Forum Geometricorum
Volume 5 (2005) 21–22.

FORUM GEOM
ISSN 1534-1178

On the Existence of Triangles with Given Lengths of One Side, the Opposite and One Adjacent Angle Bisectors

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Abstract. We give a necessary and sufficient condition for the existence of a triangle with given lengths of one sides, its opposite angle bisector, and one adjacent angle bisector.

In [1] the problem of existence of a triangle with given lengths of one side and two adjacent angle bisectors was solved. In this note we consider the same problem with one of the adjacent angle bisector replaced by the opposite angle bisector. We prove the following theorem.

Theorem 1. *Given a, ℓ_a, $\ell_b > 0$, there is a unique triangle ABC with $BC = a$ and lengths of bisectors of angles A and B equal to ℓ_a and ℓ_b respectively if and only if $\ell_b \leq a$ or*

$$a < \ell_b < 2a \quad \text{and} \quad \ell_a > \frac{4a\ell_b(\ell_b - a)}{(2a - \ell_b)(3\ell_b - 2a)}.$$

Proof. In a triangle ABC with $BC = a$ and given ℓ_a, ℓ_b, let $y = CA$ and $z = AB$. We have $\ell_b = \frac{2az}{a+z} \cos \frac{B}{2}$ and

$$z = \frac{a\ell_b}{2a \cos \frac{B}{2} - \ell_b}. \tag{1}$$

It follows that $\cos \frac{B}{2} > \frac{\ell_b}{2a}$, $\ell_b < 2a$, and

$$B < 2 \arccos \frac{\ell_b}{2a}. \tag{2}$$

Also,

$$y^2 = a^2 + z(z - 2a \cos B), \tag{3}$$

$$\ell_a^2 = yz \left(1 - \frac{a^2}{(y+z)^2}\right). \tag{4}$$

Publication Date: February 10, 2005. Communicating Editor: Paul Yiu.

Case 1: $\ell_b \leq a$. Clearly, (1) defines z as an increasing function of B on the open interval $\left(0, 2\arccos\frac{\ell_b}{2a}\right)$. As B increases from 0 to $2\arccos\frac{\ell_b}{2a}$, z increases from $\frac{a\ell_b}{2a-\ell_b}$ to ∞. At the same time, from (3), y increases from $a - \frac{a\cdot\ell_b}{2a-\ell_b} = \frac{2a(a-\ell_b)}{2a-\ell_b}$ to ∞. Correspondingly, the right hand side of (4) can be any positive number. From the intermediate value theorem, there exists a unique B for which (4) is satisfied. This proves the existence and uniqueness of the triangle.

Case 2: $a < \ell_b < 2a$. In this case, (1) defines the same increasing function z as before, but y increases from $\frac{a\cdot\ell_b}{2a-\ell_b} - a = \frac{2a(\ell_b-a)}{2a-\ell_b}$ to ∞. Correspondingly, the right hand side of (4) increases from

$$\frac{a\ell_b}{2a-\ell_b}\cdot\frac{2a(\ell_b-a)}{2a-\ell_b}\left(1-\frac{a^2}{\left(\frac{a\ell_b}{2a-\ell_b}+\frac{2a(\ell_b-a)}{2a-\ell_b}\right)^2}\right) = \frac{16a^2\ell_b^2(\ell_b-a)^2}{(2a-\ell_b)^2(3\ell_b-2a)^2}$$

to ∞. This means $\ell_a > \frac{4a\ell_b(\ell_b-a)}{(2a-\ell_b)(3\ell_b-2a)}$. Therefore, there is a unique value B for which (4) is satisfied. This proves the existence and uniqueness of the triangle. $\square$

Corollary 2. *For the existence of an isosceles triangle with equal sides a with opposite angle bisectors ℓ_a, it is necessary and sufficient that $\ell_a < \frac{4}{3}a$.*

Reference

[1] V. Oxman, On the existence of triangles with given lengths of one side and two adjacent angle bisectors, *Forum Geom.*, 4 (2004) 215–218.

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