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On the Maximal Inflation of Two Squares

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Abstract. We consider two non-overlapping congruent squares q_1, q_2 and the homothetic congruent squares q_1^k, q_2^k obtained from two similitudes centered at the centers of the squares. We study the supremum of the ratios of these similitudes for which q_1^k, q_2^k are non-overlapping. This yields a function $\psi = \psi(q_1, q_2)$ for which the squares q_1^ψ, q_2^ψ are non-overlapping although their boundaries intersect. When the squares q_1 and q_2 are not parallel, we give a 8-step construction using straight edge and compass of the intersection $q_1^\psi \cap q_2^\psi$ and we obtain two formulas for ψ. We also give an angular characterization of a vertex which belongs to $q_1^\psi \cap q_2^\psi$.

1. Introduction and notation

We study here the problem of maximizing the *inflation* of two non-overlapping congruent squares $q_1 = q_{a_1,b_1,\theta_1,c}$ and $q_2 = q_{a_2,b_2,\theta_2,c}$. The square q_i has the four vertices

$$S_j(q_i) = (a_i, b_i) + c \cdot (\cos(\theta_i + j\tfrac{\pi}{2}), \sin(\theta_i + j\tfrac{\pi}{2})).$$

Let $q^k_{a,b,\theta,c} = q_{a,b,\theta,k}$ be the homothetic of ratio k/c of the square $q_{a,b,\theta,c}$. Our problem amounts to determining the supremum $\psi = \psi(q_1, q_2)$ of the numbers $k > 0$ for which q_1^k and q_2^k are disjoint.

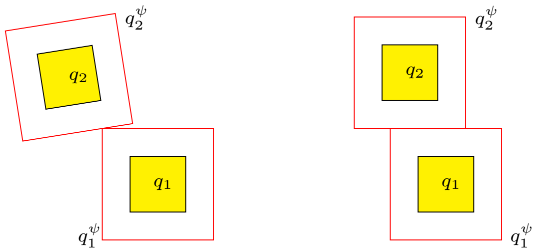

Figure 1

In [3, §4], $\psi = \psi(q_1, q_2)$ is called the *maximum inflation* of a configuration of two squares. It plays a central part in computation of dense packings of squares in a larger square. We refer to the paper of P. Erdős and R. Graham [1] who initiated the problem of maximizing the area sum of packings of an arbitrary square by unit

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squares, see also the survey of E. Friedman [2]. We note that ψ is independent of c and that

$$k \leq \psi \quad \Leftrightarrow \quad \mathrm{int}(q_1^k) \cap \mathrm{int}(q_2^k) = \emptyset, \tag{1}$$

$$k \geq \psi \quad \Leftrightarrow \quad \partial q_1^k \cap \partial q_2^k \neq \emptyset, \tag{2}$$

where as usual, we denote by $\mathrm{int}(q)$ and ∂q the interior and the boundary of a square q. An explicit formula for $\psi = \psi(q_1, q_2)$ is given in [3, Prop.2] as follows. Let us define

$$\psi_0(a, b, \theta) = \min_{i=1,\ldots,4} \left\{ \frac{|a| + |b|}{\left|1 - \sqrt{2}\mathrm{sgn}(ab)\sin(\theta + \frac{\pi}{4} + i\frac{\pi}{2})\right|} \right\},$$

and

$$\rho(q_1, q_2) = \psi_0\left(t\cos\theta_1 + t'\sin\theta_1, -t\sin\theta_1 + t'\cos\theta_1, \theta_2 - \theta_1\right),$$

with $(t, t') = (a_2 - a_1, b_2 - b_1)$. The maximal inflation of two squares q_1 and q_2 is the maximum of $\rho(q_1, q_2)$ and $\rho(q_2, q_1)$. The minimum value, say $k = \rho(q_1, q_2) < \psi$, corresponds to the belongness of a vertex E of q_2^k to a straight line AB when $q_1^k = ABCD$, but without having E between A and B. This expression of ψ gives an efficient tool for doing calculations of maximal inflation of configurations of $n \geq 2$ squares.

In this paper, the two congruent squares q_1, q_2 are such that $q_1 \cap q_2 = \emptyset$ and their centers are denoted by $C_i = C(q_i)$. We say as in [3, §4], that q_2 *strikes* q_1 if the set $q_1^\psi \cap q_2^\psi$ contains a vertex of q_2^ψ. In §§3–5, we suppose that the squares q_1, q_2 are not parallel so that $q_1^\psi \cap q_2^\psi = \{P\}$, where the vertex P of q_1 or q_2 is the *percussion point*. However, at the end of each of these sections, we discuss the parallel case in a final remark. We find in §4 a 8-step construction using straight edge and compass of P. Since P is a vertex of q_1^ψ or q_2^ψ, the construction gives immediately the other vertices of q_1^ψ, q_2^ψ. At the same time, we choose a frame in which we obtain two simpler formulas for ψ. We give in §5 an angular characterization which allows to identify which square q_1 or q_2 strikes the other.

2. Quadrants defined by squares

If $q = q_{a,b,\theta,c}$ is a square, we define the two *axes* $A_1(q)$ and $A_2(q)$ of q as the straight lines through $(a, b) \in \mathbb{R}^2$ which are parallel to the sides of q. We define the four counterclockwise consecutive *rays* $D_i(q)$ as the half-lines with origin (a, b) and which contain the vertices of q; we set $D_0(q) = D_4(q)$. A couple of consecutive rays $D_i(q)$ and $D_{i+1}(q)$ defines the i^{th} quadrant $Q_i(q)$ in $\mathbb{R}^2$ associated to the square q.

If a point M, distinct from the center of q, belongs to $\mathrm{int}(Q_i(q))$, then we note $S(q, M) = Q_i(q)$. If the point M lies on the boundaries of two consecutive quadrants $Q_{i-1}(q)$ and $Q_i(q)$, then we choose indifferently $S(q, M)$ as one of the two quadrants $Q_{i-1}(q)$ or $Q_i(q)$. Note that $M \in \mathrm{int}(S(q, N))$ iff $N \in \mathrm{int}(S(q, M))$.

Lemma 1. *If the intersection set $q_1^\psi \cap q_2^\psi$ contains a vertex P of q_2^ψ, then $P \in S(q_2, C_1)$.*

Proof. Let D be the straight line containing a diagonal of q_2 and which does not contain P. Then the disc with center P and radius ψ contains C_1 and C_2 since $d(C_1, P) \leq d(C_2, P) = \psi$. Hence there is only one half-plane $\mathcal{H}$, bounded by D, which contains this disc. Now, $\mathcal{H}$ is the union $\mathcal{S}_1 \cup \mathcal{S}_2$ of two quadrants associated to q_2 and the ray $D_i(q_2)$ through P is $\mathcal{S}_1 \cap \mathcal{S}_2$. If $C_1 \notin D_i(q_2)$, one of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $S(q_2, C_1)$; but $P \in D_i(q_2) = \mathcal{S}_1 \cap \mathcal{S}_2$ gives $P \in S(q_2, C_1)$. If $C_1 \in D_i(q_2)$, then $P \in D_i(q_2) \subset S(q_2, C_1)$. $\square$

Lemma 2. *We have*

$$q_1^\psi \cap q_2^\psi \subset S(q_1, C_2) \cap S(q_2, C_1). \tag{3}$$

The intersection of the two quadrants is depicted in Figure 2.

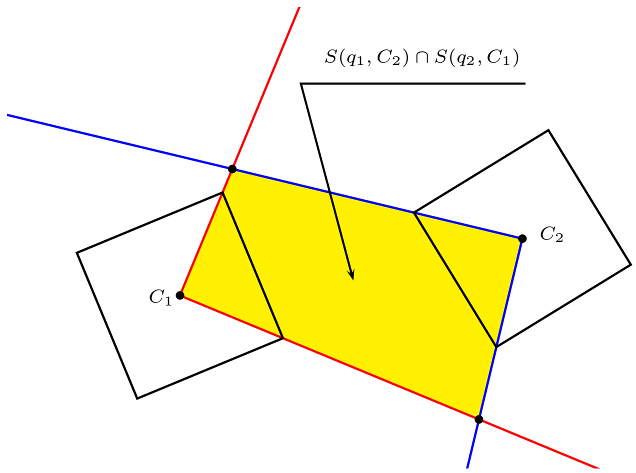

Figure 2

Proof. The proof is divided in three exclusive and exhaustive situations.
(i) First, we suppose that the intersection set $q_1^\psi \cap q_2^\psi = \{P\}$ where P is a common vertex of q_1^ψ and q_2^ψ. We readily obtain $P \in S(q_2, C_1)$ and $P \in S(q_1, C_2)$ from Lemma 1.
(ii) Second, we suppose that $q_1^\psi \cap q_2^\psi$ contains a vertex $P = (x_P, y_P)$ of q_2^ψ and that P is not a vertex of q_1^ψ. We denote by $ABCD$ the square q_1^ψ with $P \in]A, B[$ and let C_1A, C_1B be respectively the x-axis and the y-axis. For the interiors of the two squares to be disjoint, C_2 must be in $\{(x, y) \,:\, x \geq x_p \text{ and } y \geq y_p\}$ since the straight line $x + y = \psi$ separates the two squares. Hence the percussion point P and the center $C_2 = (a, b)$ of q_2 lie in the same quadrant $S(q_1, C_2)$. Due to Lemma 1, P is also in $S(q_2, C_1)$.
(iii) Third, when $q_1^\psi \cap q_2^\psi$ is a common edge of the two squares q_i^ψ, then $S(q_1, C_2) \cap S(q_2, C_1)$ is a square of size ψ and having vertices C_1, P_1, C_2, P_2. Since $q_1^\psi \cap q_2^\psi$ is a diagonal of this square, the inclusion (3) is obvious. $\square$

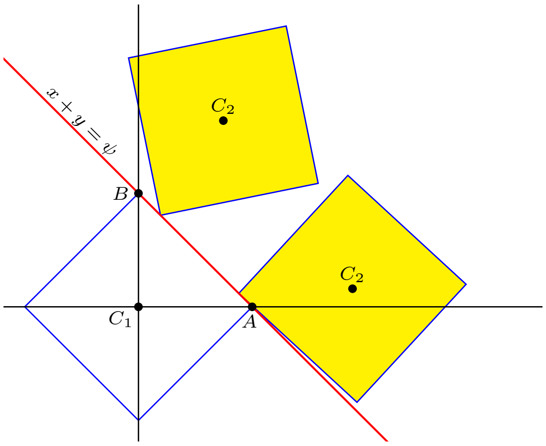

Figure 3

Remark. When the segment $[C_1, C_2]$ contains a vertex of q_1^ψ or q_2^ψ, say A, the statement in (3) can be strengthened: $q_1^\psi \cap q_2^\psi = \{A\}$ is the percussion point.

3. Location of the percussion point

We consider the integer $i_1 \in \{0,1\}$ such that the axis $A_{i_1}(q_1)$ bounds an half-plane containing $S(q_1, C_2)$. Similarly, we consider the axis $A_{i_2}(q_2)$ which bounds an half-plane containing $S(q_2, C_1)$. Since $A_{i_1}(q_1)$, $A_{i_2}(q_2)$ are not parallel, we can set $A_{i_1}(q_1) \cap A_{i_2}(q_2) = \{W\}$. We use in §4 the point V which is the intersection of the axis $A_{j_2}(q_2)$ and WC_1 and where $j_2 \in \{0,1\}$ is the integer different from i_2.

The two straight lines $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ define one dihedral angle which contains both C_1 and C_2, that we denote as $\angle C_1WC_2$. Let $\gamma = \gamma(q_1, q_2) = 2\omega = \widehat{C_1WC_2} \in [0, \pi]$ be the measure of this dihedral angle. We define now $B(q_1, q_2)$ as the half-line which bisects $\angle C_1WC_2$. We also note $\ell_1 = ||\overrightarrow{WC_1}||$ and $\ell_2 = ||\overrightarrow{WC_2}||$.

Lemma 3. *We have* $\gamma = \gamma(q_1, q_2) \in \left]0, \frac{\pi}{2}\right[$.

Proof. If $\gamma = 0$, the two axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ are equal to some straight line D. The centers C_1 and C_2 lie on D. But by construction $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ have to be perpendicular to the line D, contradiction.

If $\gamma = \frac{\pi}{2}$, the two axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ are perpendicular but this is excluded because the squares are not parallel.

We now suppose that $\frac{\pi}{2} < \gamma < \frac{3\pi}{4}$. The quadrant $S(q_2, C_1)$ intersects the axis $A_{i_1}(q_1)$ at a point M which belongs to the segment $[W, C_1]$ for C_1 lies in $S(q_2, C_1)$. Since the angle $\widehat{WMC_2} = \frac{3}{4}\pi - \gamma$ is strictly less than $\frac{\pi}{4}$, the quadrant $S(q_1, C_2)$ does not contain C_2, contradiction. See Figure 4.

The last case $\frac{3\pi}{4} \le \gamma \le \pi$ implies that $S(q_2, C_1)$ does not intersect the boundary of $\angle C_1WC_2$. This is in contradiction with $C_1 \in A_{i_1}(q_1) \cap S(q_2, C_1)$. $\square$

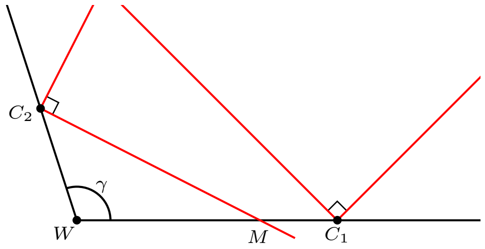

Figure 4

Lemma 4. *We have* $q_1^\psi \cap q_2^\psi \subset B(q_1, q_2)$.

Proof. Let $0 < k \leq \psi$. The homothetic square q_1^k (resp. q_2^k) has two vertices in $S(q_1, C_2)$ (resp. $S(q_2, C_1)$). The straight line passing through those vertices of q_1 (resp. q_2) is parallel at distance $k/\sqrt{2}$ to the axis $A_{i_1}(q_1)$ (resp. $A_{i_2}(q_2)$). The intersection of those two parallels belongs to $B(q_1, q_2)$ and, according to Lemma 2, allows to localize the point of percussion which is equal to $q_1^k \cap q_2^k$ when $k = \psi$. Thus $P \in B(q_1, q_2)$. □

Remark. When q_1 and q_2 are parallel, Lemma 4 remains true provided $B(q_1, q_2)$ is replaced with the straight line containing the points equidistant from the two parallel axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$.

4. Construction of the percussion point

Two rays $D_i(q_1)$ and $D_{i+1}(q_1)$ intersect $B(q_1, q_2)$ at I_1, I_3. We use the natural order on $B(q_1, q_2)$ and we can suppose that $W < I_1 < I_3$. Similarly, we define $W < I_2 < I_4$ relatively to q_2.

Lemma 5. *We have*
(a) $\ell_1 = \ell_2 \Leftrightarrow I_1 = I_2 < I_3 = I_4$.
(b) $\ell_1 < \ell_2 \Leftrightarrow I_1 < I_2 < I_3 < I_4$.
(c) $\ell_2 < \ell_1 \Leftrightarrow I_2 < I_1 < I_4 < I_3$.

Proof. If $\ell_1 = \ell_2$ then $I_1 = I_2 < I_3 = I_4$. Shifting C_1 along WC_1 towards W causes $C_1 I_1$ and $C_1 I_3$ to slide in a parallel fashion, so that $I_1 < I_2$ and $I_3 < I_4$. Since $C_1 \in S(q_2, C_1)$, the point C_1 cannot pass the intersection C_ℓ of $C_2 I_2$ and WC_1. But when $C_1 = C_\ell$, we have $\widehat{WC_1I_2} = \widehat{WC_\ell C_2} = 3\pi/4 - \gamma$. By Lemma 3, we deduce that $\pi/4 < \widehat{WC_1I_2} < 3\pi/4$ and accordingly $I_2 < I_3$. The remaining implications are straightforward. □

Theorem 6. (i) *Among the four points* $I_1, \ldots, I_4$*, the second one is the percussion point:* $P = q_1^\psi \cap q_2^\psi = \max\{I_1, I_2\}$*. We have*

$$\psi = \max\{\ell_1, \ell_2\} \frac{\sqrt{2}}{1 + \cot \omega}. \tag{4}$$

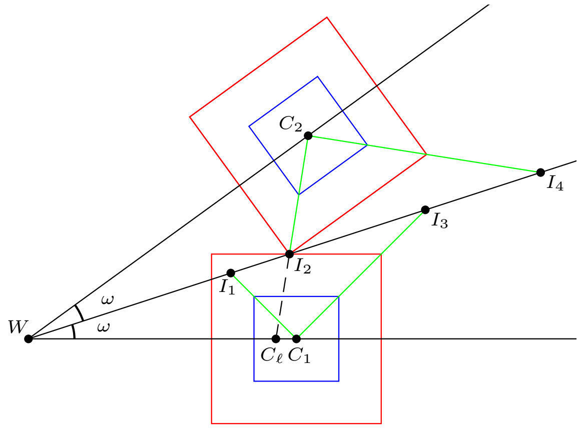

Figure 5

(ii) *If, say $\ell_2 \geq \ell_1$, then q_2 strikes q_1 at the point P which is the incenter of the triangle C_2WV.*

Proof. (i). We suppose first that $\ell_2 > \ell_1$. By Lemma 5 we have

$$d_1 = ||\overrightarrow{C_1I_1}|| < d_2 = ||\overrightarrow{C_2I_2}|| < d_3 = ||\overrightarrow{C_1I_3}|| < d_4 = ||\overrightarrow{C_2I_4}||.$$

We know from Lemma 4 that P is one of the four points $I_1, \ldots, I_4$ and thus the percussion occurs at $P = I_i$ if and only if $\psi = d_i$. It is impossible that $P = I_1$ because in that case $\psi = d_1 < d_2$ and then $P \in q_2^{d_1} \cap B(q_1, q_2) = \emptyset$. Hence $\psi > d_1$. If $\psi \geq d_3$ and since $I_2 \in]I_1, I_3[$ by Lemma 5, the point $I_2 \in q_2^{\psi}$ belongs also to the interior of q_1^{ψ} and then the two interiors are not disjoint. We get $\psi = d_2$ and $P = I_2 > I_1$. Easy calculations in the frame centered at $W = (0,0)$ and with x-axis WC_1, give $I_2 = \ell_2(1/(1+\tan\omega), \tan\omega/(1+\tan\omega))$ and (4).

The symmetric case $\ell_1 > \ell_2$ gives q_1 strikes q_2 at $P = I_1 > I_2$ and (4) again. Finally, if $\ell_1 = \ell_2$ the point $P = I_1 = I_2$ is effectively the percussion point.

(ii) If $\ell_2 \geq \ell_1$, by Lemma 4, the point $P = I_2$ belongs to the bisector ray $B(q_1, q_2)$ of the geometric angle $\angle C_1WC_2 = \angle VWC_2$. Now, since P is a vertex of q_2, we have $\widehat{VC_2P} = \widehat{PC_2W} = \pi/4$, so that P belongs to the bisector ray of the geometric angle $\angle VC_2W$. We conclude that P is the incenter of the triangle VC_2W. □

Corollary 7. *We have*

$$\begin{aligned}
\ell_1 < \ell_2 &\quad\Leftrightarrow\quad q_2 \textit{ strikes } q_1 \textit{ and } q_1 \textit{ does not strike } q_2,\\
\ell_2 < \ell_1 &\quad\Leftrightarrow\quad q_1 \textit{ strikes } q_2 \textit{ and } q_2 \textit{ does not strike } q_1,\\
\ell_1 = \ell_2 &\quad\Leftrightarrow\quad q_2 \textit{ strikes } q_1 \textit{ and } q_1 \textit{ strikes } q_2.
\end{aligned}$$

Proof. The three implications from left to right are direct consequences of Theorem 6 and its proof. Since the three cases are exclusive and exhaustive, the three converse implications readily follow. □

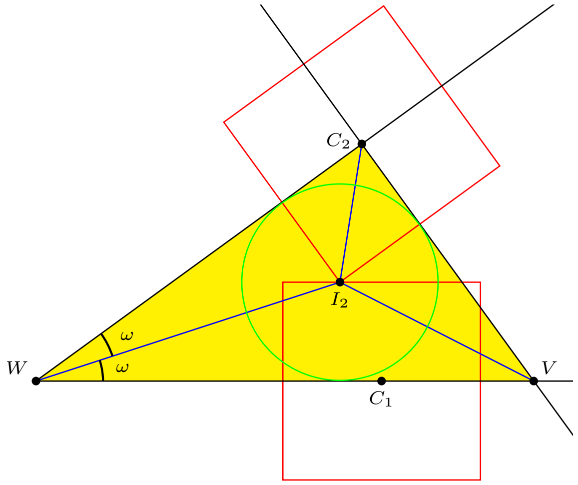

Figure 6

We now synthesize the whole preceding results. For two points M and N, we denote by $\Gamma(M,N)$ the circle with M as center and MN as radius.
Construction of P. Given the eight vertices of two congruent, non parallel and non-overlapping squares q_1 and q_2, construct
(1-2) the two centers C_1, C_2, intersection of the straight lines passing through opposite vertices of q_i, $i=1,2$,
(3-4) the axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ (this requires the determination of the quadrants $S(q_1,C_2)$ and $S(q_2,C_1)$ as much as two intermediate points),
(5) the point W, intersection of $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$,
(6) the point C_r, intersection of $\Gamma(W,C_2)$ and the half-line WC_1,
(7) the bisector $B(q_1,q_2)$ through W and $\Gamma(C_2,W)\cap\Gamma(C_r,W)$ (the four points $I_1,\cdots,I_4$ appear at this stage),
(8) the percussion point P, the second among the four points $I_1,\cdots,I_4$ on the oriented half-line $B(q_1,q_2)$.

Remarks. (1) We know that the area of the triangle VWC_2 is equal to $p\cdot r$ where p is the half-perimeter of the triangle and $r=\psi/\sqrt{2}$ the radius of the incircle. Now, we also have the formula

$$\psi=\frac{\sqrt{2}\mathrm{Area}(VWC_2)}{p}=\frac{\sqrt{2}VC_2\cdot WC_2}{VC_2+WC_2+VW}=\ell_2\frac{\sqrt{2}\sin\gamma}{\sin\gamma+\cos\gamma+1}.$$

The last value is equal to (4) when $\ell_2\geq\ell_1$.

(2) Let us suppose that the segment $[C_1,C_2]$ contains a vertex $S_i(q_2)$. This amounts to saying that $C_1=C_\ell$, so that $S(q_2,C_1)$ has been chosen as one of two quadrants $Q_{i-1}(q_2)$, $Q_i(q_2)$. But these choices lead to consider the two dihedral angles $\angle C_1WC_2$ and $\angle C_1VC_2$. Due to the second part of Theorem 6, P and the formula for ψ are not altered by this choice.

(3) When q_1 and q_2 are parallel, the construction of the four points $I_1,\cdots,I_4$ makes sense using again the straight line $B(q_1,q_2)$ equidistant from the two axes

$A_{i_1}(q_1)$ and $A_{i_2}(q_2)$. We choose an order on $B(q_1, q_2)$ and next we label those four points in such a way that $[I_2, I_3] \subset [I_1, I_4]$ and we have $q_1^\psi \cap q_2^\psi = [I_2, I_3]$. In consequence, the steps (5-8) in the above Construction are replaced with the construction of the midpoint $(C_1+C_2)/2$ (three steps), of the straight line $B(q_1, q_2)$ (three steps) and lastly of the two points I_2, I_3.

5. An angular characterization of the percussion point

We define $\alpha(q_1, q_2)$ as the minimum of $\{S_i(q_1)\widehat{C(q_1)}C(q_2), 0 \le i \le 3\}$. This set contains two acute and two obtuse angles. We have $0 \le \alpha(q_1, q_2) \le \frac{\pi}{4}$ since $\alpha(q_1, q_2) \le \frac{\pi}{2} - \alpha(q_1, q_2)$.

Theorem 8. *The square q_2 strikes q_1 if and only if $\alpha(q_2, q_1) \le \alpha(q_1, q_2)$. The percussion point is the vertex of q_1 or q_2 which realizes the minimum of the eight angles appearing in $\alpha(q_1, q_2)$ and $\alpha(q_2, q_1)$.*

Proof. Suppose that q_2 strikes $q_1 = ABCD$ at P in the interior of side AB, see Figure 7. Let AB be the x-axis and P the origin. Then for the interiors of q_1 and q_2 to be disjoint, the center C_2 of q_2 must be in $\{(x, y) \,:\, y \ge |x|\}$. Also, C_2 lies on the arc $x^2 + y^2 = \psi^2$. Let C_0, C_ℓ, C_r be the three points on this arc which intersect the lines C_1P, $y = -x$ and $y = x$ respectively.

Letting C_2 moving along the arc from C_0 to C_r, the angle $\widehat{PC_2C_1}$ increases from $\widehat{PC_0C_1} = 0$ to $\widehat{PC_rC_1} = \widehat{BC_1C_r}$ and the angle $\widehat{BC_1C_2}$ decreases from $\widehat{BC_1C_0}$ to $\widehat{BC_1C_r}$. Hence throughout the move we have $\widehat{PC_2C_1} \le \widehat{BC_1C_r} \le \widehat{BC_1C_2}$. But we have obviously $\widehat{PC_2C_1} < \widehat{AC_1C_2}$ and thus $\widehat{PC_2C_1} \le \alpha(q_1, q_2)$. The same proof holds when C_2 moves on the arc C_0C_ℓ.

Since $\widehat{PC_2C_1} \le \pi/4$, we get $\alpha(q_2, q_1) = \widehat{PC_2C_1}$ and next $\alpha(q_2, q_1) \le \alpha(q_1, q_2)$. The angle $\widehat{PC_2C_1}$ realizes effectively the minimum of the eight angles. The converse implication holds because $\alpha(q_2, q_1) = \alpha(q_1, q_2)$ is equivalent to the fact that q_1 and q_2 strike each other at a common vertex. □

Remark. In case q_1 and q_2 are parallel, q_1 strikes q_2 at P_1 and q_2 strikes q_1 at P_2. We have $\alpha(q_1, q_2) = \widehat{C_2C_1P_1} = \widehat{C_1C_2P_2} = \alpha(q_2, q_1)$. Hence the results in Theorem 8 remain true.

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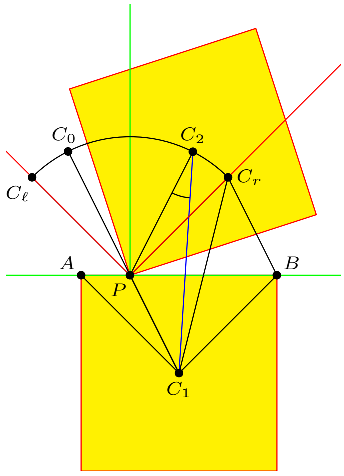

Figure 7

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