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Further Inequalities of Erdős-Mordell Type

Walther Janous

To the memory of Murray S. Klamkin

Abstract. We extend the recent generalization of the famous Erdős-Mordell inequality by Dar and Gueron in the *American Mathematical Monthly*.

1. Introduction

In the recent note [1] the following generalization of the famous Erdős - Mordell inequality has been established. (For a proof of the original inequality see for instance [2]). For a triangle $A_1A_2A_3$, we denote by a_i the length of the side opposite to A_i, $i = 1, 2, 3$. Let P be an interior point. Denote the distances of P from the vertices A_i by R_i and from the sides opposite A_i by r_i. For positive real numbers λ_1, λ_2, λ_3,

$$\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 \geq 2\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{r_1}{\sqrt{\lambda_1}} + \frac{r_2}{\sqrt{\lambda_2}} + \frac{r_3}{\sqrt{\lambda_3}}\right), \qquad (1)$$

This inequality appears in [3, p.318, Theorem 15] without proof and with an incorrect characterization for equality. In [3, Chapter XI] and [4, Chapter 12], there are quoted very many extensions and variations of the original Erdős - Mordell inequality. It is the goal of this note to prove a further generalization containing the results of [1] and to apply it to specific points in a triangle, resulting in new inequalities for several elements of triangles.

2. The inequalities

Let λ_1, λ_2, λ_3 and t denote positive real numbers, with $0 < t \leq 1$.

Theorem 1.

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geq 2^t\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{r_1^t}{\sqrt{\lambda_1}} + \frac{r_2^t}{\sqrt{\lambda_2}} + \frac{r_3^t}{\sqrt{\lambda_3}}\right). \qquad (2)$$

Equality holds if and only if $\lambda_1 : \lambda_2 : \lambda_3 = a_1^{2t} : a_2^{2t} : a_3^{2t}$ *and* P *is the circumcenter of triangle* $A_1A_2A_3$.

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Proof. As for instance in [1] we have

$$R_1 \geq \frac{a_3}{a_1} r_2 + \frac{a_2}{a_1} r_3, \qquad R_2 \geq \frac{a_1}{a_2} r_3 + \frac{a_3}{a_2} r_1, \qquad R_3 \geq \frac{a_2}{a_3} r_1 + \frac{a_1}{a_3} r_2.$$

Using the power means inequality we obtain (for $0 < t < 1$)

$$R_1^t \geq 2^t \left(\frac{\frac{a_3}{a_1} r_2 + \frac{a_2}{a_1} r_3}{2} \right)^t \geq 2^t \cdot \frac{\left(\frac{a_3}{a_1}\right)^t r_2^t + \left(\frac{a_2}{a_1}\right)^t r_3^t}{2}$$

and two similar inequalities. Applying several times the elementary estimation $x + \frac{1}{x} \geq 2$ for $x > 0$ we obtain

$$\begin{aligned} &\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \\ \geq &2^t \left(\frac{\left(\frac{a_3}{a_2}\right)^t \lambda_2 + \left(\frac{a_2}{a_3}\right)^t \lambda_3}{2} r_1^t + \frac{\left(\frac{a_1}{a_3}\right)^t \lambda_3 + \left(\frac{a_3}{a_1}\right)^t \lambda_1}{2} r_2^t + \frac{\left(\frac{a_2}{a_1}\right)^t \lambda_1 + \left(\frac{a_1}{a_2}\right)^t \lambda_2}{2} r_3^t \right) \\ \geq &2^t \left(\sqrt{\lambda_2 \lambda_3} r_1^t + \sqrt{\lambda_3 \lambda_1} r_2^t + \sqrt{\lambda_1 \lambda_2} r_3^t \right) \end{aligned}$$

as claimed. The conditions of equality are derived as in [1]. $\square$

In view of the obvious inequality $(x + y)^t > x^t + y^t$ for $x, y > 0$, we have the following theorem.

Theorem 2. *For* $t > 1$,

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geq 2\sqrt{\lambda_1 \lambda_2 \lambda_3} \left(\frac{r_1^t}{\sqrt{\lambda_1}} + \frac{r_2^t}{\sqrt{\lambda_2}} + \frac{r_3^t}{\sqrt{\lambda_3}} \right). \tag{3}$$

As a consequence of Theorem 1 we get

Theorem 3.

$$\textstyle\sum_{i=1}^3 \frac{\lambda_i}{r_i^t} \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} R_i^t}, \tag{4}$$

$$\textstyle\frac{\lambda_i}{R_i^t} \geq \frac{2^t \sqrt{\lambda_1 \lambda_2 \lambda_3}}{(R_1 R_2 R_3)^t \sum_{i=1}^3} \sum_{i=1}^3 \frac{(R_i r_i)^t}{\sqrt{\lambda_i}}, \tag{5}$$

$$\textstyle\sum_{i=1}^3 \lambda_i (R_i r_i)^t \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} (r_1 r_2 r_3)^t \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} r_i^t}, \tag{6}$$

$$\textstyle\sum_{i=1}^3 \lambda_i r_i^t \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} (r_1 r_2 r_3)^t \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} (R_i r_i)^t}, \tag{7}$$

$$\textstyle\sum_{i=1}^3 \frac{\lambda_i}{(R_i r_i)^t} \geq \frac{2^t \sqrt{\lambda_1 \lambda_2 \lambda_3}}{(R_1 R_2 R_3)^t} \sum_{i=1}^3 \frac{R_i^t}{\sqrt{\lambda_i}}. \tag{8}$$

The proofs of these inequalities follow from Theorem 1 upon application of transformations such as
(i) inversion with respect to the circle $\mathcal{C}(P, \sqrt{R_1 R_2 R_3})$ resulting in $R_i \mapsto \frac{R_1 R_2 R_3}{R_i}$ and $r_i \mapsto R_i r_i$ for $i = 1, 2, 3$,
(ii) reciprocation of $A_1 A_2 A_3$ yielding $R_i \mapsto \frac{r_1 r_2 r_3}{r_i}$ and $r_i \mapsto \frac{r_1 r_2 r_3}{R_i}$ for $i = 1, 2, 3$, and

(iii) isogonal conjugation.
For the details consult [3, pp. 293 - 295].

Remarks. (1) From (5) and (6) the following inequality is easily derived.

$$(R_1R_2R_3)^t \sum_{i=1}^{3} \frac{\lambda_i}{R_i^t} \geq 4^t \sqrt[4]{\lambda_1\lambda_2\lambda_3}(r_1r_2r_3)^t \sum_{i=1}^{3} \frac{\sqrt[4]{\lambda_i}}{r_i^t}. \tag{9}$$

whereas (7) and (8) lead to the "converse" of (9), *i.e.*,

$$\frac{1}{(r_1r_2r_3)^t} \sum_{i=1}^{3} \lambda_i r_i^t \geq \frac{4^t \sqrt[4]{\lambda_1\lambda_2\lambda_3}}{(R_1R_2R_3)^t} \sum_{i=1}^{3} \sqrt[4]{\lambda_i} R_i^t. \tag{10}$$

(2) We leave it as an exercise to the reader to derive an analogue of Theorem 2. It should be noted that the above inequalities include very many results of [3, 4] as special cases.

3. Applications to special triangle points

In this section we show that the theorems above, when specialized to suitably chosen interior points P, imply an abundance of new interesting triangle inequalities.

3.1. Let P be the incenter I of $A_1A_2A_3$. Then $r_1 = r_2 = r_3 = r$, the inradius of $A_1A_2A_3$, and $R_i = A_iI = r \csc \frac{A_i}{2}$, $i = 1, 2, 3$. Thus, from (8), we obtain, upon recalling that

$$\sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} = \frac{r}{4R},$$

the following inequality for $0 < t \leq 1$:

$$\sum_{i=1}^{3} \lambda_i \sin^t \frac{A_i}{2} \geq \sqrt{\lambda_1\lambda_2\lambda_3} \left(\frac{r}{2R}\right)^t \sum_{i=1}^{3} \frac{1}{\sqrt{\lambda_i}} \csc^t \frac{A_i}{2}. \tag{11}$$

3.2. Let P be the centroid G of $A_1A_2A_3$. Then $R_i = A_iG = \frac{2}{3}m_i$, and $r_i = \frac{h_i}{3}$, where, for $i = 1, 2, 3$, m_i and h_i denote respectively the median and altitude emanating from vertex A_i. Therefore, as an example, (4) becomes, for $0 < t \leq 1$,

$$\sum_{i=1}^{3} \frac{\lambda_i}{h_i^t} \geq \sqrt{\lambda_1\lambda_2\lambda_3} \sum_{i=1}^{3} \frac{1}{\sqrt{\lambda_i} m_i^t}. \tag{12}$$

If we put $\lambda_i = h_i^t$, $i = 1, 2, 3$, then

$$\left(\frac{\sqrt{h_2h_3}}{m_1}\right)^t + \left(\frac{\sqrt{h_3h_1}}{m_2}\right)^t + \left(\frac{\sqrt{h_1h_2}}{m_3}\right)^t \leq 3. \tag{13}$$

This inequality should be compared with the following one by Klamkin and Meir in [3, p. 215]:

$$\frac{\overline{h_1}}{m_1} + \frac{\overline{h_2}}{m_2} + \frac{\overline{h_3}}{m_3} \leq 3,$$

where $(\overline{h_1}, \overline{h_2}, \overline{h_3})$ is any permutation of (h_1, h_2, h_3).

Via the median - duality transforming an arbitrary triangle $A_1A_2A_3$ into one formed by its medians ([3, pp.109 - 111]), inequality (13) becomes

$$\left(\frac{h_1}{\sqrt{m_2m_3}}\right)^t + \left(\frac{h_2}{\sqrt{m_3m_1}}\right)^t + \left(\frac{h_3}{\sqrt{m_1m_2}}\right)^t \leq 3. \tag{14}$$

Finally, in (12), we put $\lambda_i = \frac{1}{a_i^t}$ for $i = 1, 2, 3$. A short calculation gives

$$3\left(\frac{R}{F}\right)^{\frac{t}{2}} \geq \sum_{i=1}^{3}\left(\frac{\sqrt{a_i}}{m_i}\right)^t. \tag{15}$$

Here, we make use of the identity $a_1a_2a_3 = 4RF$, where F denotes the area of $A_1A_2A_3$.

The median - dual of this inequality in turn reads

$$\sum_{i=1}^{3}\left(\frac{\sqrt{m_i}}{a_i}\right)^t \leq 3\left(\frac{\sqrt{m_1m_2m_3}}{2F}\right)^t. \tag{16}$$

Of course, if in (12) had we put $\lambda_i = \frac{\mu_i}{a_i^t}$ with $\mu_i > 0$, $i = 1, 2, 3$, we would obtain an even more general but less elegant inequality.

Remarks. (1) Clearly, many further inequalities could be deduced by the methods of this section. We leave this as an exercise to the reader.

(2) As the right hand side of inequality (1) indeed reads $2(\sqrt{\lambda_2\lambda_3}r_1 + \sqrt{\lambda_3\lambda_1}r_2 + \sqrt{\lambda_1\lambda_2}r_3)$, it is enough to assume λ_1, λ_2, λ_2 nonnegative throughout this note.

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Walther Janous: Ursulinengymnasium, Fürstenweg 86, A-6020 Innsbruck, Austria
E-mail address: walther.janous@tirol.com