A Purely Synthetic Proof of the Droz-Farny Line Theorem

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Abstract. We present a purely synthetic proof of the theorem on the Droz-Farny line, and a brief biographical note on Arnold Droz-Farny.

1. The Droz-Farny line theorem

In 1899, Arnold Droz-Farny published without proof the following remarkable theorem.

Theorem 1 (Droz-Farny [2]). If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.

Figure 1. The perpendicular lines $\mathcal{L}$ and $\mathcal{L}'$ through the orthocenter $H$ of triangle $ABC$ intersect the sidelines $BC$ at $X$, $X'$, $CA$ at $Y$, $Y'$, and $AB$ at $Z$, $Z'$ respectively. The midpoints $M_a$, $M_b$, $M_c$ of the segments $XX'$, $YY'$, $ZZ'$ are collinear.

It is not known if Droz-Farny himself has given a proof. The Droz-Farny line theorem was presented again without any proof in 1995 by Ross Honsberger [9,
It also appeared in 1986 as Problem II 206 of [16, pp.111,311-313] without references but with an analytic proof. This “remarkable theorem”, as it was named by Honsberger, has been the subject of many recent messages in the Hyacinthos group. If Nick Reingold [15] proposes a projective proof of it, he does not yet show that the considered circles intersect on the circumcircle. Darij Grinberg taking up an elegant idea of Floor van Lamoen presents a first trigonometric proof of this “rather difficult theorem” [5, 12, 3] which is based on the pivot theorem and applied on degenerated triangles. Grinberg also offers a second trigonometric proof, which starts from a generalization of the Droz-Farny’s theorem simplifying by the way the one of Nicolaos Dergiades and gives a demonstration based on the law of sines [6]. Milorad Stevanović [17] presents a vector proof. Recently, Grinberg [8] picks up an idea in a newsgroup on the internet and proposes a proof using inversion and a second proof using angle chasing. In this note, we present a purely synthetic proof.

2. Three basic theorems

**Theorem 2** (Carnot[1, p.101]). *The segment of an altitude from the orthocenter to the side equals its extension from the side to the circumcircle.*

![Figure 2](image)

**Theorem 3.** Let \( \mathcal{L} \) be a line through the orthocenter of a triangle \( ABC \). The reflections of \( \mathcal{L} \) in the sidelines of \( ABC \) are concurrent at a point on the circumcircle.

See [11, p.99] or [10, §333].
Theorem 4 (Miquel’s pivot theorem [13]). If a point is marked on each side of a triangle, and through each vertex of the triangle and the marked points on the adjacent sides a circle is drawn, these three circles meet at a point.

Figure 3.

See also [10, §184, p.131]. This result stays true in the case of tangency of lines or of two circles. Very few geometers contemporary to Miquel had realised that this result was going to become the spring of a large number of theorem.

3. A synthetic proof of Theorem 1

The right triangle case of the Droz-Farny theorem being trivial, we assume triangle $ABC$ not containing a right angle. Let $C$ be the circumcircle of $ABC$.

Let $C_a$ (respectively $C_b$, $C_c$) be the circumcircle of triangle $HXX'$ (respectively $HYY'$, $HZZ'$), and $H_a$ (respectively $H_b$, $H_c$) be the symmetric point of $H$ in the line $BC$ (respectively $CA$, $AB$). The circles $C_a$, $C_b$ and $C_c$ have centers $M_a$, $M_b$ and $M_c$ respectively.

Figure 4.

According to Theorem 2, $H_a$ is on the circle $C$. $XX'$ being a diameter of the circle $C_a$, $H_a$ is on the circle. Consequently, $H_a$ is an intersection of $C$ and $C_a$, and
the perpendicular to $BC$ through $H$. In the same way, $H_b$ is an intersection of $C$ and $C_b$, and the perpendicular to $CA$ through $H$. See Figure 4.

Consider the point $H_c$, the symmetric of $H$ in the line $AB$. According to Theorem 2, $H_a$ is on the circle $C$. Applying Theorem 3 to the line $XYZ$ through $H$, we conclude that the lines $H_aX$, $H_bY$ and $H_cZ$ intersect at a point $N$ on the circle $C$. See Figure 5.

Applying Theorem 4 to the triangle $XNY$ with the points $H_a$, $H_b$ and $H$ (on the lines $XN$, $NY$ and $YX$ respectively), we conclude that the circles $C$, $C_a$, and $C_b$ pass through a common point $M$.

*Mutatis mutandis*, we show that the circles $C$, $C_b$, and $C_c$ also pass through the same point $M$.

The circle $C_a$, $C_b$, and $C_c$, all passing through $H$ and $M$, are coaxial. Their centers are collinear. This completes the proof of Theorem 1.

4. A biographical note on Arnold Droz-Farny

Arnold Droz, son of Edouard and Louise Droz, was born in La Chaux-de-Fonds (Switzerland) on February 12, 1856. After his studies in the canton of Neuchâtel, he went to Munich (Germany) where he attended lectures given by Felix Klein, but he finally preferred geometry. In 1880, he started teaching physics and mathematics in the school of Porrentruy (near Basel) where he stayed until 1908. He is known for having written four books between 1897 and 1909, two of them about geometry. He also published in the *Journal de Mathématiques Élémentaires et
Spéciales (1894, 1895), and in L’intermédiaire des Mathématiciens and in the Educational Times (1899) as well as in Mathesis (1901). As he was very sociable, he liked to be in contact with other geometers likes the Italian Virginio Retali and the Spanish Juan Jacobo Duran Loriga. In his free time, he liked to climb little mountains and to watch horse races. He was married to Lina Farny who was born also in La Chaux-de-Fonds. He died in Porrentruy on January 14, 1912 after having suffered from a long illness. See [4, 14].

References


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