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On the Existence of Triangles with Given Lengths of One Side and Two Adjacent Angle Bisectors

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Abstract. We give a necessary and sufficient condition for the existence of a triangle with given lengths of one side and the two adjacent angle bisectors.

1. Introduction

It is known that given three lengths ℓ_1, ℓ_2, ℓ_3, there is always a triangle whose three internal angle bisectors have lengths ℓ_1, ℓ_2, ℓ_3. See [1]. In this note we consider the question of existence and uniqueness of a triangle with given lengths of one side and the bisectors of the two angles adjacent to it. Recall that in a triangle ABC with sidelengths a, b, c, the bisector of angle A (with opposite side a) has length

$$\ell = \frac{2bc}{b+c}\cos\frac{A}{2} = \sqrt{bc\left(1-\frac{a^2}{(b+c)^2}\right)}. \tag{1}$$

We shall prove the following theorem.

Theorem 1. *Given a, ℓ_1, $\ell_2 > 0$, there is a unique triangle ABC with $BC = a$, and the lengths of the bisectors of angles B, C equal to ℓ_1 and ℓ_2 if and only if*

$$\sqrt{\ell_1^2+\ell_2^2} < 2a < \ell_1+\ell_2+\sqrt{\ell_1^2-\ell_1\ell_2+\ell_2^2}.$$

2. Uniqueness

First we prove that if such a triangle exists, then it is unique.

Denote the sidelengths of the triangle by a, x, y. If the angle bisectors on the sides x and y have lengths ℓ_1 and ℓ_2 respectively, then from (1) above,

$$y =(a+x)\sqrt{1-\frac{t_2}{x}}, \tag{2}$$

$$x =(a+y)\sqrt{1-\frac{t_1}{y}}, \tag{3}$$

where $t_1 = \frac{\ell_1^2}{a}$, $t_2 = \frac{\ell_2^2}{a}$, $(t_1 < y, t_2 < x)$.

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Let $t > 0$. We consider the function $y : (t, \infty) \to (0, \infty)$ defined by

$$y(x) = (a + x)\sqrt{1 - \frac{t}{x}}.$$

Obviously, y is a continuous function on the interval (t, ∞). It is increasing and has an oblique asymptote $y = x + a - \frac{t}{2}$. It is easy to check that $y'' < 0$ in (t, ∞), so that y is a *convex* funcion and its graph is below its oblique asymptote. See Figure 1.

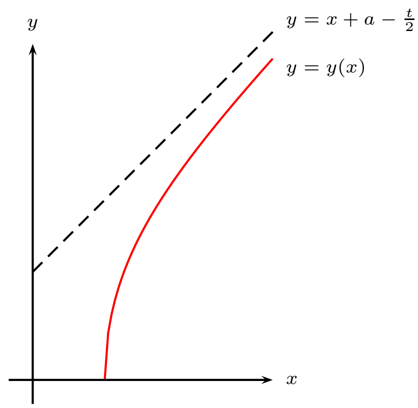

Figure 1

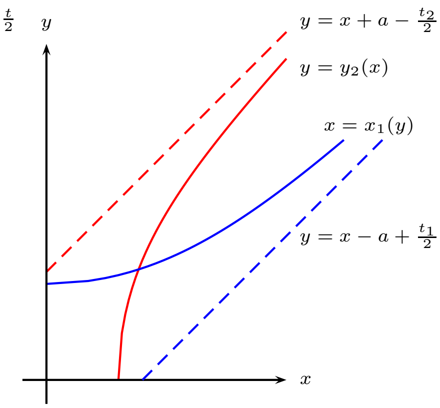

Figure 2

Now consider the system of equations

$$y = (a + x)\sqrt{1 - \frac{t}{x}}, \tag{4}$$

$$x = (a + y)\sqrt{1 - \frac{t}{y}}. \tag{5}$$

It is obvious that if a pair (x, y) satisfies (4), the pair (y, x) satisfies (5), and conversely. These equations therefore define inverse functions, and (5) defines a *concave* function $(0, \infty) \to (t, \infty)$ with an oblique asymptote $y = x - a + \frac{t}{2}$.

Applying to functions $y = y_2(x)$ and $x = x_1(y)$ defined by (2) and (3) respectively, we conclude that the system of equations (2), (3) cannot have more than one solution. See Figure 2.

Proposition 2. *If the side and the bisectors of the adjacent angles of triangle are respectively equal to the side and the bisectors of the adjacent angles of another triangle, then the triangles are congruent.*

Corollary 3 (Steiner-Lehmus theorem)**.** *If a triangle has two equal bisectors, then it is an isosceles triangle.*

Indeed, if the bisectors of the angles A and C of triangle ABC are equal, then triangle ABC is congruent to CBA, and so $AB = CB$.

3. Existence

Now we consider the question of existence of a triangle with given a, ℓ_1 and ℓ_2.

First of all note that in order for the system of equations (2), (3) to have a solution, it is necessary that $x + a - \frac{t_2}{2} > x - a + \frac{t_1}{2}$. Geometrically, this means that the asymptote of (2) is above that of (3). Thus, $2a > \frac{t_1+t_2}{2} = \frac{\ell_1^2+\ell_2^2}{2a}$, and

$$2a > \sqrt{\ell_1^2 + \ell_2^2}. \tag{6}$$

For the three lengths a, x, y to satisfy the triangle inequality, note that from (2) and (3), we have $y < a + x$ and $x < a + y$. If $x > a$ or $y > a$, then clearly $x + y > a$. We shall therefore restrict to $x < a$ and $y < a$.

Let BC be a given segment of length a. Consider a point Y in the plane such that the bisector of angle B of triangle YBC has a given length ℓ_1. It is easy to see from (1) that the length of BY is given by

$$y = \frac{a\ell_1}{2a\cos\frac{\theta}{2} - \ell_1} \quad \text{if } \angle CBY = \theta. \tag{7}$$

Let $\alpha = 2\arccos\frac{\ell_1}{2a}$. (7) defines a monotonic increasing function $y = y(\theta) : (0, \alpha) \to \left(\frac{a\ell_1}{2a-\ell_1}, \infty\right)$. It is easy to check that for $\theta \in (0, \alpha)$,

$$y > \frac{a\ell_1}{2a - \ell_1} > y\cos\theta.$$

The locus of Y is a continuous curve ξ_1 beginning at (but not including) a point M on the interval BC with $BM = \frac{a\ell_1}{2a-\ell_1}$. It has an oblique asymptote which forms an angle α with the line BC. See Figure 3. Since we are interested only in the case $y < a$, we may assume $a > \ell_1$. The angle α exceeds $\frac{2\pi}{3}$.

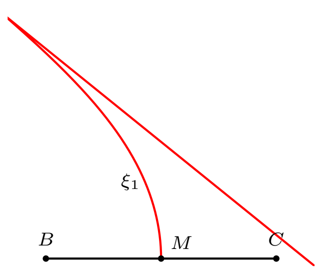

Figure 3

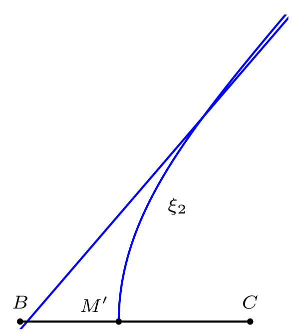

Figure 4

Consider now the locus of point Z such that the bisector of angle C of triangle ZBC has length $\ell_2 < a$. The same reasoning shows that this is a curve ξ_2 beginning at (but not including) a point M' on BC such that $M'C = \frac{a\ell_2}{2a-\ell_2}$, which has an oblique asymptote making an angle $2\arccos\frac{\ell_2}{2a}$ with CB. Again, this angle exceeds $\frac{2\pi}{3}$. See Figure 4.

The two curves ξ_1 and ξ_2 intersect if and only if $BM > BM'$, *i.e.*, $BM + M'C > a$. This gives

$$\frac{\ell_1}{2a-\ell_1} + \frac{\ell_2}{2a-\ell_2} > 1.$$

Simplifying, we have $4a^2 - 4a(\ell_1 + \ell_2) + 3\ell_1\ell_2 < 0$, or

$$\ell_1 + \ell_2 - \sqrt{\ell_1^2 - \ell_1\ell_2 + \ell_2^2} < 2a < \ell_1 + \ell_2 + \sqrt{\ell_1^2 - \ell_1\ell_2 + \ell_2^2}.$$

Since $a > \ell_1, \ell_2$, the first inequality always holds. Comparison with (6) now completes the proof of Theorem 1.

In particular, for the existence of an isosceles triangle with base a and bisectors of the equal angles of length ℓ, it is necessary and sufficient that $\frac{\sqrt{2}}{2} < \frac{a}{\ell} < \frac{3}{2}$.

Reference

[1] P. Mironescu and L. Panaitopol, The existence of a triangle with prescribed angle bisector lengths, *Amer. Math. Monthly*, 101 (1994) 58–60.

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