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A Synthetic Proof of Goormaghtigh's Generalization of Musselman's Theorem

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Abstract. We give a synthetic proof of a generalization by R. Goormaghtigh of a theorem of J. H. Musselman.

Consider a triangle ABC with circumcenter O and orthocenter H. Denote by A^*, B^*, C^* respectively the reflections of A, B, C in the side BC, CA, AB. The following interesting theorem was due to J. R. Musselman.

Theorem 1 (Musselman [2]). *The circles AOA^*, BOB^*, COC^* meet in a point which is the inverse in the circumcircle of the isogonal conjugate point of the nine point center.*

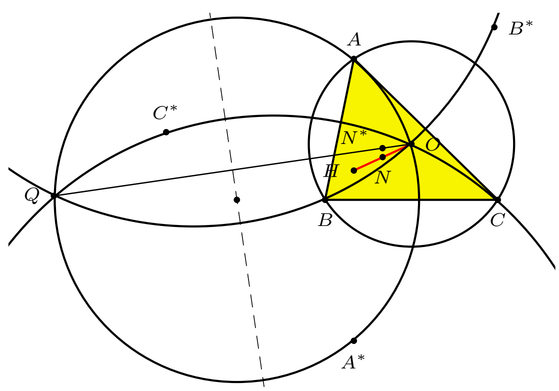

Figure 1

R. Goormaghtigh, in his solution using complex coordinates, gave the following generalization.

Theorem 2 (Goormaghtigh [2]). *Let A_1, B_1, C_1 be points on OA, OB, OC such that*

$$\frac{OA_1}{OA} = \frac{OB_1}{OB} = \frac{OC_1}{OC} = t.$$

(1) *The intersections of the perpendiculars to OA at A_1, OB at B_1, and OC at C_1 with the respective sidelines BC, CA, AB are collinear on a line ℓ.*

(2) *If M is the orthogonal projection of O on ℓ, M' the point on OM such that $OM' : OM = 1 : t$, then the inversive image of M' in the circumcircle of ABC*

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is the isogonal conjugate of the point P on the Euler line dividing OH in the ratio $OP : PH = 1 : 2t$. See Figure 1.

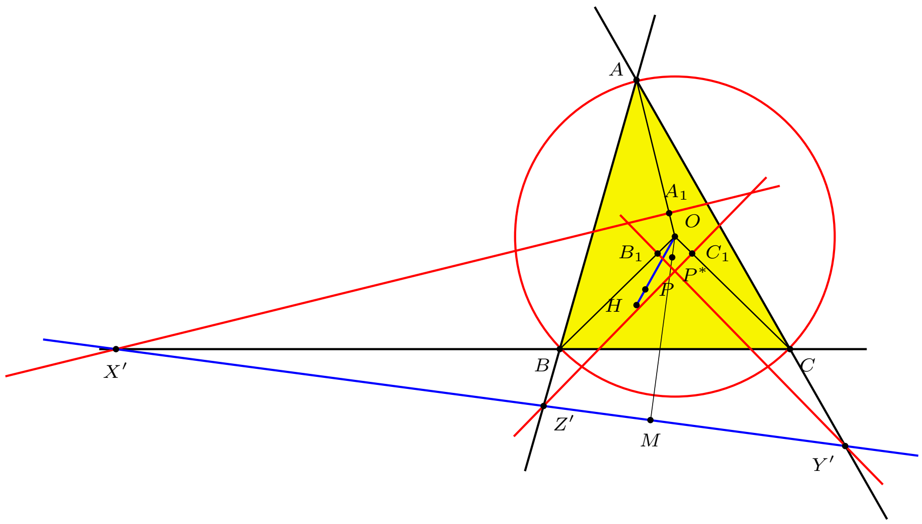

Figure 2

Musselman's Theorem is the case when $t = \frac{1}{2}$. Since the centers of the circles OAA^*, OBB^*, OCC^* are collinear, the three circles have a second common point which is the reflection of O in the line of centers. This is the inversive image of the isogonal conjugate of the nine-point center, the midpoint of OH.

By Desargues' theorem [1, pp.230–231], statement (1) above is equivalent to the perspectivity of ABC and the triangle bounded by the three perpendiculars in question. We prove this as an immediate corollary of Theorem 3 below. In fact, Goormaghtigh [2] remarked that (1) was well known, and was given in J. Neuberg's *Mémoir sur le Tétraèdre*, 1884, where it was also shown that the envelope of ℓ is the inscribed parabola with the Euler line as directrix (Kiepert parabola). He has, however, inadvertently omitted "the isogonal conjugate of " in statement (2).

Theorem 3. *Let $A'B'C'$ be the tangential triangle of ABC. Consider points X, Y, Z dividing OA', OB', OC' respectively in the ratio*

$$\frac{OX}{OA'} = \frac{OY}{OB'} = \frac{OZ}{OC'} = t. \tag{†}$$

The lines AX, BY, CZ are concurrent at the isogonal conjugate of the point P on the Euler line dividing OH in the ratio $OP : PH = 1 : 2t$.

Proof. Let the isogonal line of AX (with respect to angle A) intersect OA' at X'. The triangles OAX and $OX'A$ are similar. It follows that $OX \cdot OX' = OA^2$, and X, X' are inverse in the circumcircle. Note also that A' and M are inverse in the

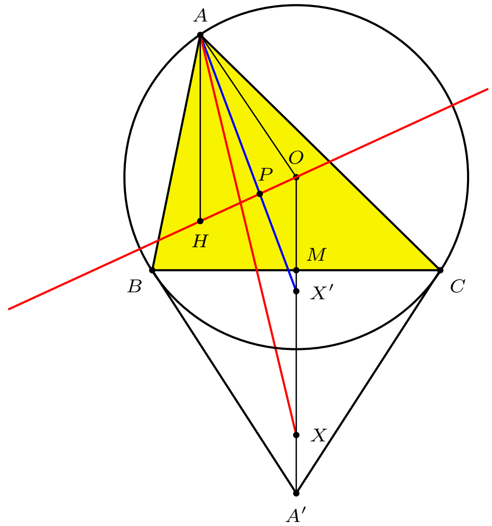

Figure 3

same circumcircle, and $OM \cdot OA' = OA^2$. If the isogonal line of AX intersects the Euler line OH at P, then

$$\frac{OP}{PH} = \frac{OX'}{AH} = \frac{OX'}{2 \cdot OM} = \frac{1}{2} \cdot \frac{OA'}{OX} = \frac{1}{2t}.$$

The same reasoning shows that the isogonal lines of BY and CZ intersect the Euler line at the same point P. From this, we conclude that the lines AX, BY, CZ intersect at the isogonal conjugate of P. □

For $t = \frac{1}{2}$, X, Y, Z are the circumcenters of the triangles OBC, OCA, OAB respectively. The lines AX, BY, CZ intersect at the isogonal conjugate of the midpoint of OH, which is clearly the nine-point center. This is Kosnita's Theorem (see [3]).

Proof of Theorem 2. Since the triangle XYZ bounded by the perpendiculars at A_1, B_1, C_1 is homothetic to the tangential triangle at O, with factor t. Its vertices X, Y, Z are on the lines OA', OB', OC' respectively and satisfy (†). By Theorem 3, the lines AX, BY, CZ intersect at the isogonal conjugate of P dividing OH in the ratio $OP : HP = 1 : 2t$. Statement (1) follows from Desargues' theorem. Denote by X' the intersection of BC and YZ, Y' that of CA and ZX, and Z' that of AB and XY. The points X', Y', Z' lie on a line ℓ.

Consider the inversion Ψ with center O and constant $t \cdot R^2$, where R is the circumradius of triangle ABC. The image of M under Ψ is the same as the inverse of M' (defined in statement (2)) in the circumcircle. The inversion Ψ clearly maps A, B, C into A_1, B_1, C_1 respectively. Let A_2, B_2, C_2 be the midpoints of BC, CA, AB respectively. Since the angles BB_1X and BA_2X are both right angles, the points B, B_1, A_2, X are concyclic, and

$$OA_2 \cdot OX = OB \cdot OB_1 = t \cdot R^2.$$

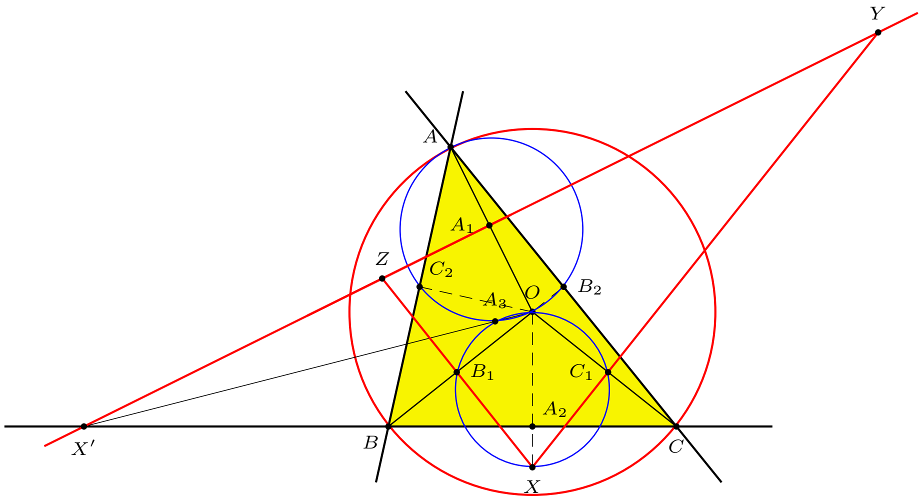

Figure 4

Similarly, $OB_2 \cdot OB_2' = OC_2 \cdot OC_2' = t \cdot R^2$. It follows that the inversion Ψ maps X, Y, Z into A_2, B_2, C_2 respectively.

Therefore, the image of X' under Ψ is the second common point A_3 of the circles OB_1C_1 and OB_2C_2. Likewise, the images of Y' and Z' are respectively the second common points B_3 of the circles OC_1A_1 and OC_2A_2, and C_3 of OA_1B_1 and OA_2B_2. Since X', Y', Z' are collinear on ℓ, the points O, A_3, B_3, C_3 are concyclic on a circle $\mathcal{C}$.

Under Ψ, the image of the line AX is the circle OA_1A_2, which has diameter OX' and contains M, the projection of O on ℓ. Likewise, the images of BY and CZ are the circles with diameters OY' and OZ' respectively, and they both contain the same point M. It follows that the common point of the lines AX, BY, CZ is the image of M under Ψ, which is the intersection of the line OM and $\mathcal{C}$. This is the antipode of O on $\mathcal{C}$.

References

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[3] D. Grinberg, On the Kosnita point and the reflection triangle, *Forum Geom.*, 3 (2003) 105–111.

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