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Inscribed Squares

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Abstract. We give simple constructions of various squares inscribed in a triangle, and some relations among these squares.

1. Inscribed squares

Given a triangle ABC, an inscribed square is one whose vertices are on the sidelines of ABC. Two of the vertices of an inscribed square must fall on a sideline. There are two kinds of inscribed squares.

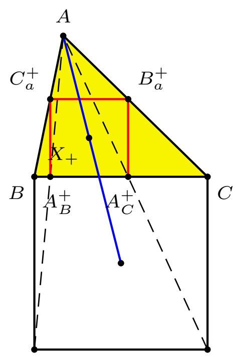

Figure 1A

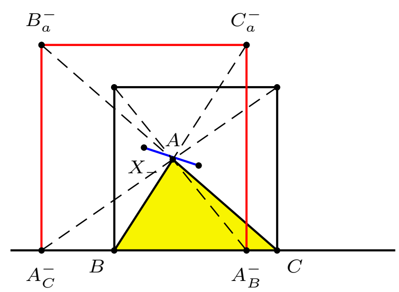

Figure 1B

1.1. *Inscribed squares of type I.* The Inscribed squares with two adjacent vertices on a sideline of ABC can be constructed easily from a homothety of a square erected on the side of ABC. Consider the two squares erected on the side BC. Their centers are the points with homogeneous barycentric coordinates $(-a^2 : S_C + \varepsilon S : S_B + \varepsilon S)$ for $\varepsilon = \pm 1$. Here, we use standard notations in triangle geometry. See, for example, [4, §1]. By applying the homothety $\mathsf{h}(A, \frac{\varepsilon S}{a^2+\varepsilon S})$, we obtain an inscribed square $\mathsf{Sq}^\varepsilon(A) = A_B^\varepsilon A_C^\varepsilon B_a^\varepsilon C_a^\varepsilon$ with center

$$\begin{aligned} X_\varepsilon =& \mathsf{h}(A, \frac{\varepsilon S}{a^2+\varepsilon S})(-a^2 : S_C + \varepsilon S : S_B + \varepsilon S) \\ =& (a^2 : S_C + \varepsilon S : S_B + \varepsilon S), \end{aligned}$$

and two vertices (A_B^ε and A_C^ε) on the sideline BC. See Figure 1. Similarly there are the inscribed squares $\mathsf{Sq}^\varepsilon(B)$ and $\mathsf{Sq}^\varepsilon(C)$.

We give the coordinates of the centers and vertices of these squares in Table 1 below.

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Table 1. Centers and vertices of inscribed squares of type I

$\mathsf{Sq}^{\varepsilon}(A)$	$\mathsf{Sq}^{\varepsilon}(B)$	$\mathsf{Sq}^{\varepsilon}(C)$
$X_\varepsilon = (a^2 : S_C + \varepsilon S : S_B + \varepsilon S)$	$Y_\varepsilon = (S_C + \varepsilon S : b^2 : S_A + \varepsilon S)$	$Z_\varepsilon = (S_B + \varepsilon S : S_A + \varepsilon S : c^2)$
$A_B^\varepsilon = (0 : S_C + \varepsilon S : S_B)$ $A_C^\varepsilon = (0 : S_C : S_B + \varepsilon S)$	$A_b^\varepsilon = (0 : b^2 : \varepsilon S)$	$A_c^\varepsilon = (0 : \varepsilon S : c^2)$
$B_a^\varepsilon = (a^2 : 0 : \varepsilon S)$,	$B_C^\varepsilon = (S_C : 0 : S_A + \varepsilon S)$ $B_A^\varepsilon = (S_C + \varepsilon S : 0 : S_A)$	$B_c^\varepsilon = (\varepsilon S : 0 : c^2)$
$C_a^\varepsilon = (a^2 : \varepsilon S : 0)$	$C_b^\varepsilon = (\varepsilon S : b^2 : 0)$	$C_A^\varepsilon = (S_B + \varepsilon S : S_A : 0)$ $C_B^\varepsilon = (S_B : S_A + \varepsilon S : 0)$

Proposition 1. *The triangle $X_\varepsilon Y_\varepsilon Z_\varepsilon$ and ABC perspective at the Vecten point*

$$V_\varepsilon = \left(\frac{1}{S_A + \varepsilon S} : \frac{1}{S_B + \varepsilon S} : \frac{1}{S_C + \varepsilon S}\right).$$

For V_+ and V_- are respectively X_{485} and X_{486} of [3].

1.2. *Inscribed squares of type II.* Another type of inscribed squares has two opposite vertices on a sideline of ABC. There are three such squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$. The square $\mathsf{Sq}^{\mathrm{d}}(A)$ has two opposite vertices on the sideline BC. Its center X can be found as follows. The perpendicular at X to BC intersects CA and AB at B_a and C_a such that $B_aX + C_aX = 0$. If $X = (0 : v : w)$, it is easy to see that

$$\begin{aligned} B_aX =& CX \cdot \tan C = \frac{av}{S_C(v+w)}, \\ C_aX =& BX \cdot \tan B = \frac{aw}{S_B(v+w)}. \end{aligned}$$

It follows that $B_aX + C_aX = 0$ if and only if $v : w = -S_C : S_B$, and the center of $\mathsf{Sq}^{\mathrm{d}}(A)$ is the point $X = (0 : -S_C : S_B)$ on the line BC. The vertices can be easily determined, as given in Table 2 below.

Table 2. Centers and vertices of inscribed squares of type II

$\mathsf{Sq}^{\mathrm{d}}(A)$	$\mathsf{Sq}^{\mathrm{d}}(B)$	$\mathsf{Sq}^{\mathrm{d}}(C)$
$X = (0 : -S_C : S_B)$	$Y = (S_C : 0 : -S_A)$	$Z = (-S_B : S_A : 0)$
$A_+ = (0 : -S_C - S : S_B + S)$ $A_- = (0 : -S_C + S : S_B - S)$	$A_b = (0 : -b^2 : 2S_A)$	$A_c = (0 : 2S_A : -c^2)$
$B_a = (-a^2 : 0 : 2S_B)$	$B_+ = (S_C + S : 0 : -S_A - S)$ $B_- = (S_C - S : 0 : -S_A + S)$	$B_c = (2S_B : 0 : -c^2)$
$C_a = (-a^2 : 2S_C : 0)$	$C_b = (2S_C : -b^2 : 0)$	$C_+ = (-S_B - S : S_A + S : 0)$ $C_- = (-S_B + S : S_A - S : 0)$

2. Some collinearity relations

Proposition 2. (a) *The centers X, Y, Z are the intercepts of the orthic axis with the sidelines of triangle ABC.*
(b) *For $\varepsilon = \pm 1$, the points A_ε, B_ε and C_ε are collinear. The line containing them is parallel to the orthic axis.*

Proof. The line containing the points A_ε, B_ε and C_ε has equation

$$(S_A + \varepsilon S)x + (S_B + \varepsilon S)y + (S_C + \varepsilon S)z = 0.$$

See Figure 2. □

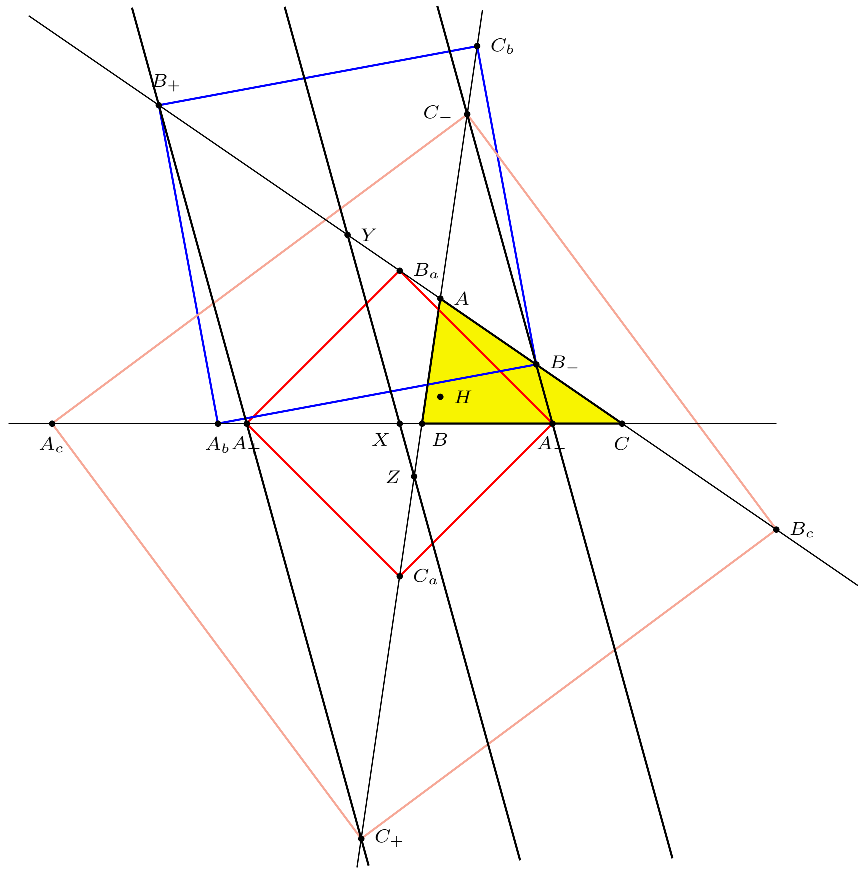

Figure 2

Proposition 3. (a) *The centers* X, Y_ε, Z_ε *of the squares* $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\varepsilon}(B)$, $\mathsf{Sq}^{\varepsilon}(C)$ *are collinear.*
(b) *The line* $B_C^\varepsilon C_B^\varepsilon$ *passes through the center* X *of* $\mathsf{Sq}^{\mathrm{d}}(A)$.
(c) *The line* $B_A^\varepsilon C_A^\varepsilon$ *passes through the point* A_ε.

Proof. (a) The line joining Y_ε and Z_ε has equation

$$-\varepsilon S x + S_B y + S_C z = 0$$

as is easily verified. This line clearly contains $X = (0 : -S_C : S_B)$.

(b) The line $B_C^\varepsilon C_B^\varepsilon$ has equation

$$-(S_A + \varepsilon S)x + S_B y + S_C z = 0.$$

It clearly passes through X.

(c) The line $B_A^\varepsilon C_A^\varepsilon$ has equation

$$-S_A x + (S_B + \varepsilon S)y + (S_C + \varepsilon S)z = 0.$$

It contains the point $A_\varepsilon = (0 : -S_C - \varepsilon S : S_B + \varepsilon S)$. See Figure 3 for $\varepsilon = 1$. $\square$

Remark. For $\varepsilon = \pm 1$, the lines in (b) and (c) above are parallel.

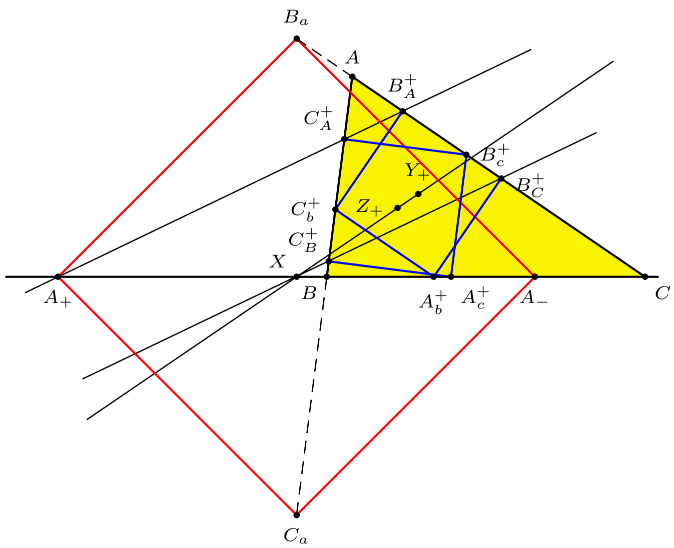

Figure 3

Let $T_A := B_A^+ C_A^+ \cap B_A^- C_A^- = (S_B - S_C : S_A : -S_A)$. The lines AT_A and BC are parallel. The three points T_A, T_B, T_C are collinear. The line connecting them has equation

$$S_A(S_B + S_C - S_A)x + S_B(S_C + S_A - S_B)y + S_C(S_A + S_B - S_C)z = 0.$$

Each of the squares of type II has a diagonal perpendicular to a sideline of triangle ABC. These diagonals clearly bound a triangle perspective to ABC with perspectrix the orthic axis. By [1] we know that the perspector lies on the circumcircle. Specifically, it is X_{74}, the Miquel perspector of the orthic axis.

The lines $B_A^\varepsilon C_A^\varepsilon$, $C_B^\varepsilon A_B^\varepsilon$, $A_C^\varepsilon B_C^\varepsilon$ bound a triangle perspective with ABC at the Kiepert perspector

$$K(\varepsilon \cdot \arctan 2) = \left(\frac{1}{2S_A + \varepsilon S} : \frac{1}{2S_B + \varepsilon S} : \frac{1}{2S_C + \varepsilon S} \right).$$

For $\varepsilon = +1$ and -1 respectively, these are X_{1131} and X_{1132} of [3]. The same perspector is found for the triangle bounded by the lines $B_C^\varepsilon C_B^\varepsilon$, $A_C^\varepsilon C_A^\varepsilon$, $A_B^\varepsilon B_A^\varepsilon$.

3. Inscribed squares and Miquel's theorem

We first recall Miquel's theorem.

Theorem 4 (Miquel)**.** *Let $A_1B_1C_1$ be a triangle inscribed in triangle ABC. There is a* pivot point *P such that $A_1B_1C_1$ is the image of the pedal triangle of P after a rotation about P followed by a homothety with center P. All inscribed triangles directly similar to $A_1B_1C_1$ have the same pivot point.*

A corollary of this theorem is for instance given in [2, Problem 8(ii), p.245].

Corollary 5. *Let X be a point defined with respect to the pedal triangle $A_PB_PC_P$ triangle of P. The images of X after the pivoting as in Miquel's theorem lie on a line.*

Proof. Let $A_2B_2C_2$ be the image of $A_PB_PC_P$ after pivoting, and let Y be the image of X. Clearly triangles PA_PA_2, PB_PB_2, PC_PC_2, and PXY are similar right triangles. This shows that Y lies on the line through X perpendicular to XP. □

Miquel's pivot theorem and Corollary 5 together give an easy explanation of Proposition 3(c). See Figure 4.

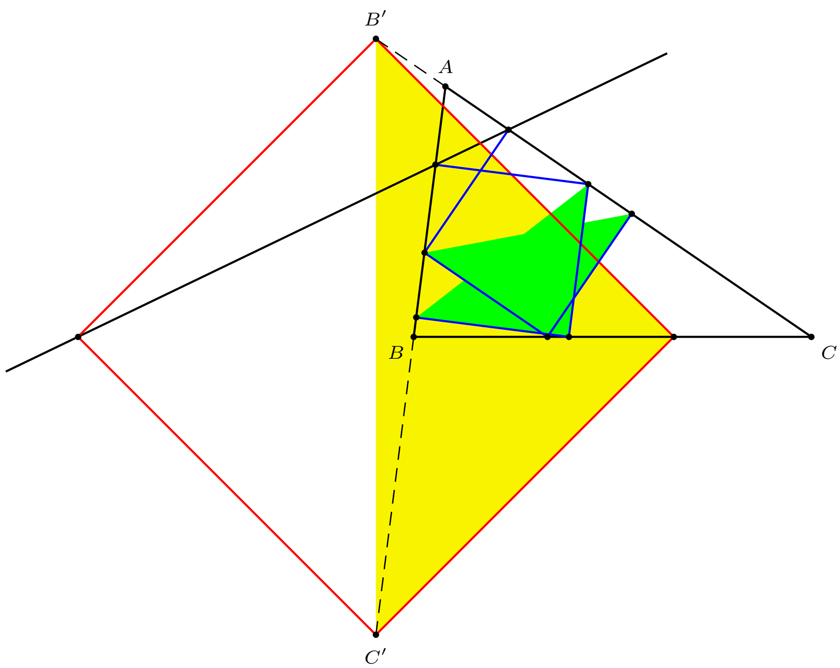

Figure 4

We have already seen that the centers of the inscribed squares of type II lie on the orthic axis. By Proposition 3(a), these centers are the intersections of the corresponding sides of the triangles $X_+Y_+Z_+$ and $X_-Y_-Z_-$ of the inscribed squares of type I. This means that the triangles $X_+Y_+Z_+$ and $X_-Y_-Z_-$ are perspective. The perspector is symmedian point $K = (a^2 : b^2 : c^2)$.

4. Squares with vertices on four given lines

Let us consider a fourth line in the plane of ABC. With the help of the inscribed squares of type I, we can construct two sets of three squares inscribing a fourline $\{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$, depending on the line containing the vertex opposite to that on d. Let ABC be the triangle bounded by the lines a, b, c. For $\varepsilon = \pm 1$, there is a square $\mathsf{Sq}^\varepsilon(\mathsf{a}) := A^\varepsilon_\mathsf{a} B^\varepsilon_\mathsf{a} D^\varepsilon_\mathsf{a} C^\varepsilon_\mathsf{a}$ with a pair of opposite vertices on a and d. The vertex on d is simply $D^\varepsilon_\mathsf{a} = B^\varepsilon_A C^\varepsilon_A \cap \mathsf{d}$. See the solution of Problem 55(a) of [5, p.146]. The other vertices of the square are determined by the same division ratio (of $B^\varepsilon_A C^\varepsilon_A$ by D^ε_a):

$$B^\varepsilon_A C^\varepsilon_A : C^\varepsilon_A D^\varepsilon_\mathsf{a} = A^\varepsilon_b A^\varepsilon_c : A^\varepsilon_c A^\varepsilon_\mathsf{a} = B^\varepsilon_C B^\varepsilon_c : B^\varepsilon_c B^\varepsilon_\mathsf{a} = C^\varepsilon_b C^\varepsilon_B : C^\varepsilon_B C^\varepsilon_\mathsf{a}.$$

See Figure 5 for $\varepsilon = +1$. In fact, if $D^\varepsilon_\mathsf{a} = (S_C + \varepsilon S, 0, S_A) + t(S_B + \varepsilon S, S_A, 0)$, then

$$\begin{aligned}
A^\varepsilon_\mathsf{a} =& (0, b^2, \varepsilon S) + t(0, \varepsilon S, c^2),\\
B^\varepsilon_\mathsf{a} =& (S_C, 0, S_A + \varepsilon S) + t(\varepsilon S, 0, c^2),\\
C^\varepsilon_\mathsf{a} =& (\varepsilon S, b^2, 0) + t(S_B, S_A + \varepsilon S, 0),
\end{aligned}$$

and the center of the square is the point

$$X^\varepsilon_\mathsf{a} = (S_C + \varepsilon S, b^2, S_A + \varepsilon S) + t(S_B + \varepsilon S, S_A + \varepsilon S, c^2).$$

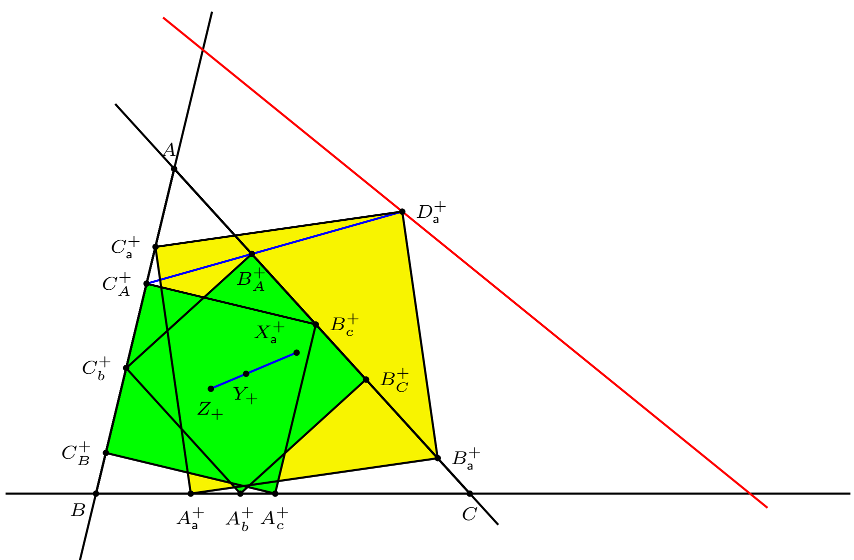

Figure 5

It is now clear that the position of A^ε_a relative to A^ε_b and A^ε_c fixes D^ε_a as well, even if we do not have a given line d. Similarly we D^ε_b and D^ε_c are fixed by B^ε_b and C^ε_c respectively. We may thus take A^ε_a, B^ε_b and C^ε_c to be the traces of a point $P = (u : v : w)$ and see if the corresponding D^ε_a, D^ε_b and D^ε_c are collinear. A

simple calculation gives

$$\begin{aligned}
D_{\mathsf{a}}^{\varepsilon} =&((S_B - \varepsilon S)v + (S_C - \varepsilon S)w : \varepsilon Sv - b^2 w : \varepsilon Sw - c^2 v),\\
D_{\mathsf{b}}^{\varepsilon} =&(\varepsilon Su - a^2 w : (S_C - \varepsilon S)w + (S_A - \varepsilon S)u : \varepsilon Sw - c^2 u),\\
D_{\mathsf{c}}^{\varepsilon} =&(\varepsilon Su - a^2 v : \varepsilon Sv - b^2 u : (S_A - \varepsilon S)u + (S_B - \varepsilon S)v).
\end{aligned}$$

Also, the centers of the squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$ are the points

$$\begin{aligned}
X_{\mathsf{a}}^{\varepsilon} =&(-(S_B - \varepsilon S)v - (S_C - \varepsilon S)w : (S_A - \varepsilon S)v + b^2 w : c^2 v + (S_A - \varepsilon S)w),\\
Y_{\mathsf{b}}^{\varepsilon} =&(a^2 w + (S_B - \varepsilon S)u : -(S_C - \varepsilon S)w - (S_A - \varepsilon S)u : (S_B - \varepsilon S)w + c^2 u),\\
Z_{\mathsf{c}}^{\varepsilon} =&((S_C - \varepsilon S)u + a^2 v : b^2 u + (S_C - \varepsilon S)v : -(S_A - \varepsilon S)u - (S_B - \varepsilon S)v).
\end{aligned}$$

Proposition 6. *Let $A_{\mathsf{a}}^{\varepsilon}$, $B_{\mathsf{b}}^{\varepsilon}$ and $C_{\mathsf{c}}^{\varepsilon}$ be the traces of a point $P = (u : v : w)$.* (a) *The three points $D_{\mathsf{a}}^{\varepsilon}$, $D_{\mathsf{b}}^{\varepsilon}$ and $D_{\mathsf{c}}^{\varepsilon}$ are collinear if and only if P lies on the circumcubic*

$$\begin{aligned}
&4a^2b^2c^2uvw + S^2 \sum_{\text{cyclic}} u((2S_A + S_B)v^2 + (2S_A + S_B)w^2)\\
=&\varepsilon S \left(2S^2 uvw + \sum_{\text{cyclic}} u((2c^2a^2 - S_{AB})v^2 + (2a^2b^2 - S_{CA})w^2)\right).
\end{aligned}$$

(b) *The centers of the squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$ are collinear if and only if*

$$\begin{aligned}
&2a^2b^2c^2uvw + S^2 \sum_{\text{cyclic}} u(c^2v^2 + b^2w^2)\\
=&\varepsilon S \left(2S^2 uvw + \sum_{\text{cyclic}} a^2 u(c^2v^2 + b^2w^2)\right).
\end{aligned}$$

Remarks. (1) The locus of P for which $D_{\mathsf{a}}^{\varepsilon}D_{\mathsf{b}}^{\varepsilon}D_{\mathsf{c}}^{\varepsilon}$ and ABC are perspective is the isogonal cubic with pivot $(a^2 + \varepsilon S : b^2 + \varepsilon S : c^2 + \varepsilon S)$.

(2) The locus of P for which $X_{\mathsf{a}}^{\varepsilon}Y_{\mathsf{b}}^{\varepsilon}Z_{\mathsf{c}}^{\varepsilon}$ and ABC are perspective is the isogonal cubic with pivot H. Here are some examples of the perspectors for P on the cubic.

Table 3. Perspectors of $X_{\mathsf{a}}^{\varepsilon}Y_{\mathsf{b}}^{\varepsilon}Z_{\mathsf{c}}^{\varepsilon}$ for $\varepsilon = \pm 1$

P	$\varepsilon = +1$	$\varepsilon = -1$
I	I	I
O	$X_{372} = (a^2(S_A - S) : \cdots : \cdots)$	$X_{371} = (a^2(S_A + S) : \cdots : \cdots)$
H	$X_{486} = \left(\frac{1}{S_A - S} : \cdots : \cdots\right)$	$X_{485} = \left(\frac{1}{S_A + S} : \cdots : \cdots\right)$
X_{485}	G	$(a^2 + S : \cdots : \cdots)$
X_{486}	$(a^2 - S : \cdots : \cdots)$	G
X_{487}	$\left(\frac{1}{b^2c^2 + S_{BC} - (S_A + S_B + S_C)S} : \cdots : \cdots\right)$	$\left(\frac{S_A - a^2}{S_A - S} : \cdots : \cdots\right)$
X_{488}	$\left(\frac{S_A - a^2}{S_A + S} : \cdots : \cdots\right)$	$\left(\frac{1}{b^2c^2 + S_{BC} + (S_A + S_B + S_C)S} : \cdots : \cdots\right)$

(3) In comparison with Proposition 6 (a), if instead of traces, we take $A_{\mathsf{a}}^{\varepsilon}$, $B_{\mathsf{b}}^{\varepsilon}$ and $C_{\mathsf{c}}^{\varepsilon}$ to be the *pedals* of a point P on the sidelines of ABC, then the locus of P for which $D_{\mathsf{a}}^{\varepsilon}$, $D_{\mathsf{b}}^{\varepsilon}$ and $D_{\mathsf{c}}^{\varepsilon}$ are collinear turns out to be a conic, though with equation too complicated to record here.

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