A Synthetic Proof of Goormaghtigh’s Generalization of Musselman’s Theorem

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Abstract. We give a synthetic proof of a generalization by R. Goormaghtigh of a theorem of J. H. Musselman.

Consider a triangle $ABC$ with circumcenter $O$ and orthocenter $H$. Denote by $A^*, B^*, C^*$ respectively the reflections of $A, B, C$ in the side $BC, CA, AB$. The following interesting theorem was due to J. R. Musselman.

**Theorem 1** (Musselman [2]). The circles $AOA^*, BOB^*, COC^*$ meet in a point which is the inverse in the circumcircle of the isogonal conjugate point of the nine point center.

R. Goormaghtigh, in his solution using complex coordinates, gave the following generalization.

**Theorem 2** (Goormaghtigh [2]). Let $A_1, B_1, C_1$ be points on $OA, OB, OC$ such that

\[
\frac{OA_1}{OA} = \frac{OB_1}{OB} = \frac{OC_1}{OC} = t.
\]

(1) The intersections of the perpendiculars to $OA$ at $A_1$, $OB$ at $B_1$, and $OC$ at $C_1$ with the respective sidelines $BC, CA, AB$ are collinear on a line $\ell$.

(2) If $M$ is the orthogonal projection of $O$ on $\ell$, $M'$ the point on $OM$ such that $OM' : OM = 1 : t$, then the inversive image of $M'$ in the circumcircle of $ABC$...
is the isogonal conjugate of the point $P$ on the Euler line dividing $OH$ in the ratio $OP : PH = 1 : 2t$. See Figure 1.

Musselman’s Theorem is the case when $t = \frac{1}{2}$. Since the centers of the circles $OAA^*, OBB^*, OCC^*$ are collinear, the three circles have a second common point which is the reflection of $O$ in the line of centers. This is the inversive image of the isogonal conjugate of the nine-point center, the midpoint of $OH$.

By Desargues’ theorem [1, pp.230–231], statement (1) above is equivalent to the perspectivity of $ABC$ and the triangle bounded by the three perpendicularg in question. We prove this as an immediate corollary of Theorem 3 below. In fact, Goormaghtigh [2] remarked that (1) was well known, and was given in J. Neuberg’s Mémoire sur le Tétraèdre, 1884, where it was also shown that the envelope of $ℓ$ is the inscribed parabola with the Euler line as directrix (Kiepert parabola). He has, however, inadvertently omitted “the isogonal conjugate of ” in statement (2).

**Theorem 3.** Let $A'B'C'$ be the tangential triangle of $ABC$. Consider points $X$, $Y$, $Z$ dividing $OA'$, $OB'$, $OC'$ respectively in the ratio

$$
\frac{OX}{OA'} = \frac{OY}{OB'} = \frac{OZ}{OC'} = t.
$$

(†)

The lines $AX$, $BY$, $CZ$ are concurrent at the isogonal conjugate of the point $P$ on the Euler line dividing $OH$ in the ratio $OP : PH = 1 : 2t$.

**Proof.** Let the isogonal line of $AX$ (with respect to angle $A$) intersect $OA$ at $X'$. The triangles $OAX$ and $OX'A$ are similar. It follows that $OX \cdot OX' = OA^2$, and $X, X'$ are inverse in the circumcircle. Note also that $A'$ and $M$ are inverse in the
same circumcircle, and $OM \cdot OA' = OA^2$. If the isogonal line of $AX$ intersects the Euler line $OH$ at $P$, then

$$\frac{OP}{PH} = \frac{OX'}{AH} = \frac{OX'}{2 \cdot OM} = \frac{1}{2} \cdot \frac{OA'}{OX} = \frac{1}{2t}.$$ 

The same reasoning shows that the isogonal lines of $BY$ and $CZ$ intersect the Euler line at the same point $P$. From this, we conclude that the lines $AX$, $BY$, $CZ$ intersect at the isogonal conjugate of $P$. \hfill \Box

For $t = \frac{1}{2}$, $X$, $Y$, $Z$ are the circumcenters of the triangles $OBC$, $OCA$, $OAB$ respectively. The lines $AX$, $BY$, $CZ$ intersect at the isogonal conjugate of the midpoint of $OH$, which is clearly the nine-point center. This is Kosnita’s Theorem (see [3]).

**Proof of Theorem 2.** Since the triangle $XYZ$ bounded by the perpendiculars at $A_1$, $B_1$, $C_1$ is homothetic to the tangential triangle at $O$, with factor $t$. Its vertices $X$, $Y$, $Z$ are on the lines $OA'$, $OB'$, $OC'$ respectively and satisfy (i). By Theorem 3, the lines $AX$, $BY$, $CZ$ intersect at the isogonal conjugate of $P$ dividing $OH$ in the ratio $OP : HP = 1 : 2t$. Statement (1) follows from Desargues’ theorem. Denote by $X'$ the intersection of $BC$ and $YZ$, $Y'$ that of $CA$ and $ZX$, and $Z'$ that of $AB$ and $XY$. The points $X'$, $Y'$, $Z'$ lie on a line $\ell$.

Consider the inversion $\Psi$ with center $O$ and constant $t \cdot R^2$, where $R$ is the circumradius of triangle $ABC$. The image of $M$ under $\Psi$ is the same as the inverse of $M'$ (defined in statement (2)) in the circumcircle. The inversion $\Psi$ clearly maps $A$, $B$, $C$ into $A_1$, $B_1$, $C_1$ respectively. Let $A_2$, $B_2$, $C_2$ be the midpoints of $BC$, $CA$, $AB$ respectively. Since the angles $BB_1X$ and $BA_2X$ are both right angles, the points $B$, $B_1$, $A_2$, $X$ are concyclic, and

$$OA_2 \cdot OX = OB \cdot OB_1 = t \cdot R^2.$$
Similarly, $OB_2 \cdot OB'_2 = OC_2 \cdot OC'_2 = t \cdot R^2$. It follows that the inversion $\Psi$ maps $X, Y, Z$ into $A_2, B_2, C_2$ respectively.

Therefore, the image of $X'$ under $\Psi$ is the second common point $A_3$ of the circles $OB_1 C_1$ and $OB_2 C_2$. Likewise, the images of $Y'$ and $Z'$ are respectively the second common points $B_3$ of the circles $OC_1 A_1$ and $OC_2 A_2$, and $C_3$ of $OA_1 B_1$ and $OA_2 B_2$. Since $X', Y', Z'$ are collinear on $\ell$, the points $O, A_3, B_3, C_3$ are concyclic on a circle $C$.

Under $\Psi$, the image of the line $AX$ is the circle $OA_1 A_2$, which has diameter $OX'$ and contains $M$, the projection of $O$ on $\ell$. Likewise, the images of $BY$ and $CZ$ are the circles with diameters $OY'$ and $OZ'$ respectively, and they both contain the same point $M$. It follows that the common point of the lines $AX, BY, CZ$ is the image of $M$ under $\Psi$, which is the intersection of the line $OM$ and $C$. This is the antipode of $O$ on $C$.

References


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