Forum Geometricorum
Volume 4 (2004) 199–201.

FORUM GEOM
ISSN 1534-1178

Another Proof of Fagnano's Inequality

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Abstract. We prove Fagnano's inequality using the scalar product of vectors.

In 1775, I. F. Fagnano, an Italian mathematician, proposed the following extremum problem.

Problem (Fagnano)**.** *In a given acute-angled triangle ABC, inscribe a triangle XYZ whose perimeter is as small as possible.*

Fagnano himself gave a solution to this problem using calculus. The second proof given in [1] repeatedly using reflections and the mirror property of the orthic triangle was due to L. Fejér. While H. A. Schwarz gave another proof in which reflection was also used, we give another proof by using the scalar product of two vectors.

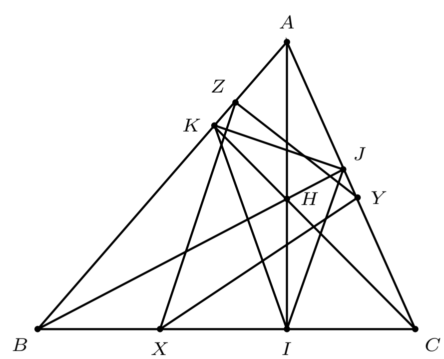

Figure 1

Let AI, BJ and CK be the altitudes of triangle ABC and H its orthocenter. Suppose that X, Y, Z are arbitrary points on the lines BC, CA and AB respectively. See Figure 1. We have

Publication Date: November 19, 2004. Communicating Editor: J. Chris Fisher.

$$
\begin{aligned}
&YZ+ZX+XY\\
=&\frac{YZ\cdot JK}{JK}+\frac{ZX\cdot KI}{KI}+\frac{XY\cdot IJ}{IJ}\\
\geqslant&\frac{\overrightarrow{YZ}\cdot\overrightarrow{JK}}{JK}+\frac{\overrightarrow{ZX}\cdot\overrightarrow{KI}}{KI}+\frac{\overrightarrow{XY}\cdot\overrightarrow{IJ}}{IJ}\\
=&\frac{(\overrightarrow{YJ}+\overrightarrow{JK}+\overrightarrow{KZ})\cdot\overrightarrow{JK}}{JK}+\frac{(\overrightarrow{ZK}+\overrightarrow{KI}+\overrightarrow{IX})\cdot\overrightarrow{KI}}{KI}+\frac{(\overrightarrow{XI}+\overrightarrow{IJ}+\overrightarrow{JY})\cdot\overrightarrow{IJ}}{IJ}\\
=&JK+KI+IJ+\overrightarrow{XI}\cdot\Big(\frac{\overrightarrow{IJ}}{IJ}+\frac{\overrightarrow{IK}}{IK}\Big)+\overrightarrow{YJ}\cdot\Big(\frac{\overrightarrow{JK}}{JK}+\frac{\overrightarrow{JI}}{JI}\Big)+\overrightarrow{ZK}\cdot\Big(\frac{\overrightarrow{KI}}{KI}+\frac{\overrightarrow{KJ}}{KJ}\Big).
\end{aligned}
$$

Since triangle ABC is acute-angled, its altitudes bisect the internal angles of its orthic triangle IJK. It follows that the vectors

$$\frac{\overrightarrow{IJ}}{IJ}+\frac{\overrightarrow{IK}}{IK},\quad \frac{\overrightarrow{JK}}{JK}+\frac{\overrightarrow{JI}}{JI},\quad \frac{\overrightarrow{KI}}{KI}+\frac{\overrightarrow{KJ}}{KJ}$$

are respectively perpendicular to the vectors $\overrightarrow{XI},\ \overrightarrow{YJ},\ \overrightarrow{ZK}$. It follows that

$$YZ+ZX+XY\geqslant JK+KI+IJ. \tag{1}$$

If the equality in (1) occurs, then the vectors $\overrightarrow{YZ},\overrightarrow{ZX},\overrightarrow{XY}$ point in the same directions of the vectors $\overrightarrow{JK},\overrightarrow{KI},\overrightarrow{IJ}$ respectively. Hence there exist positive numbers α, β and γ such that

$$\overrightarrow{YZ}=\alpha\overrightarrow{JK},\quad \overrightarrow{ZX}=\beta\overrightarrow{KI},\quad \overrightarrow{XY}=\gamma\overrightarrow{IJ}.$$

Now we have $\alpha\overrightarrow{JK}+\beta\overrightarrow{KI}+\gamma\overrightarrow{IJ}=\overrightarrow{0}$. It follows from this and the equality $\overrightarrow{JK}+\overrightarrow{KI}+\overrightarrow{IJ}=\overrightarrow{0}$ that $\alpha=\beta=\gamma$. Consequently,

$$\overrightarrow{YZ}=\alpha\overrightarrow{JK},\ \ \overrightarrow{ZX}=\alpha\overrightarrow{KI},\ \ \overrightarrow{XY}=\alpha\overrightarrow{IJ},$$

which implies that

$$YZ=\alpha JK,\ \ ZX=\alpha KI,\ \ XY=\alpha IJ,$$

and

$$YZ+ZX+XY=\alpha(JK+KI+IJ).$$

Note that the equality in (1) occurs, we have $\alpha=\beta=\gamma=1$. Then $\overrightarrow{YZ}=\overrightarrow{JK},\ \overrightarrow{ZX}=\overrightarrow{KI},\ \overrightarrow{XY}=\overrightarrow{IJ}$, which means that X,Y,Z respectively coincides with I,J,K.

Conversely, if X, Y, Z coincide with I, J, K respectively, then equality sign occurs in (1).

In conclusion, the triangle XYZ has the smallest possible perimeter when X, Y, Z coincide with I, J, K respectively.

Reference

[1] A. Bogomolny, http://www.cut-the-knot.org/Curriculum/Geometry/Fagnano.shtml

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