Generalized Mandart Conics

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Abstract. We consider interesting conics associated with the configuration of three points on the perpendiculars from a point \( P \) to the sidelines of a given triangle \( ABC \), all equidistant from \( P \). This generalizes the work of H. Mandart in 1894.

1. Mandart triangles

Let \( ABC \) be a given triangle and \( A'B'C' \) its medial triangle. Denote by \( \Delta, R, r \) the area, the circumradius, the inradius of \( ABC \). For any \( t \in \mathbb{R} \cup \{ \infty \} \), consider the points \( P_a, P_b, P_c \) on the perpendicular bisectors of \( BC, CA, AB \) such that the signed distances verify \( A'P_a = B'P_b = C'P_c = t \) with the following convention: for \( t > 0 \), \( P_a \) lies in the half-plane bounded by \( BC \) which does not contain \( A \). We call \( T_t = P_aP_bP_c \) the \( t \)-Mandart triangle with respect to \( ABC \). H. Mandart has studied in detail these triangles and associated conics ([5, 6]). We begin a modernized review with supplementary results, and identify the triangle centers in the notations of [4]. In the second part of this paper, we generalize the Mandart triangles and conics.

The vertices of the Mandart triangle \( T_t \), in homogeneous barycentric coordinates, are

\[
\begin{align*}
P_a &= -ta^2 : a\Delta + tS_C : a\Delta + tS_B, \\
P_b &= b\Delta + tS_C : -tb^2 : b\Delta + tS_A, \\
P_c &= c\Delta + tS_B : c\Delta + tS_A : -tc^2,
\end{align*}
\]

where

\[
S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.
\]

Proposition 1 ([6, §2]). The points \( P_a, P_b, P_c \) are collinear if and only if \( t^2 + Rt + \frac{1}{2}Rr = 0 \), i.e.,

\[
t = \frac{R \pm \sqrt{R^2 - 2Rr}}{2} = \frac{R \pm OI}{2}.
\]

The two lines containing those collinear points are the parallels at \( X_{10} \) (Spieker center) to the asymptotes of the Feuerbach hyperbola.
In other words, there are exactly two sets of collinear points on the perpendicular bisectors of \(ABC\) situated at the same (signed) distance from the sidelines of \(ABC\). See Figure 1.

![Figure 1. Collinear \(P_a, P_b, P_c\)](image)

**Proposition 2.** The triangles \(ABC\) and \(P_aP_bP_c\) are perspective if and only if

1. \(t = 0\): \(P_aP_bP_c\) is the medial triangle, or
2. \(t = -r\): \(P_a, P_b, P_c\) are the projections of the incenter \(I = X_3\) on the perpendicular bisectors.

In the latter case, \(P_a, P_b, P_c\) obviously lie on the circle with diameter \(IO\). The two triangles are indirectly similar and their perspector is \(X_8\) (Nagel point).

**Remark.** For any \(t\), the triangle \(Q_aQ_bQ_c\) bounded by the parallels at \(P_a, P_b, P_c\) to the sidelines \(BC, CA, AB\) is homothetic at \(I\) (incenter) to \(ABC\).

**Proposition 3.** The Mandart triangle \(T_t\) and the medial triangle \(\mathcal{A}'B'C'\) have the same area if and only if either :

1. \(t = 0\): \(T_t\) is the medial triangle,
2. \(t = -R\),
3. \(t\) is solution of: \(t^2 + Rt + Rr = 0\).
This equation has two distinct (real) solutions when \( R > 4r \), hence there are three Mandart triangles, distinct of \( A'B'C' \), having the same area as \( A'B'C' \). See Figure 2. In the very particular situation \( R = 4r \), the equation gives the unique solution \( t = -2r = -\frac{R}{2} \) and we find only two such triangles. See Figure 3.

**Proposition 4** ([5, §1]). As \( t \) varies, the line \( P_bP_c \) envelops a parabola \( P_a \).

The parabola \( P_a \) is tangent to the perpendicular bisectors of \( AB \) and \( AC \), to the line \( B'C' \) and to the two lines met in proposition 1 above. Its focus \( F_a \) is the
projection of $O$ on the bisector $AI$. Its directrix $\ell_a$ is the bisector $A'X_{10}$ of the medial triangle. See Figure 4.

![Figure 4. The parabola $\mathcal{P}_a$](image)

Similarly, the lines $P_cP_a$ and $P_aP_b$ envelope parabolas $\mathcal{P}_b$ and $\mathcal{P}_c$ respectively. From this, we note the following.

(i) The foci of $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ lie on the circle with diameter $OI$.

(ii) The directrices concur at $X_{10}$.

(iii) The axes concur at $O$.

(iv) The contacts of the lines $P_bP_c$, $P_cP_a$, $P_aP_b$ with $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ respectively are collinear. See Figure 5.

These three parabolas are generally not in the same pencil of conics since their jacobian is the union of the perpendicular at $O$ to the line $IX_{10}$ and the circle centered at $X_{10}$ having the same radius as the Fuhrmann circle: the polar lines of any point on this circle in the parabolas concur on the line and conversely.

2. Mandart conics

**Proposition 5 ([6, §7]).** The Mandart triangle $\mathbf{T}_t$ and the medial triangle are perspective at $O$. As $t$ varies, the perspectrix envelopes the parabola $\mathcal{P}_M$ with focus $X_{124}$ and directrix $X_3X_{10}$. 
We call $P_M$ the Mandart parabola. It has equation
\[ \sum_{\text{cyclic}} \frac{x^2}{(b - c)(b + c - a)} = 0. \]

Triangle $ABC$ is clearly self-polar with respect to $P_M$. The directrix is the line $X_3X_{10}$ and the focus is $X_{124}$. $P_M$ is inscribed in the medial triangle with perspector
\[ X_{1146} = ((b - c)^2(b + c - a)^2 : \cdots : \cdots), \]
the center of the circum-hyperbola passing through $G$ and $X_8$ with respect to this triangle. The contacts of $P_M$ with the sidelines of the medial triangle lie on the perpendiculars dropped from $A$, $B$, $C$ to the directrix $X_3X_{10}$. $P_M$ is the complement of the inscribed parabola with focus $X_{109}$ and directrix the line $IH$. See Figure 6.

**Proposition 6** ([5, 2, p.551]). The Mandart triangle $T_t$ and $ABC$ are orthologic. The perpendiculars from $A$, $B$, $C$ to the corresponding sidelines of $P_aP_bP_c$ are concurrent at
\[ Q_t = \left( \frac{a}{aS_A + 4\Delta t} : \cdots : \cdots \right). \]
As $t$ varies, the locus of $Q_t$ is the Feuerbach hyperbola.
Remark. The triangles $A'B'C'$ and $T_t$ are also orthologic at $Q'$, the complement of $Q_t$. Denote by $A_1B_1C_1$ the extouch triangle (see [3, p.158, §6.9]), i.e., the cevian triangle of $X_8$ (Nagel point) or equivalently the pedal triangle of $X_{40}$ (reflection of $I$ in $O$). The circumcircle $C_M$ of $A_1B_1C_1$ is called Mandart circle. $C_M$ is therefore the pedal circle of $X_{40}$ and $X_{84}$ (isogonal conjugate of $X_{40}$), the cevian circumcircle of $X_{189}$ (cyclocevian conjugate of $X_8$). $C_M$ contains the Feuerbach point $X_{11}$. Its center is $X_{1158}$, intersection of the lines $X_1X_{104}$ and $X_8X_{40}$. The second intersection with the incircle is $X_{1364}$ and the second intersection with the nine-point circle is the complement of $X_{934}$. See Figure 7. The Mandart ellipse $E_M$ (see [6, §§3,4]) is the inscribed ellipse with center $X_9$ (Mittenpunkt) and perspector $X_8$. It contains $A_1, B_1, C_1, X_{11}$ and its axes are parallel to the asymptotes of the Feuerbach hyperbola. See Figure 7.

The equation of $E_M$ is:

$$\sum_{\text{cyclic}} (c + a - b)^2(a + b - c)^2x^2 - 2(b + c - a)^2(c + a - b)(a + b - c)yz = 0$$
From this, we see that $C_M$ is the Joachimsthal circle of $X_{40}$ with respect to $E_M$: the four normals drawn from $X_{40}$ to $E_M$ pass through $A_1, B_1, C_1$ and
\[
F' = ((b + c - a)((b - c)^2 + a(b + c - 2a)) : \cdots : \cdots),
\]
the reflection $X_{11}$ in $X_9$.  

The radical axis of $C_M$ and the nine-point circle is the tangent at $X_11$ to $E_M$ and also the polar line of $G$ in $P_M$. The projection of $X_9$ on this tangent is the point $X_{1364}$ we met above. Hence, $C_M$, the nine-point circle and the circle with diameter $X_9X_{11}$ belong to the same pencil of (coaxal) circles ([6, §§8,9]).

The radical axis of $C_M$ and the incircle is the polar line of $X_{10}$ in $P_M$.

**Proposition 7.** [6, §§1,2] The Mandart triangle $T_t$ and the extouch triangle are orthologic. The perpendiculars drawn from $A_1, B_1, C_1$ to the corresponding sidelines of $T_t = P_aP_bP_c$ are concurrent at $S$. As $t$ varies, the locus of $S$ is the rectangular hyperbola $H_M$ passing through the traces of $X_8$ and $X_{190} = \left(\frac{1}{b-c} : \frac{1}{c-a} : \frac{1}{a-b}\right)$.

We call $H_M$ the Mandart hyperbola. It has equation
\[
\sum_{\text{cyclic}} (b - c) \left[(c + a - b)(a + b - c)x^2 + (b + c - a)^2yz\right] = 0
\]
\[\text{This point is not in the current edition of [4].}\]
and contains the triangle centers $X_8$, $X_9$, $X_{40}$, $X_{72}$, $X_{144}$, $X_{1145}$, $F'$, and $F''$ antipode of $X_{11}$ on $C_M$. Its asymptotes are parallel to those of the Feuerbach hyperbola. $\mathcal{H}_M$ is the Apollonian hyperbola of $X_{40}$ with respect to $E_M$. See Figure 8.

Figure 8. The Mandart hyperbola

3. Locus of some triangle centers in the Mandart triangles

We now examine the locus of some triangle centers of $T_t = P_aP_bP_c$ when $t$ varies. We shall consider the centroid, circumcenter, orthocenter, and Lemoine point.

**Proposition 8.** The locus of the centroid of $T_t$ is the parallel at $G$ to the line $OI$.

**Proposition 9.** The locus of the circumcenter of $T_t$ is the rectangular hyperbola passing through $X_1$, $X_5$, $X_{10}$, $X_{21}$ (Schiffler point) and $X_{1385}$.\(^2\)

The equation of the hyperbola is

$$
\sum_{\text{cyclic}} (b - c) [bc(b + c)x^2 + a(b^2 + c^2 - a^2 + 3bc)yz] = 0.
$$

\(^2X_{1385}\) is the midpoint of $OI$. 
It has center $X_{1125}$ (midpoint of $IX_{10}$) and asymptotes parallel to those of the Feuerbach hyperbola.

The locus of the orthocenter of $T_t$ is a nodal cubic with node $X_{10}$ passing through $O$, $X_{1385}$, meeting the line at infinity at $X_{517}$ and the infinite points of the Feuerbach hyperbola. The line through the orthocenters of the $t$-Mandart triangle and the $(-t)$-Mandart triangle passes through a fixed point.

The locus of the Lemoine point of $T_t$ is another nodal cubic with node $X_{10}$.

4. Generalized Mandart conics

Most of the results above can be generalized when $X_8$ is replaced by any point $M$ on the Lucas cubic, the isotomic cubic with pivot $X_{69}$. The cevian triangle of such a point $M$ is the pedal triangle of a point $N$ on the Darboux cubic, the isogonal cubic with pivot the de Longchamps point $X_{20}$. ³

For example, with $M = X_8$, we find $N = X_{40}$ and $M' = X_1 = I$.

Denote by $M_aM_bM_c$ the cevian triangle of such a point $M$ (on the Lucas cubic) and the pedal triangle of $N$ on the Darboux cubic. $N^*$ is the isogonal conjugate of $N$ also on the Darboux cubic. We now consider

- $\gamma_M$, inscribed conic in $ABC$ with perspector $M$ and center $\omega_M$, which is the complement of the isotomic conjugate of $M$. It lies on the Thomson cubic and on the line $KM'$ ($K = X_6$ is the Lemoine point),
- $\Gamma_M$, circumcircle of $M_aM_bM_c$ with center $\Omega_M$, midpoint of $NN^*$. $\Gamma_M$ is obviously the pedal circle of $N$ and $N^*$ and also the cevian circle of $M^\circ$, cyclocevian conjugate of $M$ (see [3, p.226, §8.12]). $M^\circ$ is a point on the Lucas cubic since this cubic is invariant under cyclocevian conjugation.

Since $\gamma_M$ and $\Gamma_M$ have already three points in common, they must have a fourth (always real) common point $Z$. Finally, denote by $Z'$ the reflection of $Z$ in $\omega_M$. See Figure 9.

Table 1 gives examples for several known centers $M$ on the Lucas cubic. ⁴ Those marked with * are indicated in Table 2; those marked with ? are too complicated to give here.

### Table 1

<table>
<thead>
<tr>
<th>$M$</th>
<th>$X_8$</th>
<th>$X_2$</th>
<th>$X_4$</th>
<th>$X_7$</th>
<th>$X_{20}$</th>
<th>$X_{69}$</th>
<th>$X_{189}$</th>
<th>$X_{253}$</th>
<th>$X_{329}$</th>
<th>$X_{1042}$</th>
<th>$X_{1034}$</th>
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<tbody>
<tr>
<td>$N$</td>
<td>$X_{40}$</td>
<td>$X_3$</td>
<td>$X_4$</td>
<td>$X_1$</td>
<td>$X_{1498}$</td>
<td>$X_{20}$</td>
<td>$X_{84}$</td>
<td>$X_{64}$</td>
<td>$X_{1490}$</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$M'$</td>
<td>$X_1$</td>
<td>$X_2$</td>
<td>$X_3$</td>
<td>$X_9$</td>
<td>$X_4$</td>
<td>$X_6$</td>
<td>$X_{223}$</td>
<td>$X_{1249}$</td>
<td>$X_{57}$</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$N^*$</td>
<td>$X_{84}$</td>
<td>$X_4$</td>
<td>$X_3$</td>
<td>$X_1$</td>
<td>*</td>
<td>$X_{64}$</td>
<td>$X_{40}$</td>
<td>$X_{20}$</td>
<td>*</td>
<td>$X_{1498}$</td>
<td>$X_{1490}$</td>
</tr>
<tr>
<td>$M^{\circ}$</td>
<td>$X_{189}$</td>
<td>$X_4$</td>
<td>$X_2$</td>
<td>$X_7$</td>
<td>$X_{1032}$</td>
<td>$X_{253}$</td>
<td>$X_5$</td>
<td>$X_{69}$</td>
<td>$X_{1034}$</td>
<td>$X_{20}$</td>
<td>$X_{329}$</td>
</tr>
<tr>
<td>$\omega_M$</td>
<td>$X_9$</td>
<td>$X_2$</td>
<td>$X_6$</td>
<td>$X_1$</td>
<td>$X_{1249}$</td>
<td>$X_3$</td>
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<td>$X_4$</td>
<td>$X_{223}$</td>
<td>$X_{1073}$</td>
<td>$X_{282}$</td>
</tr>
<tr>
<td>$\Omega_M$</td>
<td>$X_{1158}$</td>
<td>$X_5$</td>
<td>$X_5$</td>
<td>$X_1$</td>
<td>?</td>
<td>?</td>
<td>$X_{1158}$</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>$Z$</td>
<td>$X_{11}$</td>
<td>$X_{115}$</td>
<td>$X_{125}$</td>
<td>$X_{111}$</td>
<td>$X_{122}$</td>
<td>$X_{125}$</td>
<td>*</td>
<td>$X_{122}$</td>
<td>*</td>
<td>*</td>
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</tr>
<tr>
<td>$Z'$</td>
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<td>*</td>
<td>*</td>
<td>$X_{1317}$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

³ It is also known that the complement of $M$ is a point $M'$ on the Thomson cubic, the isogonal cubic with pivot $G = X_2$, the centroid.

⁴ Two isotomic conjugates on the Lucas cubic are associated to the same point $Z$ on the nine-point circle.
<table>
<thead>
<tr>
<th>Triangle center</th>
<th>First barycentric coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z'(X_4) )</td>
<td>((b + c - a)(2a^2 - a(b + c) - (b - c)^2))</td>
</tr>
<tr>
<td>( Z'(X_2) )</td>
<td>((2a^2 - b^2 - c^2)^z)</td>
</tr>
<tr>
<td>( Z'(X_4) )</td>
<td>((2a^2 - b^2 - c^2)^z)</td>
</tr>
<tr>
<td>( Z'(X_{20}) )</td>
<td>((18a^2 - 2a^2(b^2 + c^2) - (b - c)^2))</td>
</tr>
<tr>
<td>( Z'(X_{29}) )</td>
<td>((2a^2 - b^2 - c^2)^z)</td>
</tr>
<tr>
<td>( Z'(X_{20}) )</td>
<td>((18a^2 - 2a^2(b^2 + c^2) - (b - c)^2))</td>
</tr>
<tr>
<td>( Z'(X_{29}) )</td>
<td>((2a^2 - b^2 - c^2)^z)</td>
</tr>
<tr>
<td>( N^*(X_{20}) )</td>
<td>((18a^2 - 2a^2(b^2 + c^2) - (b - c)^2))</td>
</tr>
<tr>
<td>( N^*(X_{29}) )</td>
<td>((2a^2 - b^2 - c^2)^z)</td>
</tr>
<tr>
<td>( M'(X_{1032}) )</td>
<td>((18a^2 - 2a^2(b^2 + c^2) - (b - c)^2))</td>
</tr>
<tr>
<td>( N(X_{1034}) )</td>
<td>((18a^2 - 2a^2(b^2 + c^2) - (b - c)^2))</td>
</tr>
<tr>
<td>( M'(X_{1034}) )</td>
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<tr>
<td>( Z(X_{1034}) )</td>
<td>((18a^2 - 2a^2(b^2 + c^2) - (b - c)^2))</td>
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</tr>
<tr>
<td>( Z'(X_{1034}) )</td>
<td>((18a^2 - 2a^2(b^2 + c^2) - (b - c)^2))</td>
</tr>
</tbody>
</table>
**Proposition 10.** Z is a point on the nine-point circle and \( Z' \) is the foot of the fourth normal drawn from \( N \) to \( \gamma_M \).

*Proof.* The lines \( NM_a, NM_b, NM_c \) are indeed already three such normals hence \( \Gamma_M \) is the Joachimsthal circle of \( N \) with respect to \( \gamma_M \). This yields that \( \Gamma_M \) must pass through the reflection in \( \omega_M \) of the foot of the fourth normal. See Figure 9.

**Remark.** \( Z \) also lies on the cevian circumcircle of \( M' \) isotomic conjugate of \( M \) and on the inscribed conic with perspector \( M' \) and center \( M' \).

**Proposition 11.** The points \( M_a, M_b, M_c, M, N, \omega_M \) and \( Z' \) lie on a same rectangular hyperbola whose asymptotes are parallel to the axes of \( \gamma_M \).

*Proof.* This hyperbola is the Apollonian hyperbola of \( N \) with respect to \( \gamma_M \). \( \square \)

**Proposition 12.** The rectangular hyperbola passing through \( A, B, C, H \) and \( M \) is centered at \( Z \). It also contains \( M', N^*, \omega_M \) and \( M' \). Its asymptotes are also parallel to the axes of \( \gamma_M \).

*Remark.* This hyperbola is the isogonal transform of the line \( ON \) and the isotomic transform of the line \( X_{69}M \).
5. Generalized Mandart triangles

We now replace the circumcenter $O$ by any finite point $P = (u : v : w)$ not lying on one sideline of $ABC$ and we still call $A'B'C'$ its pedal triangle. For $t \in \mathbb{R} \cup \{\infty\}$, consider $P_a, P_b, P_c$ defined as follows: draw three parallels to $BC$, $CA$, $AB$ at the (signed) distance $t$ with the conventions at the beginning of the paper. $P_a, P_b, P_c$ are the projections of $P$ on these parallels. See Figure 10.

![Figure 10. Generalized Mandart triangle](image)

In homogeneous barycentric coordinates, these are the points

$$P_a = -a^3t : 2\Delta \cdot \frac{S_Cu + a^2v}{u + v + w} + taS_C : 2\Delta \cdot \frac{S_Bu + a^2w}{u + v + w} + taS_B,$$

$$P_b = 2\Delta \cdot \frac{S_Cv + b^2u}{u + v + w} + tbS_C : -b^3t : 2\Delta \cdot \frac{S_Av + b^2w}{u + v + w} + tbS_A,$$

$$P_c = 2\Delta \cdot \frac{S_Bw + c^2u}{u + v + w} + tcS_B : 2\Delta \cdot \frac{S_Aw + c^2v}{u + v + w} + tcS_A : -c^3t.$$

The triangle $T_t(P) = P_aP_bP_c$ is called $t$–Mandart triangle of $P$.

**Proposition 13.** For any $P$ distinct from the incenter $I$, there are always two sets of collinear points $P_a, P_b, P_c$. The two lines $L_1$ and $L_2$ containing the points are
parallel to the asymptotes of the hyperbola which is the isogonal conjugate of the parallel to \( IP \) at \( X_{40} \). They meet at the point:
\[
(a((b + c)bcu + cS_Cv + bS_Bw) : \cdots : \cdots).
\]

They are perpendicular if and only if \( P \) lies on \( OI \).

**Proof.** \( P_a, P_b, P_c \) are collinear if and only if \( t \) is solution of the equation:
\[
abc(a + b + c)t^2 + 2\Delta \Phi_1(u, v, w) t + 4\Delta^2 \Phi_2(u, v, w) = 0
\]
where
\[
\Phi_1(u, v, w) = \sum_{\text{cyclic}} bc(b + c)u \quad \text{and} \quad \Phi_2(u, v, w) = \sum_{\text{cyclic}} a^2vw.
\]

We notice that \( \Phi_1(u, v, w) = 0 \) if and only if \( P \) lies on the polar line of \( I \) in the circumcircle and \( \Phi_2(u, v, w) = 0 \) if and only if \( P \) lies on the circumcircle.

The discriminant of (1) is non-negative for all \( P \) and null if and only if \( P = I \). In this latter case, the points \( P_a, P_b, P_c \) are “collinear” if and only if they all coincide with \( I \).

Considering now \( P \neq I \), (1) always has two (real) solutions. \( \square \)

Figure 11 shows the case \( P = H \) with two (non-perpendicular) lines secant at \( X_{65} \) orthocenter of the intouch triangle.

Figure 12 shows the case \( P = X_{40} \) with two perpendicular lines secant at \( X_8 \) and parallel to the asymptotes of the Feuerbach hyperbola.

When \( P \) is a point on the circumcircle, equation (1) has a solution \( t = 0 \) and one of the two lines, say \( L_1 \), is the Simson line of \( P \): the triangle \( A'B'C' \) degenerates

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5\( X_{40} \) is the reflection of \( I \) in \( O \).
into this Simson line. \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) meet on the ellipse centered at \( X_{10} \) passing through \( X_{11}, \) the midpoints of \( ABC \) and the feet of the cevians of \( X_8. \) This ellipse is the complement of the circum-ellipse centered at \( I \) and has equation:

\[
\sum_{\text{cyclic}} (a + b - c)(a - b + c)x^2 - 2a(b + c - a)yz = 0.
\]

Figure 13 shows the case \( P = X_{104} \) with two lines secant at \( X_{11}, \) one of them being the Simson line of \( X_{104}. \)

Following equation (1) again, we observe that, when \( P \) lies on the polar line of \( I \) in the circumcircle, we find to opposite values for \( t: \) the two corresponding points \( P_a \) are symmetric with respect to the sideline \( BC, \) \( P_b \) and \( P_c \) similarly. The most interesting situation is obtained with \( P = X_{36} \) (inversive image of \( I \) in the circumcircle) since we find two perpendicular lines \( \mathcal{L}_1 \) and \( \mathcal{L}_2, \) parallel to the asymptotes of the Feuerbach hyperbola, intersecting at the midpoint of \( X_{36}X_{80}. \)

See Figure 14.

Construction of \( \mathcal{L}_1 \) and \( \mathcal{L}_2: \) the line \( IP \) meets the circumcircle at \( S_1 \) and \( S_2. \) The parallels at \( P \) to \( OS_1 \) and \( OS_2 \) meet \( OI \) at \( T_1 \) and \( T_2. \) The homotheties with center \( I \) which map \( O \) to \( T_1 \) and \( T_2 \) also map the triangle \( ABC \) to the triangles \( A_1B_1C_1 \) and \( A_2B_2C_2. \) The perpendiculars \( PA', PB', PC' \) at \( P \) to the sidelines of \( ABC \) meet the corresponding sidelines of \( A_1B_1C_1 \) and \( A_2B_2C_2 \) at the requested points.

---

\( ^6 \) \( X_{36} \) is the isogonal conjugate of \( X_{36}. \)

\( ^7 \) We suppose \( I \neq P. \)
Figure 13. Collinear $P_a$, $P_b$, $P_c$ with $P = X_{104}$

Figure 14. Collinear $P_a$, $P_b$, $P_c$ with $P = X_{36}$
Proposition 14. The triangles $ABC$ and $P_aP_bP_c$ are perspective if and only if $k$ is solution of:

$$
\Psi_2(u, v, w) t^2 + \Psi_1(u, v, w) t + \Psi_0(u, v, w) = 0
$$

where:

$$
\Psi_2(u, v, w) = -\frac{1}{2} abc(a + b + c)(u + v + w)^2 \sum_{\text{cyclic}} (b - c)(b + c - a)S_Au,
$$

$$
\Psi_1(u, v, w) = \frac{1}{2}(a + b + c)(u + v + w)\Delta \sum_{\text{cyclic}} \left(-2bc(b - c)(b + c - a)S_Au^2 + a^2(b - c)(a + b + c)(b + c - a)^2vw\right),
$$

$$
\Psi_0(u, v, w) = \Delta^2 \sum_{\text{cyclic}} \left(3a^4 - 2a^2(b^2 + c^2) - (b^2 - c^2)^2\right)u(c^2v^2 - b^2w^2).
$$

Remarks. (1) $\Psi_2(u, v, w) = 0$ if and only if $P$ lies on the line $IH$.

(2) $\Psi_1(u, v, w) = 0$ if and only if $P$ lies on the hyperbola passing through $I$, $H$, $X_{500}$, $X_{573}$, $X_{1742}$ and having the same asymptotic directions as the isogonal transform of the line $X_{40}X_{758}$, i.e., the reflection in $O$ of the line $X_1X_{21}$.

(3) $\Psi_0(u, v, w) = 0$ if and only if $P$ lies on the Darboux cubic. See Figure 15.

The equation (2) is clearly realized for all $t$ if and only if $P = I$ or $P = H$: all $t$–Mandart triangles of $I$ and $H$ are perspective to $ABC$. Furthermore, if $P = H$ the perspector is always $H$, and if $P = I$ the perspector lies on the Feuerbach hyperbola. In the sequel, we exclude those two points and see that there are at most two real numbers $t_1$ and $t_2$ for which $t_1$– and $t_2$–Mandart triangles of $P$ are perspective to $ABC$. Let us denote by $R_1$ and $R_2$ the (not always real) corresponding perspectors.

We explain the construction of these two perspectors with the help of several lemmas.

Lemma 15. For a given $P$ and a corresponding Mandart triangle $T_1(P) = P_aP_bP_c$, the locus of $R_a = BP_b \cap CP_c$, when $t$ varies, is a conic $\gamma_a$.

Proof. The correspondence on the pencils of lines with poles $B$ and $C$ mapping the lines $BP_b$ and $CP_c$ is clearly an involution. Hence, the common point of the two lines must lie on a conic. \hfill \Box

This conic $\gamma_a$ obviously contains $B$, $C$, $H$, $S_a = BB' \cap CC'$ and two other points $B_1$ on $AB$, $C_1$ on $AC$ defined as follows. Reflect $AB \cap PB'$ in the bisector $AI$ to get a point $B_2$ on $AC$. The parallel to $AB$ at $B_2$ meets $PC'$ at $B_3$. $B_1$ is the intersection of $AB$ and $CB_3$. The point $C_1$ on $AC$ is constructed similarly. See Figure 16.

Lemma 16. The three conics $\gamma_a$, $\gamma_b$, $\gamma_c$ have three points in common: $H$ and the (not always real) sought perspectors $R_1$ and $R_2$. Their jacobian must degenerate.

---

$^8$ $X_{500} = X_1X_{30} \cap X_3X_6$, $X_{573} = X_4X_9 \cap X_3X_6$ and $X_{1742} = X_1X_7 \cap X_3X_{238}$.
into three lines, one always real \( \mathcal{L}_P \) containing \( R_1 \) and \( R_2 \), two other passing through \( H \).

**Lemma 17.** \( \mathcal{L}_P \) contains the Nagel point \( X_8 \). In other words, \( X_8, R_1 \) and \( R_2 \) are always collinear.

With \( P = (u : v : w) \), \( \mathcal{L}_P \) has equation:

\[
\sum_{\text{cyclic}} \frac{a(cv - bw)}{b + c - a} x = 0
\]

\( \mathcal{L}_P \) is the trilinear polar of the isotomic conjugate of point \( T \), where \( T \) is the barycentric product of \( X_{57} \) and the isotomic conjugate of the trilinear pole of the line \( PI \). The construction of \( R_1 \) and \( R_2 \) is now possible in the most general case with one of the conics and \( \mathcal{L}_P \). Nevertheless, in three specific situations already mentioned, the construction simplifies as we see in the three following corollaries.

**Corollary 18.** When \( P \) lies on \( IH \), there is only one (always real) Mandart triangle \( T(P) \) perspective to \( ABC \). The perspector \( R \) is the intersection of the lines \( HX_8 \) and \( PX_{78} \).

**Proof.** This is obvious since equation (2) is at most of the first degree when \( P \) lies on \( IH \). \( \square \)
Figure 16. The three conics $\gamma_a, \gamma_b, \gamma_c$ and the perspectors $R_1, R_2$

In Figure 17, we have taken $P = X_{33}$ and $R = X_{318}$. 

Figure 17. Only one triangle $P_aP_bP_c$ perspective to $ABC$ when $P$ lies on $IH$
Remark. The line $IH$ meets the Darboux cubic again at $X_{1490}$. The corresponding Mandart triangle $T_t(P)$ is the pedal triangle of $X_{1490}$ which is also the cevian triangle of $X_{329}$.

**Corollary 19.** When $P$ (different from $I$ and $H$) lies on the conic seen above, there are two (not always real) Mandart triangles $T_t(P)$ perspective to $ABC$ obtained for two opposite values $t_1$ and $t_2$. The vertices of the triangles are therefore two by two symmetric in the sidelines of $ABC$.

In the figure 18, we have taken $P = X_{500}$ (orthocenter of the incentral triangle).

![Figure 18: Two triangles $P_aP_bP_c$ perspective with $ABC$ having vertices symmetric in the sidelines of $ABC$](image)

**Corollary 20.** When $P$ (different from $I$, $H$, $X_{1490}$) lies on the Darboux cubic, there are two (always real) Mandart triangles $T_t(P)$ perspective to $ABC$, one of them being the pedal triangle of $P$ with a perspector on the Lucas cubic.

Since one perspector, say $R_1$, is known, the construction of the other is simple: it is the “second” intersection of the line $X_8R_1$ with the conic $BCHS \alpha R_1$.

Table 3 gives $P$ (on the Darboux cubic), the corresponding persectors $R_1$ (on the Lucas cubic) and $R_2$.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$X_1$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_{20}$</th>
<th>$X_{40}$</th>
<th>$X_{64}$</th>
<th>$X_{84}$</th>
<th>$X_{1498}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>$X_7$</td>
<td>$X_2$</td>
<td>$X_4$</td>
<td>$X_{69}$</td>
<td>$X_8$</td>
<td>$X_{253}$</td>
<td>$X_{189}$</td>
<td>$X_{20}$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$X_8$</td>
<td>$X_4$</td>
<td>$X_{388}$</td>
<td>$X_{10}$</td>
<td>*</td>
<td>$X_{515}$</td>
<td>*</td>
<td></td>
</tr>
</tbody>
</table>

Table 4
In Figure 19, we have taken $P = X_{40}$ (reflection of $I$ in $O$).

### Proposition 21.
The triangles $A'B'C'$ and $P_aP_bP_c$ have the same area if and only if

1. $t = 0$, or
2. 
   $$t = -\frac{bc(b+c)u + ca(c+a)v + ab(a+b)w}{2R(a+b+c)(u+v+w)}, \quad \text{9}$$
3. $t$ is a solution of a quadratic equation \(^{10}\) whose discriminant has the same sign of

\[
    f(u, v, w) = \sum_{\text{cyclic}} b^2c^2(b+c)^2u^2 + 2a^2bc(bc - 3a(a+b+c))vw.
\]

---

9 This can be interpreted as $t = -\frac{d(P)}{d(O)} \cdot R$, where $d(X)$ denotes the distance from $X$ to the polar line of $I$ in the circumcircle.

10 $abc(a+b+c)(u+v+w)^2t^2 + 2\Delta(u+v+w)\left(\sum_{\text{cyclic}} bc(b+c)u\right) t + 8\Delta^2(a^2vw + b^2wu + c^2uv) = 0$. 

---

\[
\begin{array}{|c|c|}
\hline
\text{Triangle center} & \text{First barycentric coordinate} \\
\hline
R_2(X_{64}) & a^3 - 4a^2(b+c)^2 + 2a^2(b+c)^2(3b^2 - 4bc + 3c^2) - 4a^2(b^2 - c^2)(b+c)^2 + (b-c)^2(b+c)^2 \\
R_2(X_{1408}) & a^4 + a^2(b+c) - a(b+c)^2 - (b+c)(b-c)^2 \\
\hline
\end{array}
\]
The equation \( f(x, y, z) = 0 \) represents an ellipse \( E \) centered at \( X_{35} \) whose axes are parallel and perpendicular to the line \( OI \). See Figure 20.

According to the position of \( P \) with respect to this ellipse, it is possible to have other triangles solution of the problem. More precisely, if \( P \) is
- inside \( E \), there is no other triangle,
- outside \( E \), there are two other (distinct) triangles,
- on \( E \), there is only one other triangle.

**Proposition 22.** As \( t \) varies, each line \( P_bP_c, P_cP_a, P_aP_b \) still envelopes a parabola.

Denote these parabolas by \( P_a, P_b, P_c \) respectively. \( P_a \) has focus the projection \( F_a \) of \( P \) on \( AI \) and directrix \( \ell_a \) parallel to \( AI \) at \( E_a \) such that \( \overrightarrow{PE_a} = \cos A \overrightarrow{PF_a} \). Note that the direction of the directrix (and the axis) is independent of \( P \). \( P_a \) is still tangent to the lines \( PB', PC', B'C' \).

In this more general case, the directrices \( \ell_a, \ell_b, \ell_c \) are not necessarily concurrent. This happens if and only if \( P \) lies on the line \( OI \) and, then, their common point lies on \( IG \).

**Proposition 23.** The Mandart triangle \( T_t(P) \) and the pedal triangle of \( P \) are perspective at \( P \). As \( t \) varies, the envelope of their perspectrix is a parabola.

\[ \text{11} \]

Let \( I_a' \) be the inverse-in-circumcircle of the excenter \( I_a \), and define \( I_b' \) and \( I_c' \) similarly. The triangles \( ABC \) and \( I_a'I_b'I_c' \) are perspective at \( X_{35} \) which is a point on the line \( OI \).
The directrix of this parabola is parallel to the line $IP$. It is still inscribed in the pedal triangle $A'B'C'$ of $P$ and is tangent to the two lines $L_1$ and $L_2$ met in proposition 13.

**Remark.** Unlike the case $P = X_8$, $ABC$ is not necessary self polar with respect to this Mandart parabola.

**Proposition 24.** The Mandart triangle $T_t(P)$ and $ABC$ are orthologic. The perpendiculars from $A$, $B$, $C$ to the corresponding sidelines of $P_aP_bP_c$ are concurrent at $Q = \left( \frac{a^2}{at+2\Delta} : \cdots : \cdots \right)$. As $t$ varies, the locus of $Q$ is generally the circumconic which is the isogonal transform of the line $IP$.

This conic has equation
\[ \sum_{\text{cyclic}} a^2(cv - bw)yz = 0. \]

It is tangent at $I$ to $IP$, and is a rectangular hyperbola if and only if $P$ lies on the line $OI$ ($P \neq I$). When $P = I$, the triangles are homothetic at $I$ and the perpendiculars concur at $I$.

**References**


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