A Projective Generalization of the Droz-Farny Line Theorem

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Abstract. We give a projective generalization of the Droz-Farny line theorem.

Ayme [1] has given a simple, purely synthetic proof of the following theorem by Droz-Farny.

Theorem 1 (Droz-Farny [1]). *If two perpendicular straight lines are drawn through the orthocenter of a triangle. they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

In this note we give and prove a projective generalization. We begin with a simple observation. Given triangle ABC and a point S, the perpendiculars to AS, BS, CS at A, B, C respectively concur if and only if S lies on the circumcircle of ABC. In this case, their common point is the antipode of S on the circumcircle.

Now, consider 5 points A, B, C, I, I' lying on a conic $\mathcal{E}$ and a point S not lying on the line II'. Using a projective transformation mapping the circular points at infinity to I and I', we obtain the following.

Proposition 2. *The polar lines of S with respect to the pairs of lines (AI, AI'), (BI, BI'), (CI, CI') concur if and only if S lies on $\mathcal{E}$. In this case, their common point lies on $\mathcal{E}$ and on the line joining S to the pole of II' with respect to $\mathcal{E}$.*

The dual form of this proposition is the following.

Theorem 3. *Let ℓ and ℓ' be two lines intersecting at P, tangent to the same inscribed conic $\mathcal{E}$, and d be a line not passing through P. Let X, Y, Z (respectively X', Y', Z'; X_d, Y_d, Z_d) be the intersections of ℓ (respectively ℓ', d) with the sidelines BC, CA, AB. If X'_d is the harmonic conjugate of X_d with respect to (X, X'), and similarly for Y'_d and Z'_d, then X'_d, Y'_d, Z'_d lie on a same line d' if and only if d touches $\mathcal{E}$. In this case, d' touches $\mathcal{E}$ and the intersection of d and d' lies on the polar of P with respect to $\mathcal{E}$.*

An equivalent condition is that A, B, C and the vertices of the triangle with sidelines ℓ, ℓ', d lie on a same conic.

More generally, consider points X'_{d}, Y'_{d} and Z'_{d} such that the cross ratios

$$(X, X', X_{\mathsf{d}}, X'_{\mathsf{d}}) = (Y, Y', Y_{\mathsf{d}}, Y'_{\mathsf{d}}) = (Z, Z', Z_{\mathsf{d}}, Z'_{\mathsf{d}}).$$

These points X'_{d}, Y'_{d}, Z'_{d} lie on a line d$'$ if and only if d is tangent to $\mathcal{E}$. This follows easily from the dual of Steiner's theorem and its converse: two points P, Q lie on a conic through four given points A, B, C, D if and only if the cross ratios

$$(PA, PB, PC, PD) = (QA, QB, QC, QD).$$

If in Theorem 3 we take for d the line at infinity, we obtain the following.

Corollary 4. *The midpoints of XX', YY', ZZ' lie on a same line* d$'$ *if and only if ℓ and ℓ' touch the same inscribed parabola. In this case, if ℓ and ℓ touch the parabola at M and M',* d$'$ *is the tangent to the parabola parallel to MM'.*

An equivalent condition is that the circumhyperbola through the infinite points of ℓ and ℓ' passes through P.

We shall say that (ℓ, ℓ') is a pair of DF-lines if it satisfies the conditions of Corollary 4 above.

Now, if ℓ and ℓ' are perpendicular, we get immediately:

(a) if $P = H$, then (ℓ, ℓ') is a pair of DF-lines because H lies on any rectangular circumhyperbola, or, equivalently, on the directrix of any inscribed parabola. This is the Droz-Farny line theorem (Theorem 1 above).

(b) if $P \neq H$, then (ℓ, ℓ') is a pair of DF-lines if and only if they are the tangents from P to the inscribed parabola with directrix HP, or, equivalently, they are the parallels at P to the asymptotes of the rectangular circumhyperbola through P.

Remarks. (1) The focus of the inscribed parabola touching ℓ is the Miquel point F of the complete quadrilateral formed by AB, BC, CA, ℓ, and the directrix is the Steiner line of F. See [3].

(2) If the circle through F and with center P intersects the directrix at M, M', the tangents from P to the parabola are the perpendicular bisectors of FM and FM'.

(3) The tripoles of tangents to an inscribed parabola are collinear in a line through G.

(4) Let A_ℓ, B_ℓ, C_ℓ be the intercepts of ℓ on the sides of ABC. Let A_{r}, B_{r}, C_{r} be the reflections of these intercepts through the midpoints of the corresponding sides. Then A_{r}, B_{r}, and C_{r} are collinear on the "isotomic conjugate" of ℓ. Clearly, the isotomic conjugates of lines from a pencil are tangents to an inscribed conic and vice versa. In the case of inscribed parabolas, as above, the isotomic conjugates of the tangents are a pencil of parallel lines. It is trivial that lines dividing in equal ratios the intercepted segments by two parallel lines are again parallel. So, by isotomic conjugation of lines this holds for tangents to a parabola as well.

These remarks lead to a number of simple constructions of pairs of DF-lines satisfying a given condition.

References

[1] J.-L. Ayme, A synthetic proof of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004) 219–224.
[2] H. S. M. Coxeter, *The Real Projective Plane*, 3rd edition, Springer-Verlag, 1992.
[3] J.-P. Ehrmann, Steiner's theorems on the complete quadrilateral, *Forum Geom.*, 4 (2004) 35–56.

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