and which are the poles. In the image below F, F' are the fixed points of f, P is its pole and P' is the pole of f^{-1}. Triangles zFP, $Fz'P'$ and $zz'F'$ are similar in that orientation. This defines the recipe by which we construct geometrically $z' = f(z)$.

Moebius transformations f permute the bundle Σ of circles which pass through their fixed points F, F'. Each circle a of Σ is mapped to a circle a' of the same

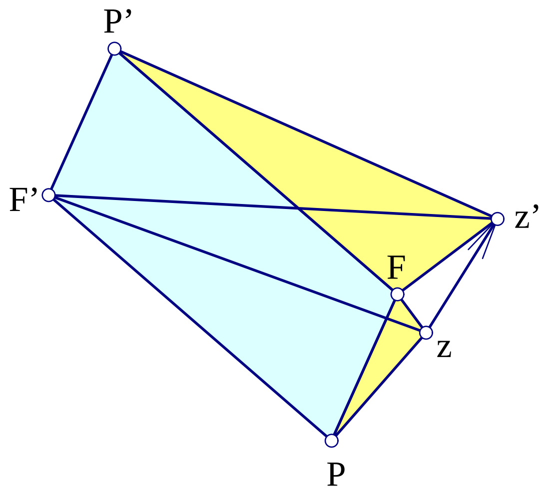

Figure 17. Building the image $z' = f(z)$

bundle, such that the angle at F is the same with the angle of the characteristic parallelogram at the pole P. In some sense the circles of Σ are *rotated* about the fixed points of f. The picture is complemented by the bundle Σ', which is orthogonal to the previous one. This is also permuted by f.

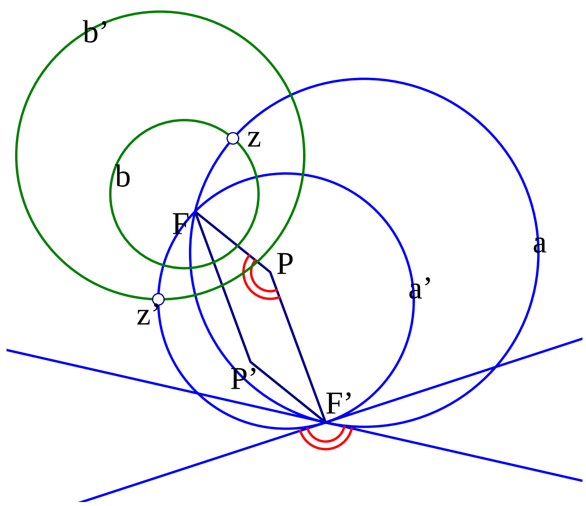

Figure 18. Characteristic bundles of a Moebius transformation

The *elliptic* Moebius transformations are characterized by their property to leave invariant a circle. The circle then belongs to the bundle Σ', whose all members remain also invariant by f. In fact, in that case f is conjugate to a rotation, and by this conjugation the two bundles correspond to the set of concentric circles about the rotation-center (Σ') and the set of lines through the rotation-center (Σ). In addition the parallelogram is then a rhombus.

Now to the proofs of the properties of Brocard rotations f of §4, preserving the notations introduced there. Since these transformations preserve the circumcircle of the triangle t, they are elliptic. Since they are conjugate, via the map F, to Rotations by $2\pi/3$, their characteristic parallelogram is a rhombus with an angle (at the pole) equal to $2\pi/3$. From the properties of F we know that the fixed points of f coincide with the isodynamic points of the triangle and the Apollonian circles are members of the bundle Σ, permuted by f. The Lemoine axis, being axis of symmetry of the isodynamic points, contains the other vertices of the rhombus. The other bundle Σ', of circles left invariant by f, coincides with the bundle generated by the circumcircle and the Lemoine axis. Later bundle contains the Brocard circle. The statement on orbital triangles follows from the corresponding property of Lemoine rotations, since the two maps coincide on the circumcircle.

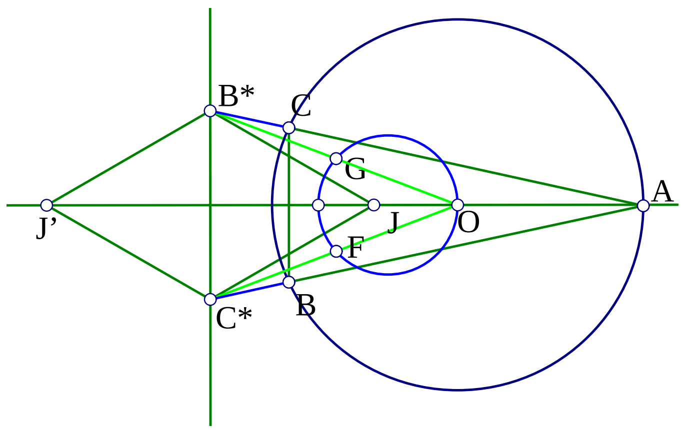

Figure 19. Projections of Brocard points on Lemoine axis

The fact that the circumcenter O, together with the two Brocard points F, G build an orbital triangle of f, follows now easily from the fact that $f = I_b \circ I_a$. In fact, from our discussion, on Brocard reflexions, we know that I_a maps the circumcenter onto A^*, the intersection of side $a = BC$ with the Lemoine axis. Then I_b, as shown there, maps A^* to one Brocard point. A similar argument proves that applying again f we get the other Brocard point. Analogously one proves that the second Brocard triangle is also an orbital triangle of f. All the statements (10-19) follow from the previous remarks.

Especially the statement about the fact that P, P' are the projections, from the circumcenter O, of the Brocard points, on the Lemoine axis, follows also easily from our arguments. In fact, the equibrocardian isosceles triangle $t = (ABC)$ of the previous picture, is also an orbital triangle of the corresponding Lemoine rotation. From there we know that its legs pass through the fixed points B^*, C^*. These points are identified as the images of the point at infinity of the Lemoine axis under the Lemoine Rotation. But this rotation coincides also with the Brocard rotation on that axis. This identifies P, P' with the other vertices of the characteristic parallelogram.

9. Remarks

(1) For every point P of the triangle's plane (e.g. some triangle center), one can define a projectivity F analogous to the one used in the two examples and establishing the conjugacy of the group G with the dihedral D_3. The projectivity F is required to map the vertices of the equilateral triangle to the vertices of the arbitrary triangle t. In addition, it is required to map the center P' of the equilateral to the selected point P. These conditions completely determine F and there are several phenomena, generalizing the previous examples. The bundle of circles centered at P' maps to a family Σ of conics. One of these conics, $c \in \Sigma$, circumcscribes t, one other being inscribed and touching the triangle's side at the feet of the cevians from P. One can define analogously the action of D_3, preserving c and permutting the vertices of the triangle. The properties of this action, reflect naturally properties of the point P with respect to triangle t. The action leaves invariant the whole family Σ.

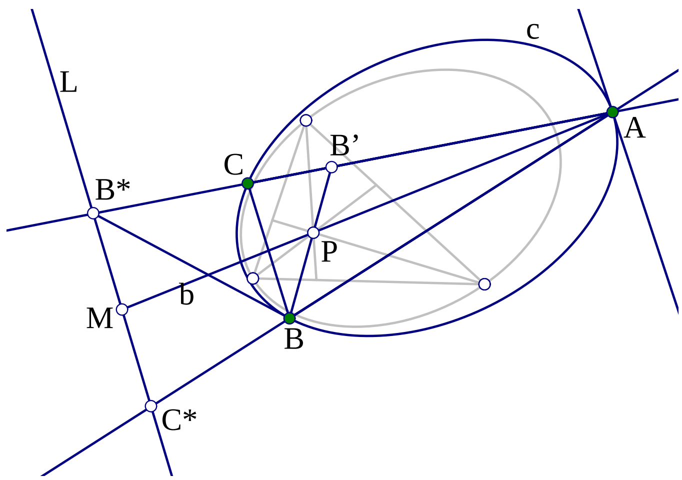

Figure 20. The limit points of the conics-family

Also, using essentially the same arguments as in the examples, one can show, that the line at infinity maps via F to the *trilinear polar* of P. The trilinear polar being then a singular member (double line) L of Σ. Besides all orbital triangles $t = (ABC)$ which have a side, BC say, parallel to this line, have the other two sides passing through two fixed points C^*, B^* of L, whereas the tangent to the member-conic c circumscribing the triangle at the other point A of the triangle is also parallel to L. The line $b = PA$, passes through the middle M of B^*C^* and is the conjugate direction to L, with respect to every conic of the family. In this case also the corresponding projective rotation f recycles points B^*, C^* and the point

at infinity of line L.
(2) The data L, P and the location of points B^*, C^* on L uniquely determine the invariant family of conics Σ and the related orbital triangles. In fact, once B^*, C^* are known, the line MP, where M is the middle of B^*, C^*, is conjugate to the direction of L, with respect to all the conics of Σ. A point A on this line can be determined, so that a special orbital triangle ABC can be constructed from the previous data. In fact, point B' on AB^* satisfies the condition that the four points $(ACB'B^*) = 1$, form a harmonic ratio. A triangle ABC is immediately constructed, so that BB^* and BB' are its bisectors and BC is parallel to L. Consequently the projectivity F can be defined, and from this the whole family is also constructed.

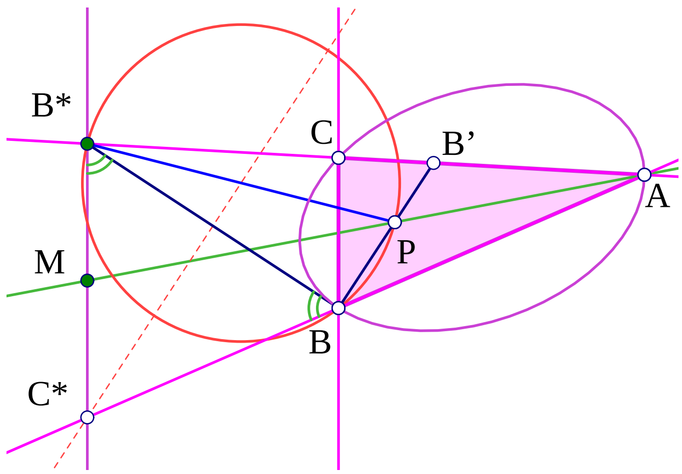

Figure 21. Special orbital triangle determined from B^*, C^*, P

(3) The previous considerations give a nice description of the set of triangles having a given line L and a given point $P \notin L$ as their trilinear polar with respect to P. They are orbital triangles of actions of the previous kind and they fall into families. Each family is characterized by the location of its limit points B^*, C^* on L.
(4) An easy calculation shows that the focal points of the members of Σ describe a singular cubic, self-intersecting at P. Besides the asymptotic line of this cubic coincides with b. When P is the Symmedian-point, the corresponding cubic coincides with the reducible one, consting of the Brocard circle and the Brocard line.
(5) Inscribed conics and corresponding actions of D_3, permutting their contact points with the sides of the triangle, could be also considered. They offer though nothing new, since they are equivalent to actions of the previous kind.
(6) In all the above groups of projectivities, the rotations are identical to the projectivities fixing the point P and cycling the vertices. One could start from such a projectivity and show the existence and invariance of the resepective family of conics. I prefer however the variant with the circumconics which introduces them

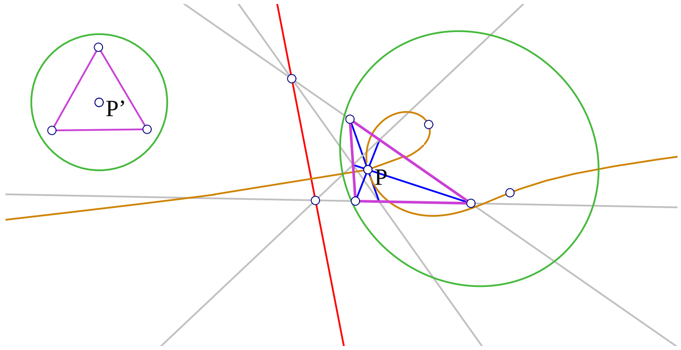

Figure 22. The focal cubic of the invariant family

into the play right from the beginning.
(7) The Brocard action is a singularium. It does not fit completely into the framework of circumconics and projectivities. As we have seen however, it has a close relationship to the Lemoine dihedral group. On Brocard Geometry there is an alternative exposition by John Conway [4], described in a letter to Hyacinthos .
(8) Finally a comment on the many figures used. They are produced with *EucliDraw*. This is a program, developed at the University of Crete, that does quickly the job of drawing interesting figures. It has many tools that do complicated jobs, reflecting the fact that it uses a conceptual granularity a bit wider than the very basic axioms. I am quite involved in its development and hope that other geometers will find it interesting, since it does quickly its job (sometimes even correctly), and new tools are continuously added. The program can be downloaded and tested from *www.euclidraw.com*.

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[4] J. H. Conway, Letter to Hyacinthos.

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