

The Twin Circles of Archimedes in a Skewed Arbelos

Hiroshi Okumura and Masayuki Watanabe

Abstract. Any area surrounded by three mutually touching circles is called a skewed arbelos. The twin circles of Archimedes in the ordinary arbelos can be generalized to the skewed arbelos. The existence of several pairs of twin circles, under certain conditions, is demonstrated.

1. Introduction

Let O be an arbitrary point on the segment AB in the plane and α , β and γ the semicircles on the same side of the diameters AO , BO and AB , respectively. The area surrounded by the three semicircles is called an arbelos or a shoemaker's knife (see Figure 1). The common internal tangent of α and β divides the arbelos into two curvilinear triangles and the incircles of these triangles are congruent. They are called the twin circles of Archimedes or Archimedean twin circles. The authors of [3] pose the following question: Is it possible to find any interesting properties of a "skewed arbelos", in which the centers of the three circles α , β and γ are not collinear (see Figure 2), without resorting to trigonometry? In this article, we show several interesting properties of the skewed arbelos, one of them being the existence, in certain situations, of up to four pairs of twin circles. This property is a generalization of the existence of the twin circles of Archimedes in the ordinary arbelos.

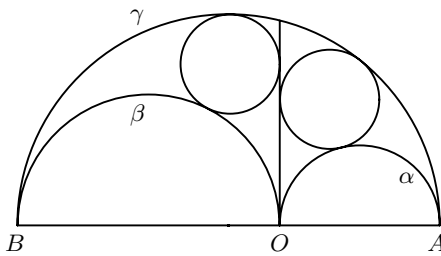


Figure 1.

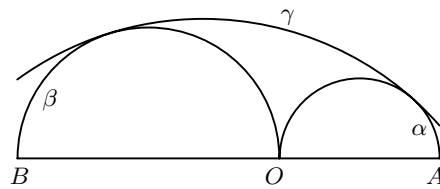


Figure 2.

Publication Date: December 29, 2004. Communicating Editor: Paul Yiu.

The authors express their sincere thanks to the referee for valuable useful comments that improved this paper.

2. The skewed arbelos

Throughout this paper, α and β are circles with centers $(a, 0)$ and $(0, -b)$ for positive real numbers a and b , touching externally at the origin O , and γ is another circle touching α and β at points different from O . We do not exclude the case, when γ touches α and β externally or when γ is one of the common external tangents of α and β . There are always two different areas surrounded by α , β and γ (if γ touches α and β externally, we still consider the exterior infinite area to be surrounded by these three circles). We select one of these areas in the following way (see Figure 3): If γ touches α and β externally from above, we choose the finite area, if γ touches α and β internally, we choose the upper area, and if γ touches α and β externally from below, we choose the infinite area. We call this area the *skewed arbelos* formed by the circles α , β and γ .

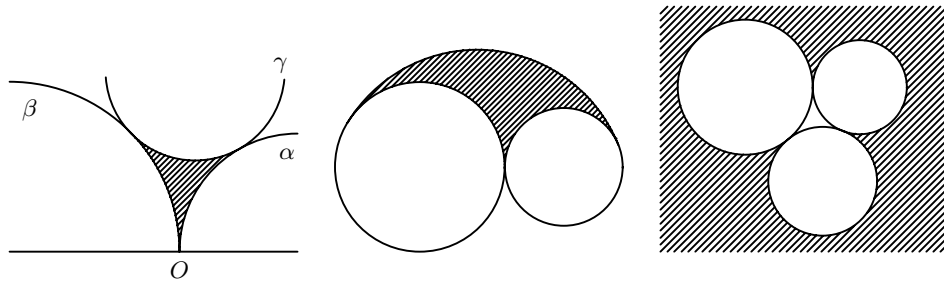


Figure 3.

Now we define four sets of tangent circles (or four chains of circles). If we include the lines parallel to the y -axis (circles of infinite radius) among the circles touching the y -axis, there are always two different circles touching γ , α and the y -axis, which do not pass through the tangency point of α and γ . We label the one inside of the skewed arbelos as α_0^+ and the other one as α_0^- . The circles β_0^+ and β_0^- touching γ , β and the y -axis are defined similarly (see Figure 4). There are also two circles touching α , α_0^+ and the y -axis, one intersecting γ and the other not. We label the former as α_{-1}^+ and the latter as α_{-1}^- . The circles α_2^+ , α_3^+ , \dots can be defined inductively in the following way: Assuming the circles α_{i-1}^+ and α_i^+ are defined, α_{i+1}^+ is the circles touching α , α_i^+ and the y -axis and different from α_{i-1}^+ . The circles α_{-2}^+ , α_{-3}^+ , \dots are defined similarly. Now the entire chain of circles

$$\{\dots, \alpha_{-2}^+, \alpha_{-1}^+, \alpha_0^+, \alpha_1^+, \alpha_2^+, \dots\}$$

is defined. The other three chains of circles

$$\begin{aligned} &\{\dots, \alpha_{-2}^-, \alpha_{-1}^-, \alpha_0^-, \alpha_1^-, \alpha_2^-, \dots\}, \\ &\{\dots, \beta_{-2}^+, \beta_{-1}^+, \beta_0^+, \beta_1^+, \beta_2^+, \dots\}, \\ &\{\dots, \beta_{-2}^-, \beta_{-1}^-, \beta_0^-, \beta_1^-, \beta_2^-, \dots\}, \end{aligned}$$

where α_{-1}^- , β_{-1}^+ and β_{-1}^- intersect γ , are defined similarly. If α_i^+ , α_i^- , β_i^+ and β_i^- are proper circles, their radii are denoted by a_i^+ , a_i^- , b_i^+ and b_i^- , respectively. If, for example, α_i^+ is a line parallel to the y -axis, we consider the reciprocal value of its radius to be zero, even though we cannot define the radius a_i^+ itself.

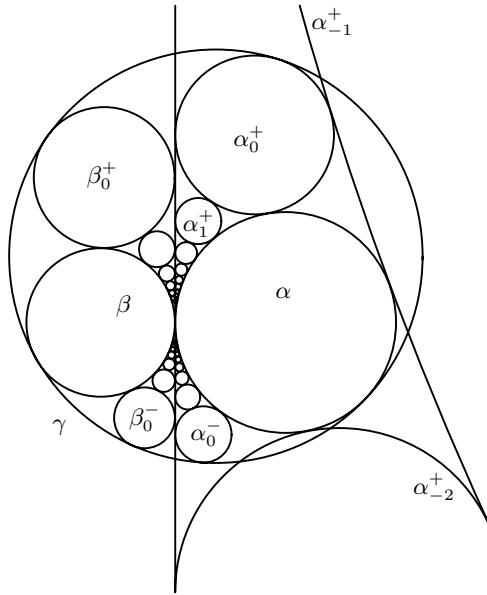


Figure 4.

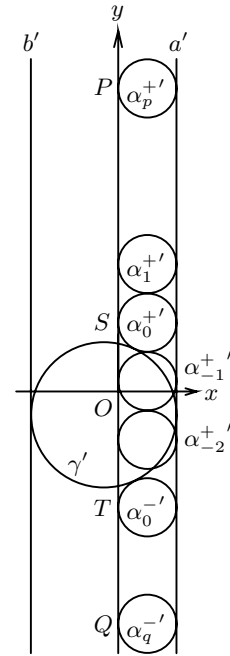


Figure 5.

If α_k^+ is a proper circle and the centers of α_k^+ and α_i^+ lie on the same side of the x -axis for all proper circles α_i^+ ($i > k$), we define $\sigma(\alpha_k^+) = 1$, otherwise we define $\sigma(\alpha_k^+) = -1$. If α_k^+ is a line parallel to the y -axis, we define $\sigma(\alpha_k^+) = 1$. The numbers $\sigma(\alpha_k^-)$, $\sigma(\beta_k^+)$, $\sigma(\beta_k^-)$ are defined similarly. If γ touches α and β internally, $\sigma(\alpha_0^+) = \sigma(\alpha_0^-) = 1$ and consequently, $\sigma(\alpha_i^+) = \sigma(\alpha_i^-) = 1$ for all non-negative integers i . Let s_i and t_j be the y -coordinates of the tangency points of the circles α_i^+ and α_j^- with the y -axis. If α_i^+ (or α_j^-) is a line, we consider $s_i = 0$ (or $t_j = 0$). We define $\sigma(\alpha_i^+, \alpha_j^-) = 1$, when $s_i t_j > 0$ and $s_i \leq t_j$, or when $s_i t_j \leq 0$ and $s_i \geq t_j$, otherwise $\sigma(\alpha_i^+, \alpha_j^-) = -1$. The number $\sigma(\beta_i^+, \beta_j^-)$ is defined similarly. If the centers of the three circles α , β and γ are collinear, we get an ordinary arbelos. In this case, the radii of the twin circles, which we denote as r_A , are equal to $ab/(a + b)$.

Theorem 1. For any integers p and q ,

$$\sigma(\alpha_p^+, \alpha_q^-) \left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}} \right) = \left| \frac{2}{\sqrt{r_A}} + \frac{p+q}{\sqrt{a}} \right|$$

and for given circles α and β , the value on the right side does not depend on the circle γ .

Proof. Let p and q be arbitrary integers. We invert the figure in the circle with center O and radius $k = 2\sqrt{ab}$, and label the images of all circles with a prime (see Figure 5). The circles $\alpha_0^{+'}$ and $\beta_0^{+'}$ always lie above the circles $\alpha_0^{-'}$ and $\beta_0^{-'}$ respectively. $\sigma(\alpha_p^+) = 1$ (resp. $\sigma(\alpha_q^-) = 1$) is equivalent to the fact that the center of $\alpha_p^{+'}$ (resp. $\alpha_q^{-'}$) lies in the region $y \geq 0$ (resp. $y \leq 0$) and $\sigma(\alpha_p^+, \alpha_q^-) = 1$ is equivalent to the fact that the y -coordinate of the center of $\alpha_p^{+'}$ is greater than or equal to the y -coordinate of the center of $\alpha_q^{-'}$. Since α' is a line parallel to the y -axis, the circles $\alpha_p^{+'}$ and $\alpha_q^{-'}$ are congruent, and we denote their common radius as a' . Similarly, we denote the common radius of the circles $\beta_p^{+'}$ and $\beta_q^{-'}$ as b' . Let us assume that $\alpha_0^{+'}$, $\alpha_0^{-'}$, $\alpha_p^{+'}$ and $\alpha_q^{-'}$ touch the y -axis at the points S , T , P and Q . If α_p^+ is a proper circle, the inversion center O is also the center of homothety of the circles α_p^+ and $\alpha_p^{+'}$ with homothety coefficient equal to the square of the radius of the inversion circle (i.e., to the power of inversion) divided by the power $O(\alpha_p^{+'})$ of the point O to the inverted circle $\alpha_p^{+'}$: $k^2/O(\alpha_p^{+'})$. Hence, the radius of α_p^+ can be expressed as $a_p^+ = k^2 a' / O(\alpha_p^{+'})$ [5, p. 50]. The reciprocal value of this radius is then $1/a_p^+ = |OP|^2 / (4aba')$. The last equation holds even if α_p^+ is a line parallel to the y -axis. Similarly, the reciprocal value of the radius of the circle α_q^- is equal to $1/a_q^- = |OQ|^2 / (4aba')$. The segment length of the common external tangent of the externally touching circles $\alpha_0^{+'}$, $\alpha_0^{-'}$, or $\alpha_p^{+'}$, $\alpha_q^{-'}$ between the tangency points is equal to $|ST|/2 = 2\sqrt{(a'+b')a'}$. Consequently,

$$\begin{aligned} \sigma(\alpha_p^+, \alpha_q^-) \left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}} \right) &= \sigma(\alpha_p^+, \alpha_q^-) \left(\frac{\sigma(\alpha_p^+) |OP| + \sigma(\alpha_q^-) |OQ|}{2\sqrt{ab}\sqrt{a'}} \right) \\ &= \frac{|PQ|}{2\sqrt{ab}\sqrt{a'}} = \frac{||ST| + 2pa' + 2qa'|}{2\sqrt{ab}\sqrt{a'}} = \frac{|4\sqrt{(a'+b')a'} + 2(p+q)a'|}{2\sqrt{ab}\sqrt{a'}}. \end{aligned}$$

Since $4aa' = 4bb' = 4ab$ by the definition of inversion, we get $a' = b$ and $b' = a$, and we finally obtain

$$\sigma(\alpha_p^+, \alpha_q^-) \left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}} \right) = \left| 2\sqrt{\frac{1}{a} + \frac{1}{b}} + \frac{p+q}{\sqrt{a}} \right|.$$

The proof of the theorem is now complete. □

We can get a similar expression for the radii of the circles β_r^+ and β_s^- for any integers s and r . According to the proof of Theorem 1, the circles α_p^+ and α_q^- coincide if and only if $P = Q$ and this is also equivalent to

$$\sqrt{1 + \frac{a}{b}} = -\frac{p + q}{2}.$$

Hence, we obtain the following corollary:

Corollary 2. *The two chains $\{\dots, \alpha_{-2}^+, \alpha_{-1}^+, \alpha_0^+, \alpha_1^+, \alpha_2^+, \dots\}$ and $\{\dots, \alpha_{-2}^-, \alpha_{-1}^-, \alpha_0^-, \alpha_1^-, \alpha_2^-, \dots\}$ coincide if and only if there is an integer n such that*

$$\frac{a}{b} = \frac{n^2}{4} - 1.$$

In this event, $\alpha_p^+ = \alpha_{-|n|-p}^-$ for any integer p . For given circles α and β , this property does not depend on the circle γ .

From the inverted skewed arbelos (see Figure 5), it is easy to see that the circles $\alpha_p^+, \alpha_p^-, \beta_q^+$ and β_q^- have two common tangent circles for any integers p and q . The line passing through the center $O_{\gamma'}$ of the circle γ' and perpendicular to the y -axis is also perpendicular to the lines α' and β' and to the circle γ' . Let δ be the circle, which is inverted into this line. Since inversion preserves angles between circles or lines, the circle δ is centered on the y -axis and perpendicular to the circles α, β and γ . Consequently, the inversion in δ with positive power leaves the y -axis and these circles in place and exchanges α_p^+, α_p^- and β_q^+ and β_q^- , respectively. Since the inversion center is also the center of homothety of a circle and its image (external, if the inversion center is outside of the circle, and internal in the opposite case), the external center of similitude of the circles α_p^+ and α_p^- is the same point on the y -axis (the center of the circle δ) for any integer p . This point is also the external center of similitude of β_q^+ and β_q^- for any integer q .

Since $\sigma(\alpha_p^+, \alpha_{-p}^-) = \sigma(\beta_q^+, \beta_{-q}^-) = 1$ for any integers p and q , we get the following corollary:

Corollary 3. *For any integers p and q ,*

$$\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_{-p}^-)}{\sqrt{a_{-p}^-}} = \frac{\sigma(\beta_q^+)}{\sqrt{b_q^+}} + \frac{\sigma(\beta_{-q}^-)}{\sqrt{b_{-q}^-}} = \frac{2}{\sqrt{r_A}}$$

and for given circles α and β , the constant value on the right side does not depend on the circle γ .

Corollary 4. *If γ touches α and β internally,*

$$\frac{1}{\sqrt{a_0^+}} + \frac{1}{\sqrt{a_0^-}} = \frac{1}{\sqrt{b_0^+}} + \frac{1}{\sqrt{b_0^-}} = \frac{2}{\sqrt{r_A}}$$

and for given circles α and β , the constant value on the right side does not depend on the circle γ .

From the last corollary, it is obvious that Theorem 1 is a generalization of the existence of the twin circles of Archimedes in the ordinary arbelos.

3. The n -th twin circles of Archimedes (symmetrical case)

In this section, we demonstrate that in certain situations, a skewed arbelos also has a twin circle property, which is a generalization of the twin circles of Archimedes in an ordinary arbelos. We use the same notations as in the previous section. If one circle of the set $\{\alpha_n^+, \alpha_{-n}^-, \alpha_{-n}^+, \alpha_n^-\}$ is congruent to one circle from the set $\{\beta_n^+, \beta_{-n}^-, \beta_{-n}^+, \beta_n^-\}$ for some integer n , the congruent pair is called a *pair of the n -th twin circles of Archimedes*. The twin circles of Archimedes in the ordinary arbelos are represented by one pair of the 0-th twin circles.

If the circles α , β and γ form an ordinary arbelos, the intersection of γ with the y -axis in the region $y > 0$ has the coordinates $(0, 2\sqrt{ab})$. For a real number z , the point $(0, 2\sqrt{ab}/z)$ is denoted by V_z and we consider V_0 to be the point at infinity on the y -axis. We show that $V_{n\pm 1}$ are closely related to some pairs of the n -th twin circles of Archimedes. There are also other points on the y -axis, related to pairs of the n -th twin circles of Archimedes. For a real number z , consider the following points with the y -coordinates

$$\begin{aligned} W_z^{++} &: \frac{-2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{z(\sqrt{a}+\sqrt{b})+2\sqrt{a+b}}, \\ W_z^{--} &: \frac{-2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{z(\sqrt{a}+\sqrt{b})-2\sqrt{a+b}}, \\ W_z^{+-} &: \frac{-2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{z(\sqrt{a}-\sqrt{b})+2\sqrt{a+b}}, \\ W_z^{-+} &: \frac{-2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{z(\sqrt{a}-\sqrt{b})-2\sqrt{a+b}}. \end{aligned}$$

Reflecting the points V_z , W_z^{++} and W_z^{+-} in the x -axis, we get the points V_{-z} , W_{-z}^{--} and W_{-z}^{-+} . Since $\sqrt{2} \leq 2\sqrt{a+b}/(\sqrt{a}+\sqrt{b}) < 2$, W_n^{++} and W_n^{--} cannot be the point at infinity on the y -axis for any integer n , but it can happen that each of W_n^{+-} and W_n^{-+} is identical with the point at infinity for some a , b and integer n . If the circle γ passes, for example, through both V_{n+1} and V_{n-1} , we say that γ passes through $V_{n\pm 1}$.

Theorem 5. *Let n be an integer and $a \neq b$.*

(i) $1/a_n^+ = 1/b_n^+$ if and only if the circle γ passes through $V_{n\pm 1}$ or $W_{n\pm 1}^{++}$. If γ passes through $V_{n\pm 1}$,

$$\frac{1}{a_n^+} = \frac{1}{b_n^+} = \left(n \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right)^2 \tag{1}$$

and if γ passes through $W_{n\pm 1}^{++}$,

$$\frac{1}{a_n^+} = \frac{1}{b_n^+} = \left(\left(n \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right) \left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} \right) \right)^2. \tag{2}$$

(ii) $1/a_{-n}^- = 1/b_{-n}^-$ if and only if the circle γ passes through $V_{n\pm 1}$ or $W_{n\pm 1}^-$. If γ passes through $V_{n\pm 1}$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_{-n}^-} = \left(-n \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right)^2$$

and if γ passes through $W_{n\pm 1}^-$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_{-n}^-} = \left(\left(-n \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right) \left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}} \right) \right)^2.$$

(iii) $1/a_{-n}^+ = 1/b_{-n}^-$ if and only if the circle γ passes through $V_{n\pm 1}$ or $W_{n\pm 1}^{+-}$. If γ passes through $V_{n\pm 1}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(-n \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right)^2$$

and if γ passes through $W_{n\pm 1}^{+-}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(\left(-n \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right) \left(\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} \right) \right)^2.$$

(iv) $1/a_n^- = 1/b_{-n}^+$ if and only if the circle γ passes through $V_{n\pm 1}$ or $W_{n\pm 1}^{-+}$. If γ passes through $V_{n\pm 1}$,

$$\frac{1}{a_n^-} = \frac{1}{b_{-n}^+} = \left(n \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right)^2$$

and if γ passes through $W_{n\pm 1}^{-+}$,

$$\frac{1}{a_n^-} = \frac{1}{b_{-n}^+} = \left(\left(n \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right) \left(\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} \right) \right)^2.$$

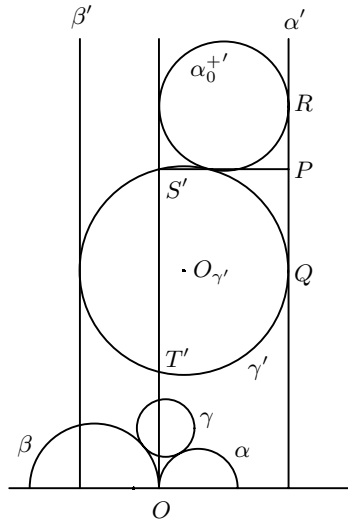


Figure 6.

Proof. Let S and T be the intersections of γ and the y -axis, where S lies on the arc or the line forming the boundary of the skewed arbelos. We denote the y -coordinates of S and T by s and t . If the circle γ touches α and β internally, $t < 0 < s$, otherwise $s < t$. We invert the figure in the circle centered at O and with radius $2\sqrt{ab}$ as in the proof of Theorem 1 (see Figure 6), label the images of all circles and points with a prime and denote the radii of $\alpha_n^{+'}$ and $\beta_n^{+'}$ by a' and b' . Then we obtain $a' = b$ and $b' = a$. Let the line parallel to the x -axis and passing through S' intersect the line α' at the point P . Let γ' and $\alpha_0^{+'}$ touch α' at the points Q and R , respectively, and let $O_{\gamma'}$ be the center of the circle γ' . From the right triangle formed by the lines $O_{\gamma'}S'$, $S'P$ and the line through $O_{\gamma'}$ parallel to the y -axis, we get $|PQ| = 2\sqrt{a'b'}$. The segment length of the common external tangent of the touching circles γ' , $\alpha_0^{+'}$ between the tangency points is equal to $|QR| = 2\sqrt{(a' + b')a'}$. Hence, the reciprocal radius of α_n^+ is equal to

$$\begin{aligned} \frac{1}{a_n^+} &= \frac{O(\alpha_n^{+'})}{4aba'} = \frac{(s' - |PQ| + |QR| + 2na')^2}{4aba'} \\ &= \frac{(s' - 2\sqrt{a'b'} + 2\sqrt{(a' + b')a'} + 2na')^2}{4aba'} \\ &= \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a + b)b} + 2nb)^2}{4ab^2}, \end{aligned}$$

where s' is the y -coordinate of the point S' and $O(\alpha_n^{+'})$ is the power of the point O to the inverted circle $\alpha_n^{+'}$. Therefore, $1/a_n^+ = 1/b_n^+$ is equivalent to

$$\frac{(s' - 2\sqrt{ab} + 2\sqrt{(a + b)b} + 2nb)^2}{4ab^2} = \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a + b)a} + 2na)^2}{4a^2b}.$$

This quadratic equation for s' has two roots:

$$s' = 2(n+1)\sqrt{ab}. \quad (3)$$

and

$$s' = -2(n-1)\sqrt{ab} - \frac{4\sqrt{ab(a+b)}}{\sqrt{a} + \sqrt{b}}. \quad (4)$$

Since $ss' = 4ab$, these are equivalent to

$$s = \frac{2\sqrt{ab}}{n+1}$$

and

$$s = \frac{-2\sqrt{ab}(\sqrt{a} + \sqrt{b})}{(n-1)(\sqrt{a} + \sqrt{b}) + 2\sqrt{a+b}}.$$

Hence, $1/a_n^+ = 1/b_n^+$ is equivalent to $S = V_{n+1}$ or $S = W_{n-1}^{++}$. If $S = V_{n+1}$, then

$$t' = s' - 2|PQ| = 2(n-1)\sqrt{ab},$$

where t' is the y -coordinate of the point T' . Hence,

$$t = \frac{4ab}{t'} = \frac{2\sqrt{ab}}{n-1},$$

and we obtain $T = V_{n-1}$. Similarly, $S = W_{n-1}^{++}$ implies $T = W_{n+1}^{++}$. Assume now that the circle γ passes through $V_{n\pm 1}$. If $S = V_{n-1}$ and $T = V_{n+1}$, we would have

$$s' - t' = \frac{4ab}{s} - \frac{4ab}{t} = -4\sqrt{ab} < 0,$$

which contradicts to the fact $s' > t'$. Therefore, $S = V_{n+1}$ and s' is given by equation (3). Consequently, we arrive to equation (1):

$$\frac{1}{a_n^+} = \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2} = \left(n \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right)^2.$$

If γ passes through $W_{n\pm 1}^{++}$, $S = W_{n-1}^{++}$. For if $S = W_{n+1}^{++}$, we would again have

$$s' - t' = \frac{4ab}{s} - \frac{4ab}{t} = -4\sqrt{ab} < 0,$$

which is a contradiction. Using equation (4), we arrive to equation (2):

$$\begin{aligned} \frac{1}{a_n^+} &= \frac{\left(-2n\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb - \frac{4\sqrt{ab(a+b)}}{\sqrt{a} + \sqrt{b}} \right)^2}{4ab^2} \\ &= \left(\left(n \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) + \frac{1}{\sqrt{r_A}} \right) \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}} \right)^2. \end{aligned}$$

Cases (ii), (iii) and (iv) can be proved similarly as case (i). The reciprocal radii $1/a_{-n}^-$, $1/a_{-n}^+$ and $1/a_n^-$ are equal to

$$\frac{1}{a_{-n}^-} = \frac{(s' - |PQ| - |QR| + 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} - 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2},$$

$$\frac{1}{a_{-n}^+} = \frac{(s' - |PQ| + |QR| - 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} - 2nb)^2}{4ab^2},$$

$$\frac{1}{a_n^-} = \frac{(s' - |PQ| - |QR| - 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} - 2\sqrt{(a+b)b} - 2nb)^2}{4ab^2}.$$

One root of the quadratic equations corresponding to cases (ii), (iii) and (iv) is always given by equation (3) and the other roots are

$$s' = -2(n-1)\sqrt{ab} + \frac{4\sqrt{ab(a+b)}}{\sqrt{a} + \sqrt{b}}, \tag{5}$$

$$s' = -2(n-1)\sqrt{ab} - \frac{4\sqrt{ab(a+b)}}{\sqrt{a} - \sqrt{b}}, \tag{6}$$

$$s' = -2(n-1)\sqrt{ab} + \frac{4\sqrt{ab(a+b)}}{\sqrt{a} - \sqrt{b}}. \tag{7}$$

□

If the circle γ passes through the point $V_{n\pm 1}$, we label the arbelos as $(V_{n\pm 1})$. The arbeloi $(W_{n\pm 1}^{++})$, $(W_{n\pm 1}^{--})$, $(W_{n\pm 1}^{+-})$ and $(W_{n\pm 1}^{-+})$ are defined similarly. Reflecting the arbeloi $(V_{n\pm 1})$, $(W_{n\pm 1}^{++})$, $(W_{n\pm 1}^{+-})$ in the x -axis yields the arbeloi $(V_{-n\pm 1})$, $(W_{-n\pm 1}^{--})$, $(W_{-n\pm 1}^{-+})$, respectively. Equation (3) is obtained, when the signs of the expressions $s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb$ and $s' - 2\sqrt{ab} + 2\sqrt{(a+b)a} + 2na$ are the same. This implies that in $(V_{n\pm 1})$, the centers of the circles α_n^+ and β_n^+ lie on the same side of the x -axis. On the other hand, equation (4) is obtained, when the signs of these expressions are different from each other. Consequently, in $(W_{n\pm 1}^{++})$, the centers of α_n^+ and β_n^+ lie on the opposite sides of the x -axis. Similarly, we can find, on which sides of the x -axis lie the centers of the n -th twin circles of Archimedes in the remaining arbeloi. These results are arranged in Table 1.

	$(V_{n\pm 1})$	$(W_{n\pm 1}^{++})$	$(W_{n\pm 1}^{--})$	$(W_{n\pm 1}^{+-})$	$(W_{n\pm 1}^{-+})$
same side	α_n^+, β_n^+ $\alpha_{-n}^-, \beta_{-n}^-$			α_{-n}^+, β_n^-	α_n^-, β_{-n}^+
opposite side	α_{-n}^+, β_n^- α_n^-, β_{-n}^+	α_n^+, β_n^+	$\alpha_{-n}^-, \beta_{-n}^-$		

Table 1.

According to Theorem 5, there are four different pairs of the n -th twin circles of Archimedes in $(V_{n\pm 1})$, for any non-zero integer n (see Figure 9). In this case, γ touches α and β externally from below for $n \leq -1$, internally for $n = 0$, externally from above for $n \geq 1$. The twin circles of Archimedes in the ordinary arbelos $(V_{0\pm 1})$ and their radii are obtained for $n = 0$. Figures 7 and 8 show the other pairs of the 0-th twin circles of Archimedes in the arbeloi $(W_{0\pm 1}^{++})$ and $(W_{0\pm 1}^{+-})$. The 0-th twin circles of Archimedes in $(W_{0\pm 1}^{--})$ and $(W_{0\pm 1}^{-+})$ are obtained by reflecting these figures in the x -axis and exchanging all plus and minus signs in the notation.

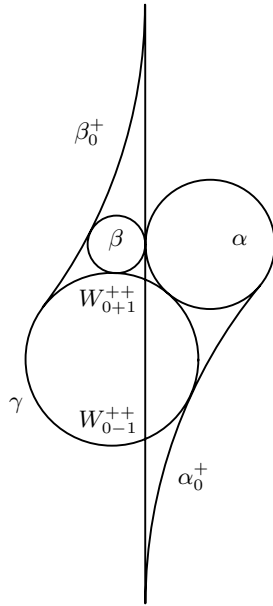


Figure 7. $a_0^+ = b_0^+$ for $(W_{0\pm 1}^{++})$

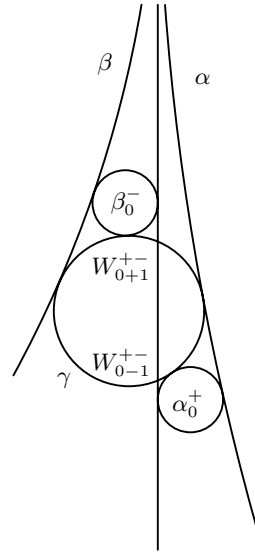


Figure 8. $a_0^+ = b_0^-$ for $(W_{0\pm 1}^{+-})$

If γ is the common external tangent of α and β touching these circles from above, it passes through $V_{1\pm 1}$, because this tangent bisects the segment OV_1 [2]. Hence, we get the following corollary (see Figure 9):

Corollary 6. *If γ is the common external tangent of α and β , touching these circles from above, then (i) $a_1^+ = b_1^+$, (ii) $a_{-1}^- = b_{-1}^-$, (iii) $a_{-1}^+ = b_1^-$, (iv) $a_1^- = b_{-1}^+$, and*

$$(v) \frac{1}{\sqrt{a_1^+}} = \frac{1}{\sqrt{a_1^-}} + \frac{1}{\sqrt{a_{-1}^+}} + \frac{1}{\sqrt{a_{-1}^-}} = \frac{1}{\sqrt{b_1^+}} = \frac{1}{\sqrt{b_1^-}} + \frac{1}{\sqrt{b_{-1}^+}} + \frac{1}{\sqrt{b_{-1}^-}}.$$

Proof. Since $1/\sqrt{a}$, $1/\sqrt{b}$, $1/\sqrt{r_A}$ satisfy the triangle inequality, relation (v) immediately follows from Theorem 5. □

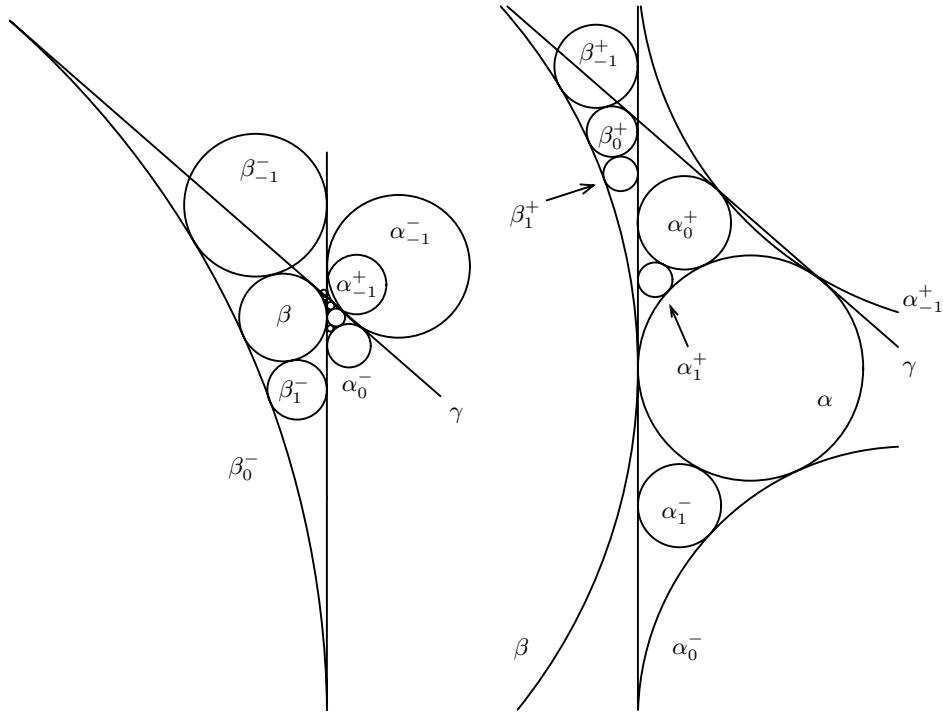


Figure 9. $a_{-1}^+ = b_1^-$, $a_{-1}^- = b_{-1}^-$ for $(V_{1\pm 1})$ Magnified, $a_1^+ = b_1^+$, $a_1^- = b_{-1}^+$

Theorem 7. Any circle touching α and β at points different from O passes through $V_{z\pm 1}$ for some real number z . The proper circle touching α and β at points different from O and passing through $V_{z\pm 1}$ for a real number $z \neq \pm 1$ can be given by the equation

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{8}$$

and conversely. The common external tangents of α and β can be expressed by the equations

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0, \tag{9}$$

which are obtained from equation (8) by approaching z to ± 1 .

Proof. We again invert the circles α , β and γ in the circle centered at O and with radius $2\sqrt{ab}$ as in the proofs of Theorems 1 and 5 and use the same notation. The circle γ is then carried into the circle γ' with radius $c' = a + b$, because $a' = b$ and $b' = a$. The intersection of the skewed arbelos boundary and the y -axis can be expressed as V_{z+1} for some real number z . Let t be the y -coordinate of the other intersection of γ and the y -axis. These intersections are carried into the intersections of γ' and the y -axis with the y -coordinates $s' = 4ab/s = 2(z+1)\sqrt{ab}$ and $t' = s' - 4\sqrt{ab} = 2(z-1)\sqrt{ab}$ (see the proof of Theorem 5), leading to $t = 4ab/t' = 2\sqrt{ab}/(z-1)$. Hence, the other intersection of γ and the y -axis

is identical with the point V_{z-1} . Assume that γ is a proper circle passing through $V_{z\pm 1}$ for a real number $z \neq \pm 1$ and let (x_0, y_0) be the coordinates of the center of γ . Obviously, $y'_0 = (s' + t')/2 = 2z\sqrt{ab}$ and $x'_0 = (2a' - 2b')/2 = b - a$, where (x'_0, y'_0) are the coordinates of the center of γ' . The inversion center at the coordinate origin O is also the center of homothety of the circles γ and γ' , with homothety coefficient equal to $h = 4ab/O(\gamma')$. Since $O(\gamma') = s't' = 4(z^2 - 1)ab$, this homothety coefficient is equal to $h = 1/(z - 1)^2$. Hence, $x_0 = x'_0h = (b - a)/(z^2 - 1)$, $y_0 = y'_0h = 2z\sqrt{ab}/(z^2 - 1)$ and the radius of the circle γ is $c = c'h = (a + b)/|z^2 - 1|$, which leads to equation (8). The converse follows from the fact that (8) determines a circle touching α and β at points different from O and passing through V_{z+1} at the skewed arbelos boundary and this circle is then expressed by (8) again as we have already demonstrated. If $z \rightarrow \pm 1$ and we neglect the terms quadratic in $z^2 - 1$ in (8), the remaining factors $z^2 - 1$ cancel out and we arrive to equation (9). \square

4. Relationship of two skewed arbeloi

In this section, we analyze further properties of the skewed arbeloi $(V_{n\pm 1})$, $(W_{n\pm 1}^{++})$, $(W_{n\pm 1}^{--})$, $(W_{n\pm 1}^{+-})$ and $(W_{n\pm 1}^{-+})$ for an arbitrary integer n and also consider properties of the circle orthogonal to α and β . We assume that the circles α and β are fixed. For these arbeloi, the circles formerly denoted by α_m^+ for an integer m are now labeled explicitly as $\alpha_{n,m}^+$ and their radii as $a_{n,m}^+$. Similarly, we relabel the circles formerly denoted by α_m^- , β_m^+ and β_m^- and their radii. The circle passing through $V_{z\pm 1}$ and touching α and β at points different from O is denoted by γ_z for a real number z . If γ_z is a proper circle, it is expressed by (8), and the circle γ_n forms $(V_{n\pm 1})$ with α and β . Reflecting the arbeloi $(V_{n\pm 1})$, $(W_{n\pm 1}^{++})$ and $(W_{n\pm 1}^{+-})$ in the x -axis yields the arbeloi $(V_{-n\pm 1})$, $(W_{-n\pm 1}^{--})$, $(W_{-n\pm 1}^{-+})$, respectively. Therefore $1/a_{n,m}^\pm = 1/a_{-n,m}^\mp$ and $1/b_{n,m}^\pm = 1/b_{-n,m}^\mp$ in the arbelos pairs $(V_{n\pm 1})$ and $(V_{-n\pm 1})$; $(W_{n\pm 1}^{++})$ and $(W_{-n\pm 1}^{--})$; $(W_{n\pm 1}^{+-})$ and $(W_{-n\pm 1}^{-+})$, but this is trivial.

Since the y -coordinates of the points $V_{n\pm 1}$, $W_{n\pm 1}^{++}$ and $W_{n\pm 1}^{--}$ are symmetrical in a and b , the radii $b_{n,m}^\pm$ can be obtained from $a_{n,m}^\pm$ by replacing a with b and b with a in the arbeloi $(V_{n\pm 1})$, $(W_{n\pm 1}^{++})$ and $(W_{n\pm 1}^{--})$. On the other hand, the y -coordinates of the points $W_{n\pm 1}^{+-}$ and $W_{n\pm 1}^{-+}$ are not symmetrical in a and b . Hence, we cannot draw the same conclusion for the arbeloi $(W_{n\pm 1}^{+-})$ and $(W_{n\pm 1}^{-+})$. Using the same notations as in the proof of Theorem 5, from equation (3) for the arbelos $(V_{n\pm 1})$, we get

$$\frac{1}{a_{n,m}^\pm} = \frac{(s' - 2\sqrt{ab} \pm 2\sqrt{(a+b)b} \pm 2mb)^2}{4ab^2} = \left(\frac{n}{\sqrt{b}} \pm \frac{m}{\sqrt{a}} \pm \frac{1}{\sqrt{r_A}} \right)^2.$$

Using equation (4) for the arbelos $(W_{n\pm 1}^{++})$,

$$\frac{1}{a_{n,m}^+} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} + \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{a_{n,m}^-} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} + \frac{3\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2.$$

Using equation (5) for the arbelos ($W_{n\pm 1}^-$),

$$\frac{1}{a_{n,m}^+} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} - \frac{3\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{a_{n,m}^-} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} - \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2.$$

Using equation (6) for the arbelos ($W_{n\pm 1}^{+-}$),

$$\frac{1}{a_{n,m}^+} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} + \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{a_{n,m}^-} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} + \frac{3\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{b_{n,m}^+} = \left(\frac{n}{\sqrt{a}} - \frac{m}{\sqrt{b}} - \frac{3\sqrt{b} - \sqrt{a}}{\sqrt{b} - \sqrt{a}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{b_{n,m}^-} = \left(\frac{n}{\sqrt{a}} + \frac{m}{\sqrt{b}} - \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}} \frac{1}{\sqrt{r_A}} \right)^2.$$

Using equation (7) for the arbelos ($W_{n\pm 1}^{-+}$),

$$\frac{1}{a_{n,m}^+} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} - \frac{3\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{a_{n,m}^-} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} - \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{b_{n,m}^+} = \left(\frac{n}{\sqrt{a}} - \frac{m}{\sqrt{b}} + \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}} \frac{1}{\sqrt{r_A}} \right)^2,$$

$$\frac{1}{b_{n,m}^-} = \left(\frac{n}{\sqrt{a}} + \frac{m}{\sqrt{b}} + \frac{3\sqrt{b} - \sqrt{a}}{\sqrt{b} - \sqrt{a}} \frac{1}{\sqrt{r_A}} \right)^2.$$

By comparing the above equations, we obtain the following theorem (see Figure 10):

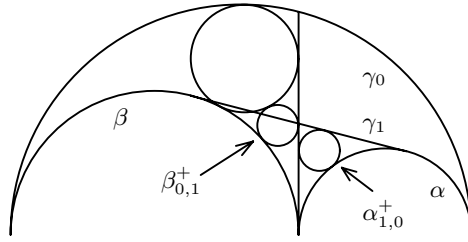


Figure 10. $a_{1,0}^+ = b_{0,1}^+$ for $(V_{1\pm 1})$ and $(V_{0\pm 1})$

Theorem 8. *Let n and m be integers.*

- (i) *For $(V_{n\pm 1})$ and $(V_{m\pm 1})$, we have $1/a_{n,m}^+ = 1/b_{m,n}^+$, $1/b_{n,m}^+ = 1/a_{m,n}^+$, $1/a_{n,-m}^- = 1/b_{m,-n}^-$, and $1/b_{n,-m}^- = 1/a_{m,-n}^-$.*
- (ii) *For $(W_{n\pm 1}^{++})$ and $(W_{m\pm 1}^{++})$, we have $1/a_{n,m}^+ = 1/b_{m,n}^+$ and $1/b_{n,m}^+ = 1/a_{m,n}^+$.*
- (iii) *For $(W_{n\pm 1}^{--})$ and $(W_{m\pm 1}^{--})$, we have $1/a_{n,-m}^- = 1/b_{m,-n}^-$ and $1/b_{n,-m}^- = 1/a_{m,-n}^-$.*
- (iv) *For $(W_{n\pm 1}^{+-})$ and $(W_{m\pm 1}^{+-})$, we have $1/a_{n,-m}^+ = 1/b_{m,n}^-$.*
- (v) *For $(W_{n\pm 1}^{-+})$ and $(W_{m\pm 1}^{-+})$, we have $1/a_{n,m}^- = 1/b_{m,-n}^+$.*
- (vi) *For $(W_{n\pm 1}^{--})$ and $(W_{m\pm 1}^{++})$, we have $1/a_{n,m}^- = 1/b_{m,-n}^+$ and $1/b_{n,m}^- = 1/a_{m,-n}^+$.*
- (vii) *For $(W_{n\pm 1}^{+-})$ and $(W_{m\pm 1}^{-+})$, we have $1/a_{n,m}^+ = 1/b_{m,n}^+$ and $1/b_{n,-m}^- = 1/a_{m,-n}^-$.*

For different real numbers z and w , $\zeta_{z,w}^\alpha$ is the circle touching α , γ_z and γ_w and passing through neither the tangency point of α and γ_z nor the tangency point of α and γ_w and different from β . Similarly the circle $\zeta_{z,w}^\beta$ is defined. In the figure formed by $(V_{0\pm 1})$ and $(V_{1\pm 1})$, two other congruent pairs of inscribed circles can be found (see Figure 11).

Theorem 9. *The circle inscribed in the curvilinear triangle formed by γ_0 , the y -axis, and one of the twin circles of Archimedes touching β is congruent to $\zeta_{0,1}^\alpha$.*

To prove this theorem, we use the following result of the old Japanese geometry [7] (see Figure 12):

Lemma 10. *Assume that the circle C with radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t . If two externally touching circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h , r , r_1 and r_2 are related as*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

Proof. The centers of C_1 and C_2 can be on the opposite sides of the normal dropped on t from the center of C or on the same side of this normal. From the right triangles formed by the centers of C and C_i ($i = 1, 2$), the line parallel to t through the center of C , and the normal dropped on t from the center of C_i , we have

$$|\sqrt{(r - r_1)^2 - (h + r_1 - r)^2} \pm \sqrt{(r - r_2)^2 - (h + r_2 - r)^2}| = 2\sqrt{r_1 r_2},$$

where we used the fact that the segment length of the common external tangent of C_1 and C_2 between the tangency points is equal to $2\sqrt{r_1 r_2}$. The formula of the lemma follows from this equation. \square

Now we can prove Theorem 9. The distance between the common external tangent of α and β and the midpoint of the minor arc of the circle γ_0 formed by this tangent is $2r_A$ [2]. According to Lemma 10, the radii of the two inscribed circles are the root of the same quadratic equation

$$\frac{1}{r} + \frac{1}{a} + \frac{a+b}{ab} = 2\sqrt{\frac{(a+b)^2}{a^2 b r}}$$

From Figure 11, it is obvious that one root of this quadratic equation is equal to b . The other root is then $a^2 b / (a + 2b)^2$. \square

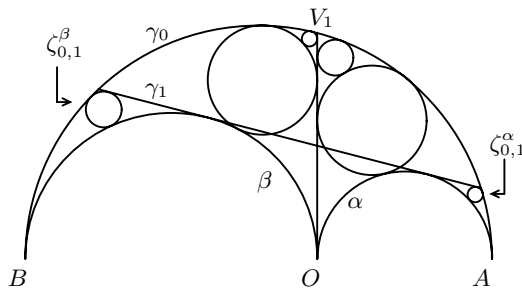


Figure 11. Two small congruent pairs

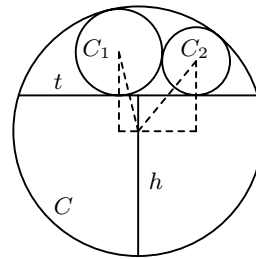


Figure 12.

Now we consider circles orthogonal to α and β . Let $t = (a + b) / \sqrt{ab}$ and let ϵ_z be the circle with a diameter OV_z for a real number z , where we consider ϵ_0 is identical with the x -axis. The mapping $\gamma_z \rightarrow \epsilon_z$ gives a one to one correspondence between the circles touching α and β at points different from O and the circles orthogonal to α and β . The circle ϵ_1 intersects α and γ_1 perpendicularly at their tangency point and the line segment AV_1 also passes through this point [2].

Theorem 11. *Let z and w be real numbers.*

- (i) *The circle ϵ_z intersects α and γ_z perpendicularly at their tangency point and the line segment AV_z also passes through this point.*
- (ii) *Let $w \neq 0$. The circle ϵ_z is orthogonal to any circle touching γ_{z-w} and γ_{z+w} . In particular ϵ_z intersects α and $\zeta_{z-w, z+w}^\alpha$ perpendicularly at their tangency point. If the two circles γ_{z-w} and γ_{z+w} intersect, ϵ_z also passes through their intersection.*
- (iii) *The two circles γ_z and γ_w touch if and only if $z - w = \pm t$. The circle ϵ_z touches $\gamma_{z-t/2}$ and $\gamma_{z+t/2}$ at their tangency point.*
- (iv) *The reciprocal radius of ϵ_{zt} is $|z| / r_A$.*

Proof. We once again invert the circles in the circle centered at O and with radius $2\sqrt{ab}$ as in the proofs of Theorems 1, 5 and 7 and use the same notation.

The circle γ_z is then carried into the circle γ_z' touching α' at a point with the y -coordinate $2z\sqrt{ab}$ as shown in the proof of Theorem 7 and ϵ_z is carried into the line ϵ_z' : $y = 2z\sqrt{ab}$. This implies that ϵ_z intersects α and γ_z at their tangency point perpendicularly. The last part of (i) follows from the fact that the three points A' , the tangency point of α' and γ_z' and V_z' lie on a circle passing through O in this order. (ii) follows from the fact that the two circles γ_{z-w}' and γ_{z+w}' are symmetrical in the line ϵ_z' . The two circles γ_z' and γ_w' touch if and only if $2z\sqrt{ab} - 2w\sqrt{ab} = \pm 2(a + b)$ and this is equivalent to $z - w = \pm t$. This gives the first half part of (iii). The remaining part of (iii) and (iv) are now obvious. \square

The circle $\zeta_{z-w, z+w}^\alpha$ touches α at a fixed point for any non-zero real number w , which is the intersection of α and ϵ_z by (ii) of the theorem. For any chain of circles touching α and β , the reciprocals of the radii of their associated circles orthogonal to α and β and the circles in this chain form a geometric progression by the first half part of (iii) and (iv) of the theorem, where we assume that the radius of the associated circle touching the x -axis from below has minus sign. In particular, starting with the ordinary arbelos, we get the chain of circles

$$\{\dots, \gamma_{-2t}, \gamma_{-t}, \gamma_0, \gamma_t, \gamma_{2t}, \dots\}$$

and the reciprocal radius of the circle ϵ_{nt} associated with γ_{nt} in this chain is n/r_A . In the case $n = 1$, we get the well-known fact that the circle orthogonal to α, β and the inscribed circle of the ordinary arbelos is congruent to the twin circles of Archimedes in the ordinary arbelos [1]. Now let us consider some other special cases of Theorem 11. In Figure 11, the circle with center V_1 passing through O , i.e., $\epsilon_{1/2}$, intersects α and $\zeta_{0,1}^\alpha$ (also β and $\zeta_{0,1}^\beta$) perpendicularly at their tangency point and also intersects γ_0 and γ_1 at their intersections. These results are obtained by letting $z = w = 1/2$ in (ii). The circle $\epsilon_{(n+1/2)t}$ with radius $r_A/(n + \frac{1}{2})$ touches γ_{nt} and $\gamma_{(n+1)t}$ at their tangency point by (iii) and (iv). In particular the circle $\epsilon_{t/2}$, which is double the size of the twin circles of Archimedes in the ordinary arbelos, intersects α and $\zeta_{0,t}^\alpha$ (also β and $\zeta_{0,t}^\beta$) perpendicularly at their tangency point and also touches γ_0 and γ_t at their tangency point (see Figure 13).

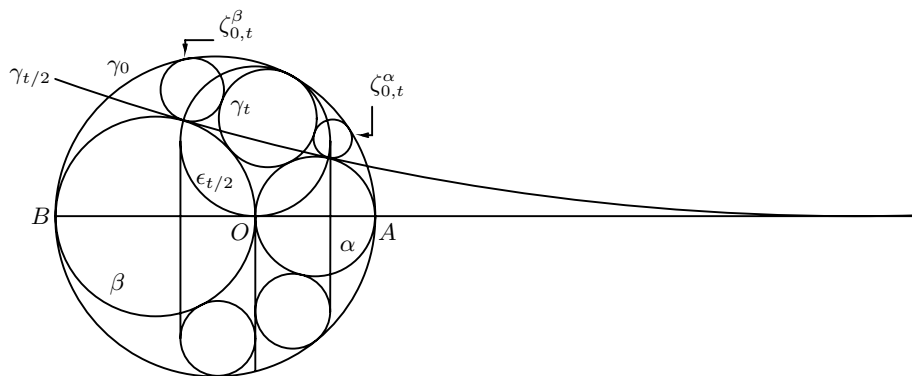


Figure 13.

There is a tangent between $\epsilon_{t/2}$ and each of the twin circles of Archimedes in the ordinary arbelos which is parallel to the y -axis. In order to avoid the overlapping circles, reflected twin circles of Archimedes in the x -axis are drawn in Figure 13. From (8) we can see that the circle $\gamma_{t/2}$ (also $\gamma_{-t/2}$) touches the x -axis.

5. The n -th twin circles of Archimedes (asymmetrical case)

To investigate further possibilities of the existence of pairs of the n -th twin circles of Archimedes, we define several other points on the y -axis, which are also related to some of those pairs. Consider the following points on the y -axis with given y -coordinates:

$$\begin{aligned} X_{n,+} &: \frac{-2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{n(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}, \\ X_{n,-} &: \frac{-2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{n(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}; \\ Y_{n,+} &: \frac{-2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{n(\sqrt{a}-\sqrt{b})+(\sqrt{a}+\sqrt{b})}, \\ Y_{n,-} &: \frac{-2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{n(\sqrt{a}-\sqrt{b})-(\sqrt{a}+\sqrt{b})}. \end{aligned}$$

Also,

$$\begin{aligned} Z_{n,+}^{++} &: \frac{2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{n(\sqrt{a}-\sqrt{b})+(\sqrt{a}+\sqrt{b})-2\sqrt{a+b}}, \\ Z_{n,-}^{++} &: \frac{2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{n(\sqrt{a}-\sqrt{b})-(\sqrt{a}+\sqrt{b})-2\sqrt{a+b}}, \\ Z_{n,+}^{--} &: \frac{2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{n(\sqrt{a}-\sqrt{b})+(\sqrt{a}+\sqrt{b})+2\sqrt{a+b}}, \\ Z_{n,-}^{--} &: \frac{2\sqrt{ab}(\sqrt{a}+\sqrt{b})}{n(\sqrt{a}-\sqrt{b})-(\sqrt{a}+\sqrt{b})+2\sqrt{a+b}}, \\ Z_{n,+}^{+-} &: \frac{2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{n(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})-2\sqrt{a+b}}, \\ Z_{n,-}^{+-} &: \frac{2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{n(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})-2\sqrt{a+b}}, \\ Z_{n,+}^{-+} &: \frac{2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{n(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})+2\sqrt{a+b}}, \\ Z_{n,-}^{-+} &: \frac{2\sqrt{ab}(\sqrt{a}-\sqrt{b})}{n(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})+2\sqrt{a+b}}. \end{aligned}$$

Reflecting the points $X_{n,+}$, $X_{n,-}$, $Y_{n,+}$, $Y_{n,-}$, $Z_{n,+}^{++}$, $Z_{n,-}^{++}$, $Z_{n,+}^{+-}$ and $Z_{n,-}^{+-}$ in the x -axis, we get the points $X_{-n,-}$, $X_{-n,+}$, $Y_{-n,-}$, $Y_{-n,+}$, $Z_{-n,-}^{--}$, $Z_{-n,+}^{--}$, $Z_{-n,-}^{-+}$ and $Z_{-n,+}^{-+}$, respectively. Since $-1 < (\sqrt{a}-\sqrt{b})/(\sqrt{a}+\sqrt{b}) < 1$, $X_{n,+}$ and $X_{n,-}$ cannot be the point at infinity on the y -axis for any integer n , if $a \neq b$. However, any of the other points can be identical with the point at infinity for some a and b and integer n . The proof of the next theorem is similar to the proof of Theorem 5.

Theorem 12. *Let n be an arbitrary integer and $a \neq b$.*

(i) $1/a_n^+ = 1/b_{-n}^+$ if and only if the circle γ passes through $X_{n,\pm}$ or $Z_{n,\pm}^{++}$. If γ passes through $X_{n,\pm}$,

$$\frac{1}{a_n^+} = \frac{1}{b_{-n}^+} = \left(n \frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} - 1 \right)^2 \frac{1}{r_A}$$

and if γ passes through $Z_{n,\pm}^{++}$,

$$\frac{1}{a_n^+} = \frac{1}{b_{-n}^+} = \left(\left(n \frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} - 1 \right) \left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} \right) \right)^2 \frac{1}{r_A}.$$

(ii) $1/a_{-n}^- = 1/b_n^-$ if and only if the circle γ passes through $X_{n,\pm}$ or $Z_{n,\pm}^{--}$. If γ passes through $X_{n,\pm}$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_n^-} = \left(n \frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} + 1 \right)^2 \frac{1}{r_A}$$

and if γ passes through $Z_{n,\pm}^{--}$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_n^-} = \left(\left(n \frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} + 1 \right) \left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} \right) \right)^2 \frac{1}{r_A}.$$

(iii) $1/a_{-n}^+ = 1/b_{-n}^-$ if and only if the circle γ passes through $Y_{n,\pm}$ or $Z_{n,\pm}^{+-}$. If γ passes through $Y_{n,\pm}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(n \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - 1 \right)^2 \frac{1}{r_A}$$

and if γ passes through $Z_{n,\pm}^{+-}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(\left(n \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - 1 \right) \left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} \right) \right)^2 \frac{1}{r_A}.$$

(iv) $1/a_n^- = 1/b_n^+$ if and only if the circle γ passes through $Y_{n,\pm}$ or $Z_{n,\pm}^{-+}$. If γ passes through $Y_{n,\pm}$,

$$\frac{1}{a_n^-} = \frac{1}{b_n^+} = \left(n \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} + 1 \right)^2 \frac{1}{r_A}$$

and if γ passes through $Z_{n,\pm}^{-+}$,

$$\frac{1}{a_n^-} = \frac{1}{b_n^+} = \left(\left(n \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} + 1 \right) \left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} \right) \right)^2 \frac{1}{r_A}.$$

Each of the propositions (i), (ii), (iii) and (iv) in Theorems 5 and 11 asserts the existence of two different pairs of the n -th twin circles of Archimedes in two different arbeloi, but the ratio of their radii is independent of n and the circle γ and always equal to $\left((\sqrt{a} + \sqrt{b}) / (\sqrt{a} - \sqrt{b}) \right)^{\pm 2}$.

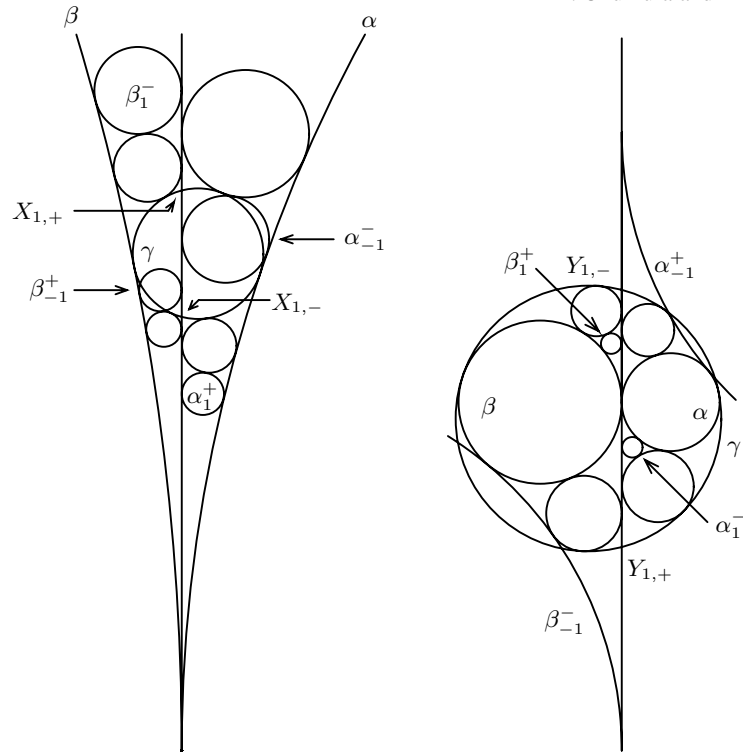


Figure 14. $a_1^+ = b_{-1}^+, a_{-1}^- = b_1^-$ for $(X_{1,\pm})$ $a_{-1}^+ = b_{-1}^-, a_1^- = b_1^+$ for $(Y_{1,\pm})$

If the circle γ passes through the points $X_{n,\pm}$, we label the arbelos as $(X_{n,\pm})$. The arbeloi $(Y_{n,\pm}), (Z_{n,\pm}^{++}), (Z_{n,\pm}^{--}), (Z_{n,\pm}^{+-})$ and $(Z_{n,\pm}^{-+})$ are defined similarly. Reflecting $(X_{n,\pm}), (Y_{n,\pm}), (Z_{n,\pm}^{++})$ and $(Z_{n,\pm}^{+-})$ in the x -axis, we get $(X_{-n,\pm}), (Y_{-n,\pm}), (Z_{-n,\pm}^{--})$ and $(Z_{-n,\pm}^{-+})$, respectively. Table 2 shows, on which sides of the x -axis lie the centers of the n -th twin circles of Archimedes in these arbeloi. According to Theorem 12, there are two pairs of the n -th twin circles of Archimedes in the arbeloi $(X_{n,\pm})$ and $(Y_{n,\pm})$ (see Figure 14).

	$(X_{n,\pm})$	$(Y_{n,\pm})$	$(Z_{n,\pm}^{++})$	$(Z_{n,\pm}^{--})$	$(Z_{n,\pm}^{+-})$	$(Z_{n,\pm}^{-+})$
same side	α_n^+, β_{-n}^+ α_{-n}^-, β_n^-				$\alpha_{-n}^+, \beta_{-n}^-$	α_n^-, β_n^+
opposite side		$\alpha_{-n}^+, \beta_{-n}^-$ α_n^-, β_n^+	α_n^+, β_{-n}^+	α_{-n}^-, β_n^-		

Table 2.

6. Another twin circle property

We demonstrate the existence of another pair of twin circles in the case, when the circle γ and the line joining the centers of α and β intersect. This pair of twin circles is a generalization of the circles W_6 and W_7 in [4]. A related result can be seen in [6]. We start by proving the following lemma:

Lemma 13. *Let A_0B_0 be the diameter of the circle γ parallel to the x -axis and intersecting the y -axis at the point O . Let $a_0 = |A_0O'|$ and $b_0 = |B_0O'|$, where A_0 and B_0 lie on the same sides of the y -axis as the circles α and β , respectively. If γ touches α and β internally, $a/b = a_0/b_0$ and if γ touches α and β externally, $a/b = b_0/a_0$.*

Proof. Assume that γ touches α and β internally and $a < b$ (see Figure 15). Let O_α, O_β and O_γ be the centers of α, β and γ and F the foot of the normal dropped from O_γ to the x -axis. By Pythagorean theorem we get

$$|O_\gamma O_\alpha|^2 - |O_\alpha F|^2 = |O_\gamma O_\beta|^2 - |O_\beta F|^2.$$

Substituting $|O_\gamma O_\alpha| = (a_0 + b_0)/2 - a$, $|O_\gamma O_\beta| = (a_0 + b_0)/2 - b$, $|O_\alpha F| = a + |O_\gamma O'|$, $|O_\beta F| = b - |O_\gamma O'|$ and $|O_\gamma O'| = (a_0 + b_0)/2 - a_0$, we obtain $a/b = a_0/b_0$. The case, when γ touches α and β externally, can be proved in a similar way. \square

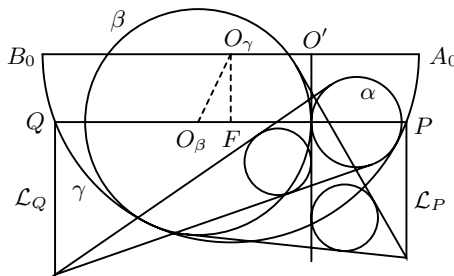


Figure 15.

Theorem 14. *Let AO and BO be the diameters of the circles α and β on the x -axis. Let P and Q be the intersections of the circle γ with the x -axis, choosing P and Q so that A, P, Q, B follow in this order on the x -axis, if we regard it as a circle of infinite radius closed through the point at infinity. Let \mathcal{L}_P and \mathcal{L}_Q be the lines through P and Q perpendicular to the x -axis. The circle touching the y -axis from the side opposite to β and the tangents to β from an arbitrary point on \mathcal{L}_P is congruent to the circle touching the y -axis from the side opposite to α and the tangents to α from an arbitrary point on \mathcal{L}_Q .*

Proof. We use the same notation as in Lemma 13 and its proof. Assume that γ touches α and β internally and $a < b$. According to Lemma 13, there is a real number k , such that $a = a_0k$ and $b = b_0k$. Hence,

$$\begin{aligned} |O_\gamma F|^2 &= ((a_0 + b_0)/2 - b_0k)^2 - (b_0k - d)^2, \\ |QF|^2 &= |O_\gamma Q|^2 - |O_\gamma F|^2 = 2a_0b_0k + d^2, \end{aligned}$$

where $d = |OF| = (b_0 - a_0)/2$. Let r_b be the radius of the circle touching the y -axis from the side opposite to α and the common external tangents of α from an arbitrary point on \mathcal{L}_Q . Similarly, let r_a be the radius of the circle touching the y -axis from the side opposite to β and the common external tangent of β from an arbitrary point on \mathcal{L}_P . From the similarity of the circle with radius r_b and the circle α , we have

$$\begin{aligned} \frac{\sqrt{d^2 + 2a_0b_0k} + d - r_b}{r_b} &= \frac{\sqrt{d^2 + 2a_0b_0k} + d + a_0k}{a_0k}, \\ \frac{1}{r_b} &= \frac{1}{a_0k} + \frac{\sqrt{d^2 + 2a_0b_0k} - d}{a_0b_0k}. \end{aligned}$$

Similarly we obtain

$$\frac{1}{r_a} = \frac{1}{b_0k} + \frac{\sqrt{d^2 + 2a_0b_0k} + d}{a_0b_0k}.$$

But we can easily show that $1/r_a - 1/r_b = 0$ or $r_a = r_b$. The case, when γ touches α and β externally, can be proved in a similar way. \square

Theorem 14 holds even in the case, when γ is one of the common external tangents of the circles α and β , if we consider γ to intersect the x -axis at the point at infinity. In this case, if $a < b$, these twin circles are congruent to α . If γ touches α and β internally, the minimum radii of these twin circles are equal to r_A , which is the case of the ordinary arbelos. If γ touches α and β externally, the radii of the twin circles are maximum in the case, when γ touches the x -axis. Let r be the maximum radius of the twin circles, c the radius of γ and d the distance of the tangency point of γ with the x -axis from the origin O and assume $a < b$. In this case

$$c^2 = (c + a)^2 - (d - a)^2 = (c + b)^2 - (d + b)^2.$$

Eliminating c and solving this equation for d , we get $d = 4ab/(b - a)$. From the similarity of the circle α and the corresponding twin circle, $(d - a)/a = (d + r)/r$, which implies $r = 2r_A$. Consequently, we obtain that if $a < b$, $r_A < a < 2r_A$, and the the common radii of the twin circles take the minimum value r_A for the ordinary arbelos, a when γ is one of the common external tangents of α and β , and the maximum value $2r_A$ when γ touches the x -axis. Since the circle γ touching the x -axis is identical with $\gamma_{\pm t/2}$ as mentioned at the end of §4, there is one more circle congruent to the twin circles in the last case, which is the circle $\epsilon_{\pm t/2}$ associated to $\gamma_{\pm t/2}$ by (iv) of Theorem 11 (see Figure 13).

7. Conclusion

We have demonstrated several interesting properties of the skewed arbelos, which could not have been found by consider the ordinary one. Since we confined our discussion largely to a generalization of the twin circles of Archimedes, it appears to be worth the effort to investigate other topics related to the skewed arbelos. We conclude our paper by proposing a problem. Let α , β and γ be three circles forming a skewed arbelos, i.e., γ is given by equations (8) or (9), and let δ be a circle touching α and β at their tangency point O and intersecting γ . The circle δ divides the skewed arbelos into two curvilinear triangles. Find (or construct) the circle δ , such that the incircles of the two curvilinear triangles are congruent (see Figure 16).

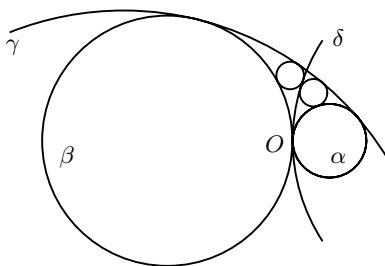


Figure 16.

References

- [1] L. Bankoff, Are the twin circles of Archimedes really twins?, *Math. Mag.*, 47 (1974) 214–218.
- [2] L. Bankoff, The marvelous arbelos, in *The lighter side of mathematics* 247–253, ed. R. K. Guy and R. E. Woodrow, Mathematical Association of America, 1994.
- [3] L. Brother, F. S. C. Raphael, The shoemaker’s knife, *Mathematics Teacher*, (1973) 319–323.
- [4] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
- [5] R. A. Johnson, *Advanced Euclidean geometry*, Dover, NY, 1960.
- [6] H. Okumura, Two similar triangles, *Math. Gazette*, 79 (1995) 569–571.
- [7] G. Yamamoto, *Sampō Jojutsu*, 1841.

Hiroshi Okumura: Department of Information Engineering, Maebashi Institute of Technology,
460-1 Kamisadori Maebashi Gunma 371-0816, Japan
E-mail address: okumura@maebashi-it.ac.jp

Masayuki Watanabe: Department of Information Engineering, Maebashi Institute of Technology,
460-1 Kamisadori Maebashi Gunma 371-0816, Japan
E-mail address: watanabe@maebashi-it.ac.jp