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Where are the Conjugates?

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Abstract. The positions and properties of a point in relation to its isogonal and isotomic conjugates are discussed. Several families of self-conjugate conics are given. Finally, the topological implications of conjugacy are stated along with their implications for pivotal cubics.

1. Introduction

The edges of a triangle divide the Euclidean plane into seven regions. For the projective plane, these seven regions reduce to four, which we call the central region, the a region, the b region, and the c region (Figure 1). All four of these regions, each distinguished by a different color in the figure, meet at each vertex. Equivalent structures occur in each, making the projective plane a natural background for fundamental triangle symmetries. In the sense that the projective plane can be considered a sphere with opposite points identified, the projective plane divided into four regions by the edges of a triangle can be thought of as an octahedron projected onto this sphere, a remark that will be helpful later.

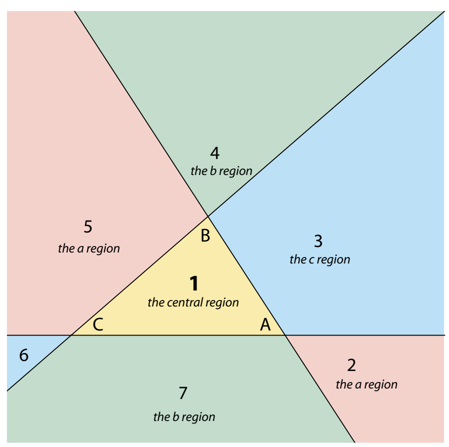

Figure 1. The plane of the triangle, Euclidean and projective views

A point $\mathbf{P}$ in any of the four regions has an harmonic associate in each of the others. Cevian lines through $\mathbf{P}$ and/or its harmonic associates traverse two of the these regions, there being two such possibilities at each vertex, giving 6 Cevian (including exCevian) lines. These lines connect the harmonic associates with the vertices in a natural way.

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Given two points in the plane there are two central points (a non-projective concept), the midpoint and a point at infinity. Given two lines there are two central lines, the angle bisectors. Where there is a sense of center, there is a sense of deviation from that center. For each point not at a vertex of the triangle there is a conjugate point defined using each of these senses of center. The isogonal conjugate is the one defined using angles and the isotomic conjugate is defined using distances. This paper is about the relation of a point to its conjugates.

We shall use the generic term *conjugate* when either type is implied. Other types of conjugacy are possible [2], and our remarks will hold for them as well.

Notation. Points and lines will be identified in bold type. John Conway's notation for points is used. The four incenters (the incenter and the three excenters) are $\mathbf{I}_o$, $\mathbf{I}_a$, $\mathbf{I}_b$, $\mathbf{I}_c$. The four centroids (the centroid and its harmonic associates) are $\mathbf{G}$, $\mathbf{A^G}$, $\mathbf{B^G}$, $\mathbf{C^G}$. We shall speak of equivalent structures around the four incenters or the four centroids. An angle bisector is identifed by the two incenters on it and a median by the two centroids on it as in "ob", or "ac". $\mathbf{A_P}$ is the Cevian trace of line $\mathbf{AP}$ and $\mathbf{A^P}$ is a vertex of the pre-Cevian triangle of $\mathbf{P}$. We shall often refer to this point as an "ex-"version of $\mathbf{P}$ or as an harmonic associate of $\mathbf{P}$. Coordinates are barycentric. $\mathrm{t}\mathbf{P}$ is the isotomic conjugate of $\mathbf{P}$, $\mathrm{g}\mathbf{P}$ the isogonal conjugate.

The isogonal of a line through a vertex is its reflection across either bisector through that vertex. The isogonal lines of the three Cevian lines of a point $\mathbf{P}$ concur in its conjugate $\mathrm{g}\mathbf{P}$. In the central region of a triangle, the relation of a point to its conjugate is simple. This region of the triangle is divided into 6 smaller regions by the three internal bisectors. If $\mathbf{P}$ is on a bisector, so is $\mathrm{g}\mathbf{P}$, with the incenter between them, making the bisectors fixed lines under isogonal conjugation. If $\mathbf{P}$ is not on a bisector, then $\mathrm{g}\mathbf{P}$ is in the one region of the six that is on the opposite side of each of the three bisectors. This allows us to color the central region with three colors so that a point and its conjugate are in regions of the same color (Figure 2). The isotomic conjugate behaves analogously with the medians serving as fixed or self-conjugate lines.

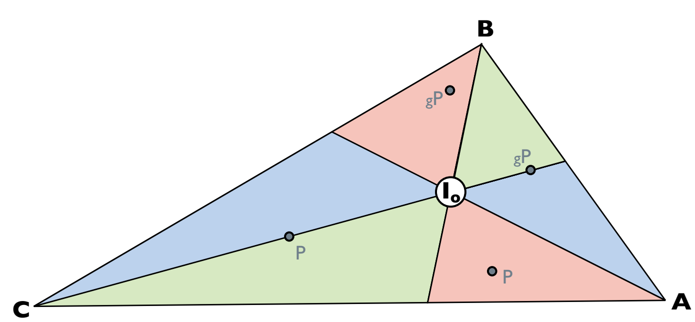

Figure 2. Angle bisctors divide the central region of the triangle into co-isotomic regions. The isogonal conjugate of a point on a bisector is also on that bisector. The conjugate of a point in one of the colored regions is in the other region of the same color.

2. Relation of conjugates to self-conjugate lines

The central region is all well and good, but the other three regions are locally identical in behavior and are to be considered structurally equivalent. Figure 3 shows the triangle with the incentral quadrangle. Each vertex of $\mathbf{ABC}$ hosts two bisectors, traditionally called internal and external. It is important to realize that an isogonal line through any vertex can be created by reflection in either bisector. This means that the three particular bisectors through any of the four incenters (one from each vertex) can be used to define the isogonal conjugate. Hence the behavior of conjugates around $\mathbf{I}_b$, say, is locally identical to that around $\mathbf{I}_o$, as shown in Figure 4.

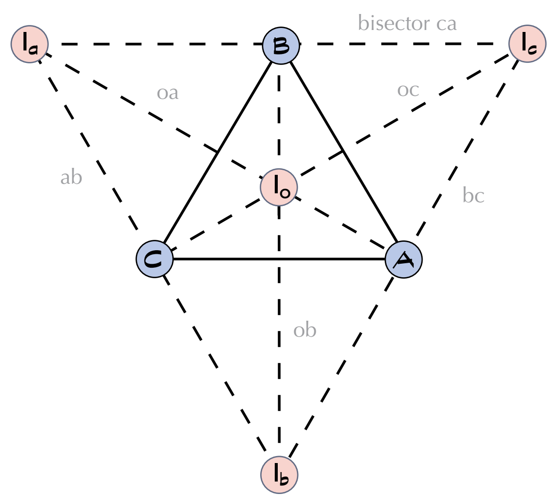

Figure 3. The triangle and its incentral quadrangle

If $\mathbf{P}$ is in the central region, the conjugate $\mathrm{g}\mathbf{P}$ is also; both are on the same side (the interior side) of each of the three external bisectors. So in the central region a point and its conjugate are on opposite sides of three bisectors (the internal ones) and on the same side of three others (the external ones). This is also true in the neighborhood of $\mathbf{I}_b$, although the particular bisectors have changed. No matter where in the plane, a point not on a bisector is on the opposite of three bisectors from its conjugate and on the same side for the other three bisectors. To some extent this statement is justified by the local equivalence of conjugate behavior mentioned above, but this assertion will be fully justified later in §10 on topological properties.

3. Formal properties of the conjugacy operation

Each type of conjugate has special fixed points and lines in the plane. As these properties are generally known, they will be stated without proof. Figures 5 and 8 show the mentioned structures.

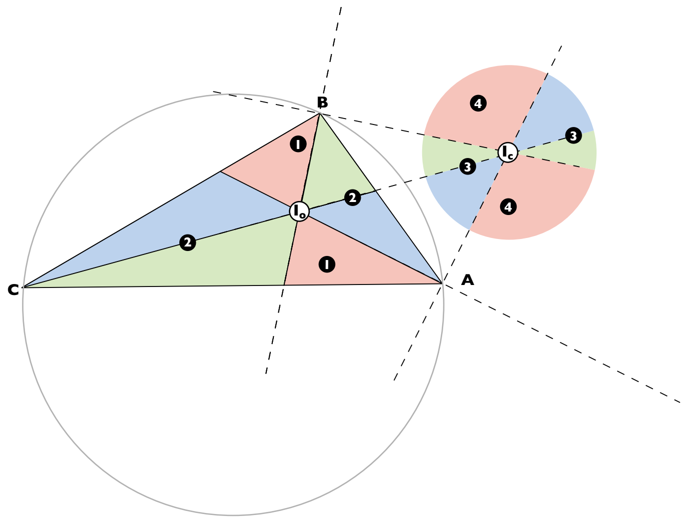

Figure 4. This picture shows the local equivalence of the region around $\mathbf{I}_o$ to that around $\mathbf{I}_c$. This equivalence appears to end at the circumcircle. Numbered points are co-conjugal, each being the conjugate of the other. For each region a pair of points both on and off a bisector is given.

conjugacy	fixed points	fixed lines	special curves	singularities
isotomic	centroid, its harmonic conjugates	medians and ex-medians	line at infinity, Steiner ellipse	vertices
isogonal	incenter, its harmonic conjugates	internal and external bisectors	line at infinity, circumcircle	vertices

For each type of conjugacy there are 4 points in the plane, harmonically related, that are fixed points under conjugacy. For isogonal conjugacy these are the 4 in/excenters. For isotomic conjugacy these are the centroid and its harmonic associates.In each case the six lines that connect the 4 fixed points are the fixed lines.

Special curves: Each point on the Steiner ellipse has the property that its isotomic Cevians are parallel, placing the isotomic conjugate at infinity. Similarly for any point on the circumcircle, its isogonal Cevians are parallel, again placing the isogonal conjugate at infinity. These special curves are very significant in the Euclidean plane, but not at all significant in the projective plane.

The conjugate of a point on an edge of $\mathbf{ABC}$ is at the corresponding vertex, an ∞ to 1 correspondence. This implies that the conjugate at a vertex is not defined, making the vertices the three points in the plane where this is true. This leads to a complicated partition of the Euclidean plane, as the behavior the conjugate of a point inside the Steiner ellipse or the circumcircle is different from that outside. We

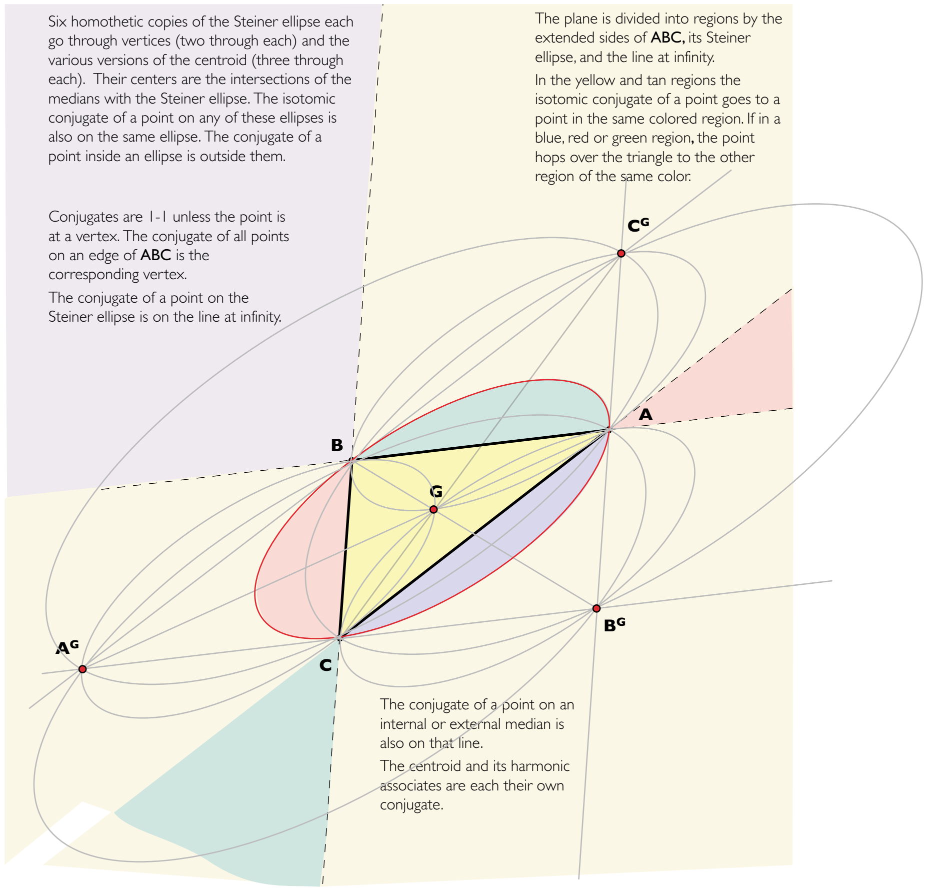

Figure 5. Isotomic conjugates

thus have the pictures of the regions of the plane in terms of conjugates as shown in Figures 5 and 8.

The colors in these two pictures show regions of the plane which are shared by the conjugates. The boundaries of these regions are the sides of the triangle, the circumconic and the line at infinity. The conjugate of a point in a region of a certain color is a region of the same color. For the red, green, and blue regions the conjugate is always in the other region of the same color.

These properties are helpful in locating a point in relation to the position of its conjugate, but there is more to this story.

4. Conjugate curves

4.1. *Lines.* The conjugate of a curve is found by taking the conjugate of each point on the curve. In general the conjugate of a straight line is a circumconic, but there are some exceptions.

Theorem 1. *If a line goes through a vertex of the reference triangle* $\mathbf{ABC}$*, the conjugate of this line is a line through the same vertex.*

Proof. Choose vertex $\mathbf{B}$. A line through this vertex has the form $nz - \ell x = 0$. The isotomic conjugate is $\frac{n}{z} - \frac{\ell}{x} = 0$, which is the same as $nx - \ell z = 0$, a line through the same vertex. The isogonal conjugate works analogously. □

This result is structurally useful. If a point approaches a vertex on a straight line (or a smooth curve, which must approximate one) its conjugate crosses an edge by the conjugate line ([3]).

4.2. *Self conjugate conics (isotomic case).* The isotomic conjugate of the general conic is a quartic curve, but again there are some interesting exceptions.

Theorem 2. *Conics through* $\mathbf{AGCB^G}$ *and* $\mathbf{ACC^GA^G}$ *are self-isotomic.*

Proof. The general conic is $\ell x^2 + my^2 + nz^2 + Lyz + Mzx + Nxy = 0$. Choosing the case $\mathbf{AGCB^G}$, since $\mathbf{A}$ and $\mathbf{C}$ are on the conic, we have that $\ell = n = 0$. From $\mathbf{G}$ and $\mathbf{B^G}$ we get the two equations $m \pm L + M \pm N = 0$, from which we get $M = -m$ and $N = -L$ giving $y^2 - zx + \lambda y(z - x) = 0$ as the family of conics through these two points. Replacing each coordinate with its reciprocal and assuming that $xyz \neq 0$, we see that this equation is self-isotomic.

For the case $\mathbf{CAA^GC^G}$ the equation is $y^2 + zx + \lambda y(z + x) = 0$, also self-isotomic.

□

Each family has one special conic homothetic to the Steiner ellipse and of special interest: $y^2 - zx = 0$, which goes through $\mathbf{AGCB^G}$, and $y^2 + zx + 2y(z+x) = 0$, which goes through $\mathbf{ACC^GA^G}$. Conics homothetic to the Steiner ellipse can be written as $yz + zx + xy + (Lx + My + Nz)(x + y + z) = 0$. Choosing $L = N = 0$ and $M = \pm 1$ gives the two conics of interest. The first of these has striking properties.

Theorem 3. *The ellipse* $y^2 - zx = 0$

(1) *goes through* $\mathbf{C}$, $\mathbf{A}$, $\mathbf{G}$, $\mathbf{A^G}$,

(2) *is tangent to edges* a *and* c,

(3) *contains the isotomic conjugate* $\mathrm{t}\mathbf{P}$ *of every point* $\mathbf{P}$ *on it,* (*and if one of* $\mathbf{P}$ *and* $\mathrm{t}\mathbf{P}$ *is inside, then the other is outside the ellipse; the line connecting a point on the ellipse with its conjugate is parallel to the* b *edge* [3]),

(4) *contains the* $\mathbf{B}$*-harmonic associate of every point on it,*

(5) *has center* $(2 : -1 : 2)$ *which is the intersection of the Steiner ellipse with the b-median,*

(6) *is the translation of the Steiner ellipse by the vector from* $\mathbf{B}$ *to* $\mathbf{G}$,

(7) *contains* $\mathbf{P}^n = (x^n : y^n : z^n)$ *for integer values of* n *if* $\mathbf{P} = (x : y : z)$, $(xyz \neq 0)$*, is on the curve,*

(8) *is the inverse in the Steiner ellipse of the b-edge of* **ABC** .

These last two properties are included for their interest, but have little to do with the topic at hand (other than that $n = -1$ is the isotomic conjugate). A second paper will be devoted to these properties of this curve.

Proof. (1) can be verified by substituting coordinates as done above.

(2) is true by the general principle that if an equation has the form (line 2)^2 = (line 1)$\cdot$(line 3), then the curve has a double intersection at the intersection of line 1 and line 2 and at the intersection of line 3 and line 2 and is tangent to lines 1 and 2 at those points.

For (3) we take the isotomic conjugage of a point on the curve to obtain $\frac{1}{y^2} - \frac{1}{zx} = 0$, which, since this curve only exists where the product zx is positive, is the same as $zx - y^2 = 0$, so that $\mathrm{t}\mathbf{P}$ is on the curve if $\mathbf{P}$ is, which also implies that the point and the conjugate are on different sides of the ellipse. $(yz : zx : xy)$ is the conjugate. If on the ellipse $zx = y^2$ we have $(yz : y^2 : xy) \sim (z : y : x)$. The vector from this point to $(x : y : z)$ is proportional to $(-1 : 0 : 1)$, which is in the direction of the b-edge.

(4) can be verified by noting that if (x, y, z) is on the ellipse, so is its harmonic associate $(x, -y, z)$.

(5) The center is found as the polar of the line at infinity.

(6) is verified by computing the translation $T : \mathbf{B} \to \mathbf{G}$, and computing $S(T^{-1}\mathbf{P})$, where $S(\mathbf{P})$ is the Steiner ellipse in terms of a point $\mathbf{P}$ on the curve.

(7) is verified since $(y^n)^2 - z^n x^n$ has $y^2 - zx$ as a factor, so that $\mathbf{P}^n$ is on the curve if $\mathbf{P}$ is.

(8) $(\cdots : y : \cdots) \to (\cdots : y^2 - zx : \cdots)$ is the Steiner inversion and takes $y = 0$ into $y^2 - zx = 0$. $\square$

5. The isotomic ellipses

Consider the three curves

$$x^2 - yz = 0,$$
$$y^2 - zx = 0,$$
$$z^2 - xy = 0,$$

which are translations of the Steiner ellipse, each through two vertices, and tangent to the edges of $\mathbf{ABC}$. Exactly as the three medians are self-isotomic and separate the central region of the triangle, so too do these ellipses. If a point is inside one, its conjugate is outside. The line from a point on one of these curves to its conjugate its parallel to a side of the triangle, or perhaps stated more correctly, to the an ex-median.

Consider the three curves

$$x^2 + yz + 2x(y + z) = 0,$$
$$y^2 + zx + 2y(z + x) = 0,$$
$$z^2 + xy + 2z(x + y) = 0$$

each homothetic to the Steiner ellipse. Each goes through two ex-centroids and two vertices and is centered at the other vertex. These are the exterior versions of the above three, rather as the ex-medians are external versions of the medians. They are self-isotomic and the line from a point to its conjugate is parallel to a median (proved below). These ellipses go through the ex-centroids and serve to define regions about them just as the others do for the central regions. They can also be seen in Figure 5. These six isotomic ellipses are all centered on the Steiner circumellipse of $\mathbf{ABC}$. Their tangents at the vertices are either parallel to the medians or the exmedians. For any point in the plane where the conjugate is defined, the point and its conjugate are on the same side (inside or outside) for three ellipses and on opposite sides for the other three (just as for the medians).

6. $\mathbf{P} - \mathrm{t}\mathbf{P}$ lines

For points on the interior versions (those that pass through $\mathbf{G}$) of these conics, the lines from a point to its conjugate are parallel to the ex-medians (and hence to the sides of $\mathbf{ABC}$). For points on the exterior ellipses, the line joining a point to its conjugate is parallel to a median of $\mathbf{ABC}$. This is illustrated in Figure 6.

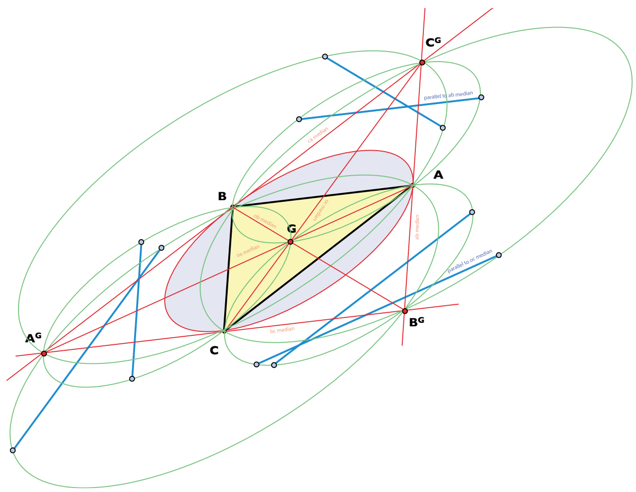

Figure 6. Points paired with their conjugates are connected by blue lines, each of which is parallel to a median or an ex-median of $\mathbf{ABC}$. The direction of the lines for the two ellipses through $\mathbf{A}$ and $\mathbf{B}$ are noted.

For the interior ellipses, this property has been proved. For the exterior ones the math is a bit harder. Note that a point and its conjugate can be written as $(x : y : z)$ and $(yz : zx : xy)$. The equation of the ellipse can be written as $zx = y^2 + 2y(z + x)$, so that the conjugate becomes

$$(yz : y^2 + 2y(z + x) : xy) \sim (z : y + 2(z + x) : x).$$

The vector between these two (normalized) points is

$$(x + y + z : -2(x + y + z) : x + y + z) \sim (1 : -2 : 1)$$

which is the direction of a median.

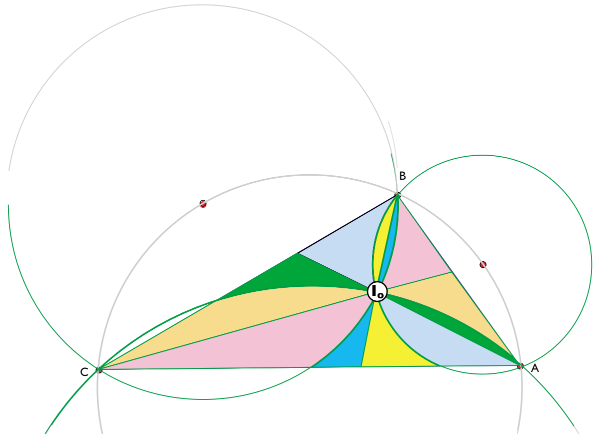

Figure 7. The central region divided by three bisectors and three self-isogonal circles.

7. The self-isogonal circles

Just as the ellipse homothetic to the Steiner ellipse through $\mathbf{CGAG^B}$ is isotomically self-conjugate, the circle through the corresponding set of points $\mathbf{CI}_o\mathbf{AI}_b$ is isogonally self-conjugate, a very pretty result. Just as the there are six versions of the isotomic ellipses, each with a center on the Steiner ellipse, there are 6 isogonal circles, each centered on the circumcircle, also a pretty result (Figure 8).

We note that $\mathbf{I}_o\mathbf{CI}_b\mathbf{A}$ is cyclic because the bisector $\mathbf{AI}_b$ is perpendicular to the bisector $\mathbf{I}_o\mathbf{A}$. The angles at $\mathbf{A}$ and $\mathbf{C}$ are right angles so that opposite angles of the quadrilateral are supplementary. Hence there is a circle through $\mathbf{CI}_o\mathbf{AI}_b$. It is in fact the diametral circle on $\mathbf{I}_o\mathbf{I}_b$.

The equation of a general circle is

$$a^2yz + b^2zx + c^2xy + (\ell x + my + nz)(x + y + z) = 0.$$

Demanding that it go through the above 4 points, we get

$$cay^2 - b^2zx - (a - c)(ayz - cxy) = 0$$

with center $(a(a + c) : -b^2 : c(a + c))$, the midpoint of $\mathbf{I}_o\mathbf{I}_a$. There are six such circles, each through 2 vertices and two incenters. Each pair of incenters determines one of these circles hence there are 6 of them. Just as each bisector goes through 2 incenters, so does each of these circles. Just as the bisectors separate a point from its conjugate, so do these circles, giving an even more detailed view of conjugacy in the neighborhood of an incenter (see Figure 7).

If a point on one of these six circles is connected to its conjugate, the line is parallel to one of the six bisectors, the circles through $\mathbf{I}_o$ pairing with exterior bisectors. The tangent lines at the vertices are also parallel to a bisector. These statements are proved just as for the isotomic ellipses.

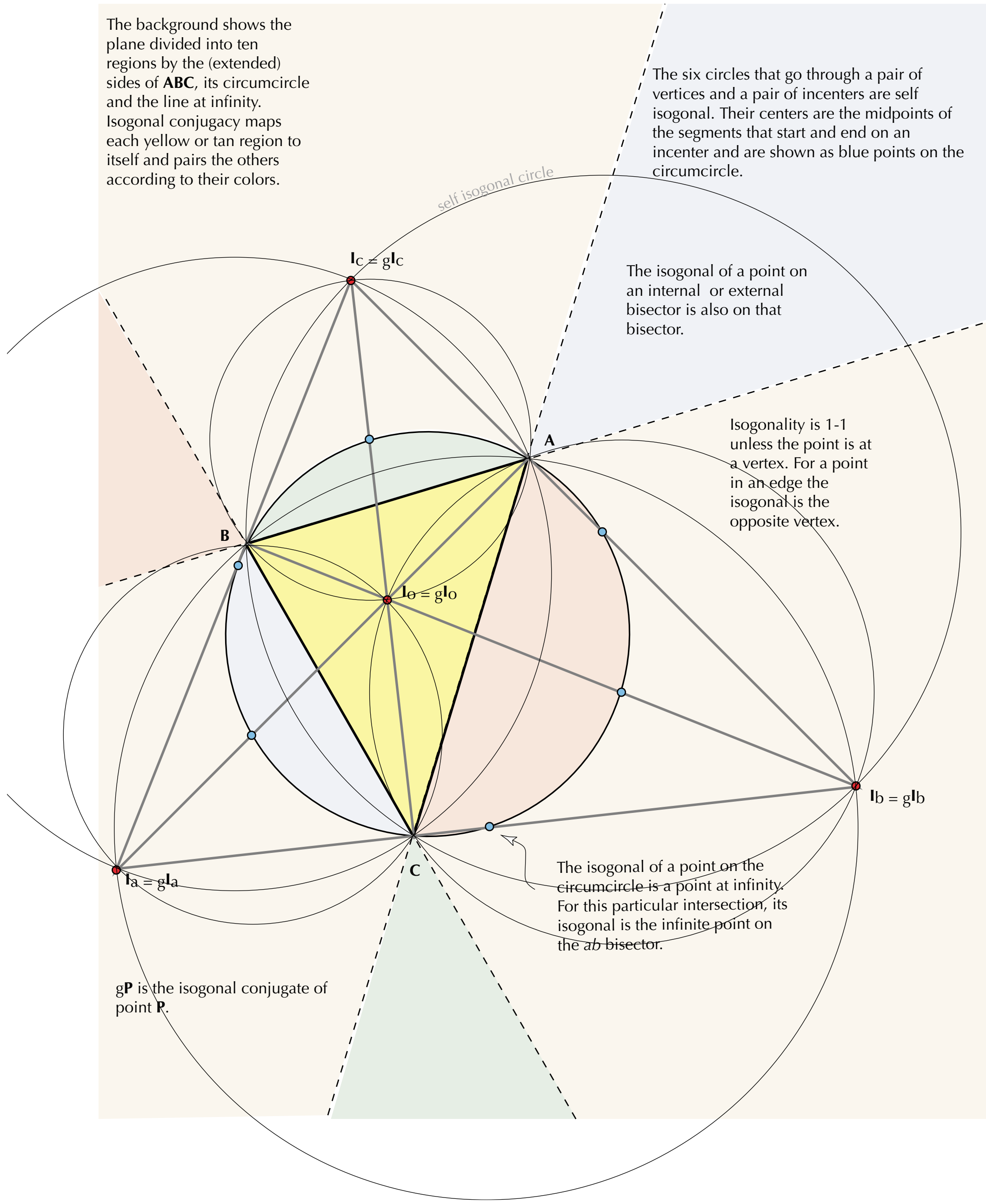

Figure 8. Isogonal conjuates

8. Self-isogonal conics

Demanding a conic go through $\mathbf{CI}_o\mathbf{AI}_b$, we get $cay^2 - b^2zx + \lambda y(az - cx) = 0$, which can be verified to be self- isogonal. Those through $\mathbf{CAI}_a\mathbf{I}_c$ have equation $cay^2 + b^2zx + \lambda y(az + cx) = 0$, and are similarly isogonal.

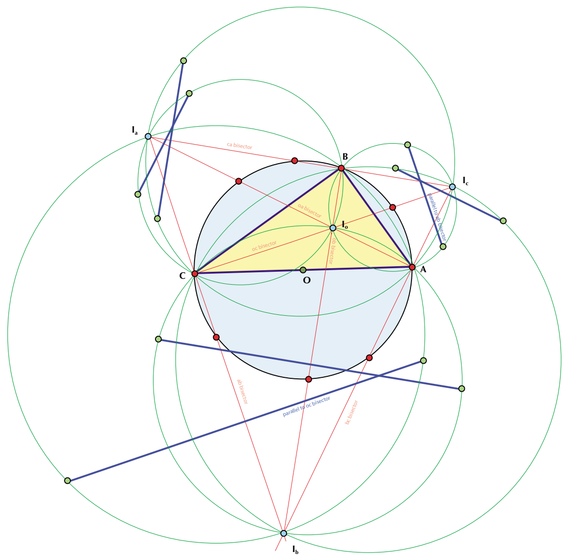

Figure 9. $\mathbf{P} - \mathrm{g}\mathbf{P}$ lines. On each isogonal circle the line from a point to its conjugate is parallel to one of the angle bisectors. If the circle goes through $\mathbf{I}_o$ the line is parallel to the corresponding external bisector. The red points on the circumcircle are the centers of the isogonal circles. For the two circles through $\mathbf{A}$ and $\mathbf{B}$, the directions of the $\mathbf{P} - \mathrm{g}\mathbf{P}$ line is noted.

9. The central region - an enhanced view

These self-conjugate circles thus help us place the isogonal conjugate of $\mathbf{P}$ just as do the median lines. If a point is on one of these circles, then so is its conjugate. If inside, the conjugate is outside and vice versa. This division of the plane into regions is very effective at giving the general location of the conjugate of a point (Figure 7). Of course this behavior around $\mathbf{I}_o$ is mimicked by that around the other incenters.

10. Topological considerations

There is a complication to the above analysis which leads to a very pretty picture of conjugacy in the projective plane. Conjugacy is 1-1 both ways except at the vertices where it blows up. This is in fact a topological blowup. To see this, let $\mathbf{P}$ move out of the central region across the b-edge, say. Near both $\mathbf{I}_o$ and $\mathbf{I}_b$, the behavior of a point to its conjugate is simple and known. In the central region, $\mathbf{P}$ and its conjugate $\mathbf{Q}$ were on opposite sides of the b-bisector; once $\mathbf{P}$ passed through the b-edge, $\mathbf{Q}$ passed through the $\mathbf{B}$-vertex, after which it is on the same side of the b-bisector as $\mathbf{P}$. We say that the plane of the triangle, underwent a Möbius-like twist at the $\mathbf{B}$-vertex. Continuing $\mathbf{P}$'s journey out of the central region through the b-edge towards $\mathbf{I}_b$, we encounter the second problem. As $\mathbf{P}$ nears the circumcircle, $\mathbf{Q}$ goes to infinity. As $\mathbf{P}$ crosses the circumcircle, $\mathbf{Q}$ crosses the line at infinity as well as the bisector, giving another twist to the plane as it passes. As

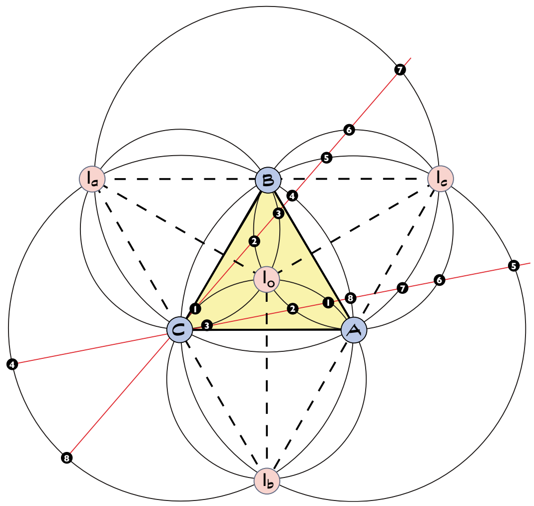

Figure 10. Here points numbered 18 are arranged on a line through **C**. The conjugates, numbered equally, are on the isogonal line through **C**, but are spaced wildly. The isogonal circles show and explain the unusual distribution of the conjugates.

P moves near $\mathbf{I}_b$, the center of the b-excircle, **Q** moves towards it, now again on the opposite side of the bisector. (This emphasis on topological properties is a result of a conversation about conjugacy with John Conway, one of the most interesting conversations about triangle geometry that I have ever had).

The isotomic conjugate behaves analogously at the vertices and at infinity with the Steiner ellipse taking the place of the circumcircle and the six medians replacing the six bisectors.

There is a way to tame the conjugacy operation at the three points in the plane which are not 1-1, and to throw light on the behavior of conjugates at the same time.

As a point approaches a vertex along a line, its conjugate goes to the point on the edge intersected by the isogonal line. Hence although the conjugate at a vertex is undefined, each direction into the vertex corresponds to a point on an edge.We represent this by letting the point “blowup”, becoming a small disc. Each point on the edge of the disc represents a direction with respect to the center. Its antipodal point is on the same line so the disc has opposite points identified. This topological blowup replaces the vertex with a Möbius-like surface (a cross-cap), explaining the shift of the conjugate from the opposite side of a bisector to the same side.

Figure 11 shows the plane of the triangle from this point of view for the isogonal case. It is a very different view indeed. The important lines are the six bisectors and the important points are the three vertices and the four incenters. The edges of the triangle are only shown for orientation and the circumcircle is not relevant to the picture. The colors show co-isogonal regions - if a point is in a region of a certain color, so is its conjugate. The twists of the plane occur at the vertices

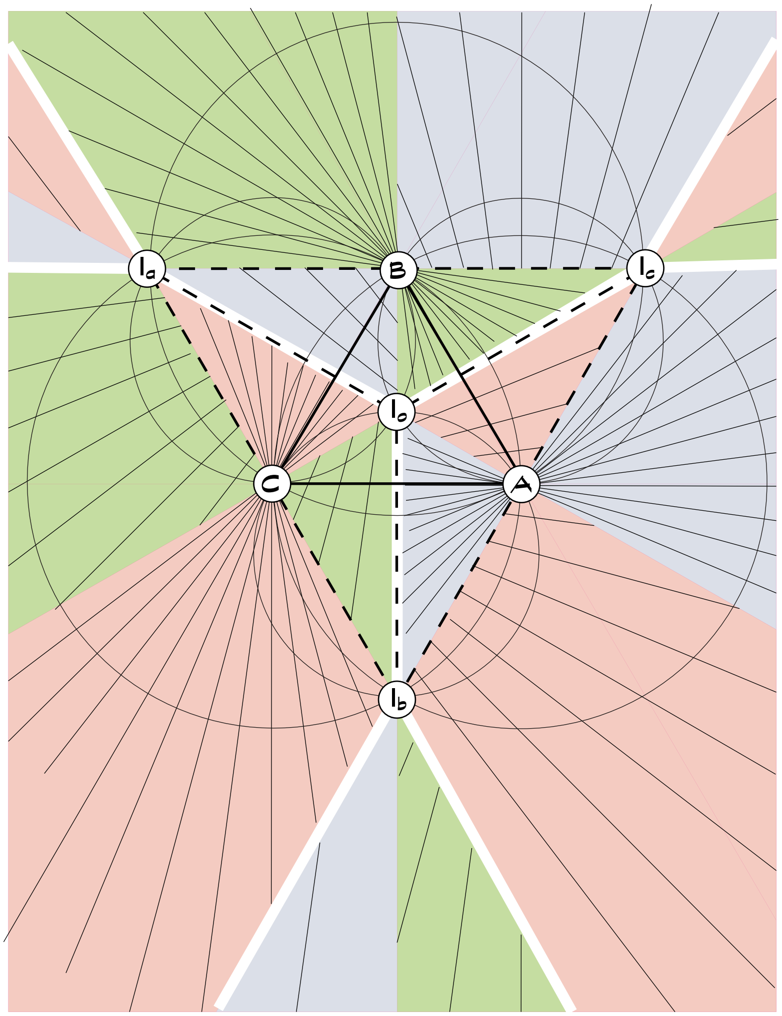

Figure 11. (Drawn with John Conway). Topological view of the location of conjugates The colors show co-isogonal regions. The lines issuing from the vertices show isogonal lines. The isogonal circles are shown. The white lines are the boundries of the three faces of a projective cube.

as shown by the colored regions converging on the vertices. In fact this figure forms a projective cube where the incenters are the four vertices that remain after anitpodes are identified. The view shown is directly toward the "vertex" $\mathbf{I}_o$ with the lines $\mathbf{I}_o\mathbf{I}_a$, $\mathbf{I}_o\mathbf{I}_b$, $\mathbf{I}_o\mathbf{I}_c$ being the three edges from that vertex. $\mathbf{I}_o\mathbf{I}_b\mathbf{I}_c\mathbf{I}_a$ form a face. The white lines are the edges of the cube. In the middle of each face is a cross-cap structure at a vertex. The final picture is of a projective cube with each face containing a crosscap singularity. The triangle $\mathbf{ABC}$ and its sides can be considered the projective octahedron inscribed to the cube with the four regions identified in the introductory paragraph being the four faces.

This leads to a nice view of pivotal cubics which are defined in terms of conjugates. The cubics go through all 7 relevant points.

11. Cubics

We can learn a bit about the shape of pivotal cubics from this topological picture of the conjugates. Pivotal cubics include both a point and its conjugate, so that each branch of the cubic must stay in co-isogonal regions, which are of a definite color on our topological picture.

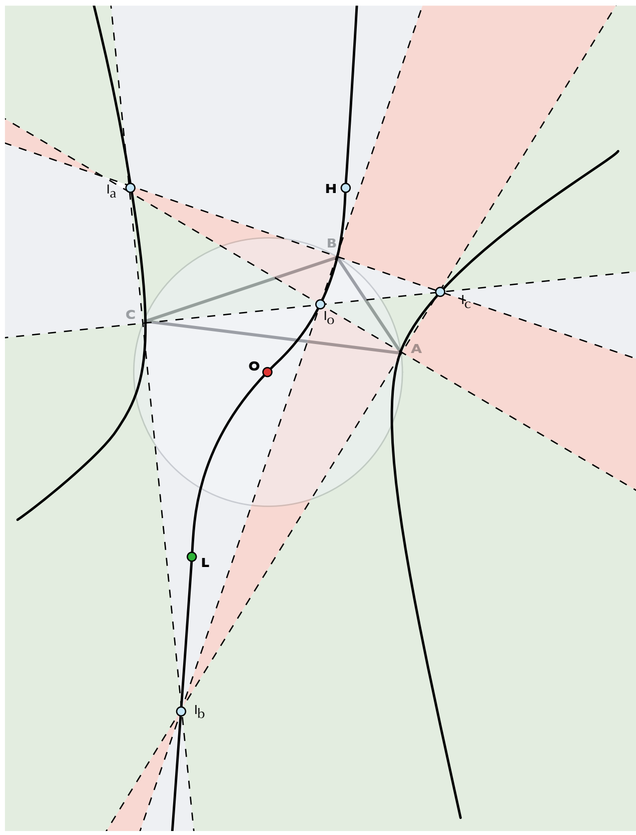

Figure 12. The Darboux cubic is a pivotal isogonal cubic, meaning that the isogonal conjugate of each point is on the cubic and colinear with the pivot point, which in this case is the deLongchamps point. The colored regions show the pattern of the conjugates. If a point is in a region of a certain color, so is its conjugate. This picture shows that the branches of the cubic turn to stay in regions of a particular color.

The Darboux cubic (Figure 12) has two branches, one through a single vertex, $\mathbf{I}_o$ and, in the illustration, $\mathbf{I}_b$. The other goes through $\mathbf{I}_c$, $\mathbf{I}_a$ and two vertices, wrapping around through the line at infinity. The Neuberg cubic (Figure 13) does the same. Its “circular component” being more visible since it does not pass through the line at infinity. We can understand the various “wiggles” of these cubics as necessary to stay in a self-conjugal region. Also we can see that a conjugate of a point on one branch cannot be on the other branch.

Geometry is fun.

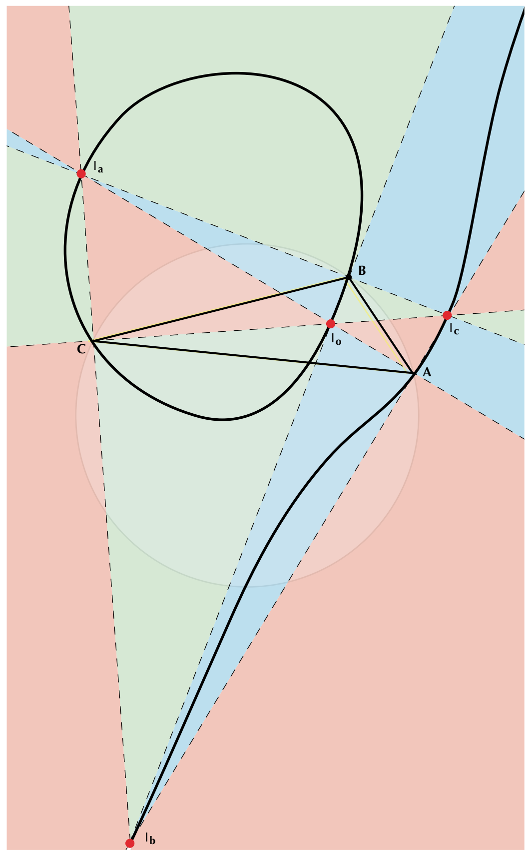

Figure 13. The Neuberg cubic is a pivotal isogonal cubic, meaning that the isogonal conjugate of each point is on the cubic and colinear with the pivot point, which in this case is the Euler infinity point. The colored regions show the pattern of the conjugates. If a point is in a region of a certain color, so its conjugate. This picture shows that the branches of the cubic turn to stay in regions of a particular color.

References

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