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The Arbelos in n-Aliquot Parts

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Abstract. We generalize the classical arbelos to the case divided into many chambers by semicircles and construct embedded patterns of such arbelos.

1. Introduction and preliminaries

Let $\{\alpha,\ \beta,\ \gamma\}$ be an arbelos, that is, $\alpha,\ \beta,\ \gamma$ are semicircles whose centers are collinear and erected on the same side of this line, $\alpha,\ \beta$ are tangent externally, and γ touches α and β internally. In this paper we generalize results on the Archimedean circles of the arbelos. We take the line passing through the centers of $\alpha,\ \beta,\ \gamma$ as the x-axis and the line passing through the tangent point O of α and β and perpendicular to the x-axis as the y-axis. Let $\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta$ be $n+1$ distinct semicircles touching α and β at O, where $\alpha_1, \dots, \alpha_{n-1}$ are erected on the same side as α and β, and intersect with γ. One of them may be the line perpendicular to the x-axis (i.e. y-axis). If the n inscribed circles in the curvilinear triangles bounded by $\alpha_{i-1},\ \alpha_i,\ \gamma$ are congruent we call this configuration of semicircles $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ an arbelos in n-aliquot parts, and the inscribed circles the Archimedean circles in n-aliquot parts. In this paper we calculate the radii of the Archimedean circles in n-aliquot parts and construct embedded patterns of arbelos in aliquot parts.

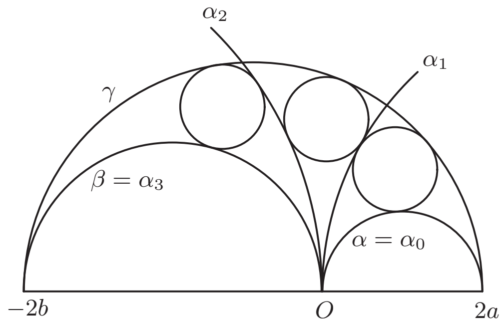

Figure 1. The case $n = 3$

For the arbelos $\{\alpha,\ \beta,\ \gamma\}$ we denote by $\Phi(\alpha, \beta, \gamma)$ the family of semicircles through O, having the common point with γ in the region $y \geq 0$ and with centers on the x-axis, together with the line perpendicular to the x-axis at O. Renaming if necessary we assume α in the region $x \geq 0$. Let $a,\ b$ be the radii of $\alpha,\ \beta$. The semicircle γ meets the x-axis at $-2b$ and $2a$.

For a semicircle $\alpha_i \in \Phi(\alpha, \beta, \gamma)$, let a_i be the x-coordinate of its center. Define $\mu(\alpha_i)$ as follows.

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If $a \neq b$,

$$\mu(\alpha_i) = \begin{cases} \dfrac{a_i - a + b}{a_i}, & \text{if } \alpha_i \text{ is a semi-circle,} \\ 1, & \text{if } \alpha_i \text{ is the line.} \end{cases}$$

If $a = b$,

$$\mu(\alpha_i) = \begin{cases} \dfrac{1}{a_i}, & \text{if } \alpha_i \text{ is a semi-circle,} \\ 0, & \text{if } \alpha_i \text{ is the line.} \end{cases}$$

In both cases $\mu(\alpha_i)$ depends only on α_i and the center of γ, but not on the radius of γ. For $\alpha_i,\ \alpha_j \in \Phi(\alpha,\beta,\gamma)$, the equality $\mu(\alpha_i) = \mu(\alpha_j)$ holds if and only if $\alpha_i = \alpha_j$. For any $\alpha_i \in \Phi(\alpha,\beta,\gamma)$,

$$\begin{aligned} \frac{b}{a} &= \mu(\alpha) \geq \mu(\alpha_i) \geq \mu(\beta) = \frac{a}{b} \text{ if } a < b, \\ \frac{1}{a} &= \mu(\alpha) \geq \mu(\alpha_i) \geq \mu(\beta) = -\frac{1}{a} \text{ if } a = b, \\ \frac{b}{a} &= \mu(\alpha) \leq \mu(\alpha_i) \leq \mu(\beta) = \frac{a}{b} \text{ if } a > b. \end{aligned}$$

For $\alpha_i,\ \alpha_j \in \Phi(\alpha,\beta,\gamma)$, define the order

$$\alpha_i < \alpha_j \text{ if and only if } \begin{cases} \mu(\alpha_i) > \mu(\alpha_j) & \text{if } a \leq b, \\ \mu(\alpha_i) < \mu(\alpha_j) & \text{otherwise.} \end{cases}$$

This means that α_i is nearer to α than α_j is. Throughout this paper we shall adopt these notations and assumptions.

2. An arbelos in aliquot parts

Lemma 1. *If α_i and α_j are semicircles in $\Phi(\alpha,\beta,\gamma)$ with $\alpha_i < \alpha_j$, the radius of the inscribed circle in the curvilinear triangle bounded by α_i, α_j and γ is*

$$\frac{ab(a_j - a_i)}{a_i a_j - a a_i + b a_j}\ .$$

Proof. Let $\mathcal{C}$ be the inscribed circle with radius r. First we invert $\{\alpha_i,\ \alpha_j,\ \gamma,\ \mathcal{C}\}$ in the circle with center O and radius k. Then α_i and α_j are inverted to the lines $\overline{\alpha_i}$ and $\overline{\alpha_j}$ perpendicular to the x-axis , γ is inverted to the semicircle $\overline{\gamma}$ erected on the x-axis and $\mathcal{C}$ is inverted to the circle $\overline{\mathcal{C}}$ tangent to $\overline{\gamma}$ externally. We write the x-coordinates of the intersections of $\overline{\alpha_i}$, $\overline{\alpha_j}$ and $\overline{\gamma}$ with the x-axis as $s,\ t$ and $p,\ q$ with $q < p$. Then $t < s$ since $a_i < a_j$.

By the definition of inversion we have

$$s = \frac{k^2}{2a_i}\ ,\ t = \frac{k^2}{2a_j}\ ,\ p = \frac{k^2}{2a}\ ,\ q = -\frac{k^2}{2b}\ . \tag{1}$$

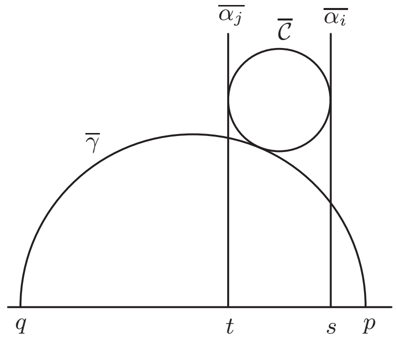

Figure 2

Since the x-coordinates of the center and the radius of $\overline{\mathcal{C}}$ are $\dfrac{s+t}{2}$ and $\dfrac{s-t}{2}$, and those of $\overline{\gamma}$ are $\dfrac{p+q}{2}$ and $\dfrac{p-q}{2}$, we have

$$\left(\frac{s+t}{2}-\frac{p+q}{2}\right)^2+d^2=\left(\frac{s-t}{2}+\frac{p-q}{2}\right)^2,$$

where d is the y-coordinate of the center of $\overline{\mathcal{C}}$. From this,

$$st-sp-tq+pq+d^2=0\,. \tag{2}$$

Since O is outside $\overline{\mathcal{C}}$, we have

$$r=\frac{k^2}{|\left(\frac{s+t}{2}\right)^2+d^2-\left(\frac{s-t}{2}\right)^2|}\cdot\frac{s-t}{2}=\frac{k^2}{\left(\frac{s+t}{2}\right)^2+d^2-\left(\frac{s-t}{2}\right)^2}\cdot\frac{s-t}{2}.$$

By using (1) and (2) we get the conclusion. □

Lemma 2. *If α_i (resp. α_j) is the line, then the radius of the inscribed circle is*

$$\frac{-ab}{a_j-a}(resp.\ \frac{ab}{a_i+b}).$$

Proof. Even in this case (2) in the proof of Lemma 1 holds with $s=0$ (resp. $t=0$), and we get the conclusion. □

Theorem 3. *Assume $a\neq b$, and let $\alpha_i,\ \alpha_j\in\Phi(\alpha,\beta,\gamma)$ with $\alpha_i<\alpha_j$. The radius of the circle inscribed in the curvilinear triangle bounded by $\alpha_i,\ \alpha_j$ and γ is*

$$\frac{ab(\mu(\alpha_i)-\mu(\alpha_j))}{b\mu(\alpha_i)-a\mu(\alpha_j)}.$$

Proof. If α_i and α_j are semicircles, then

$$\frac{ab(\mu(\alpha_i)-\mu(\alpha_j))}{b\mu(\alpha_i)-a\mu(\alpha_j)}=\frac{ab\left(\dfrac{a_i-a+b}{a_i}-\dfrac{a_j-a+b}{a_j}\right)}{b\cdot\dfrac{a_i-a+b}{a_i}-a\cdot\dfrac{a_j-a+b}{a_j}}=\frac{ab(a_j-a_i)}{a_ia_j-aa_i+ba_j}.$$

Hence the theorem follows from Lemma 1. If one of α_i, α_j is the line, the result follows from Lemma 2. □

Similarly we have

Theorem 4. *Assume $a = b$, and let $\alpha_i,\ \alpha_j \in \Phi(\alpha, \beta, \gamma)$ with $\alpha_i < \alpha_j$. The radius of the circle inscribed in the curvilinear triangle bounded by α_i, α_j and γ is*

$$\frac{a^2(\mu(\alpha_j) - \mu(\alpha_i))}{a(\mu(\alpha_j) - \mu(\alpha_i)) - 1}.$$

The functions $x \mapsto \dfrac{ab(1-x)}{b-ax}$, $a \neq b$ and $x \mapsto \dfrac{a^2x}{ax-1}, a > 0$ are injective. Therefore, we have

Corollary 5. *Let $\alpha_0,\ \alpha_1, \dots, \alpha_n \in \Phi(\alpha,\ \beta,\ \gamma)$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. The circles inscribed in the curvilinear triangle bounded by α_{i-1}, α_i and γ $(i = 1, 2, \dots n)$ are all congruent if and only if $\mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n)$ is a geometric sequence if $a \neq b$, or an arithmetic sequence if $a = b$.*

Theorem 6. *Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts. The common radius of the Archimedean circles in n-aliquot parts is*

$$\begin{cases} \dfrac{ab\left(b^{\frac{2}{n}} - a^{\frac{2}{n}}\right)}{b^{\frac{2}{n}+1} - a^{\frac{2}{n}+1}}, & \text{if } a \neq b, \\[2ex] \dfrac{2a}{n+2}, & \text{if } a = b. \end{cases}$$

Proof. First we consider the case $a \neq b$. We can assume $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ by renaming if necessary. The sequence $\dfrac{b}{a} = \mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n) = \dfrac{a}{b}$ is a geometric sequence by Corollary 5. If we write its common ratio as d, we have $\dfrac{a}{b} = d^n\left(\dfrac{b}{a}\right)$, and then $d = \left(\dfrac{a}{b}\right)^{\frac{2}{n}}$. By Theorem 3 the radius of the Archimedean circle is

$$\frac{ab(1-d)}{b-ad} = \frac{ab\left(1 - \left(\dfrac{a}{b}\right)^{\frac{2}{n}}\right)}{b - a\left(\dfrac{a}{b}\right)^{\frac{2}{n}}} = \frac{ab\left(b^{\frac{2}{n}} - a^{\frac{2}{n}}\right)}{b^{\frac{2}{n}+1} - a^{\frac{2}{n}+1}}.$$

Similarly we can get the second assertion. □

Note that the second assertion is the limiting case of the first assertion when $b \to a$.

Theorem 7. *Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Then α_i is the line in $\Phi(\alpha, \beta, \gamma)$ if n is even and $i = \frac{n}{2}$.*

Otherwise it is a semicircle with radius

$$\begin{cases} \left|\dfrac{b^{\frac{2i}{n}-1}(a-b)}{a^{\frac{2i}{n}-1}-b^{\frac{2i}{n}-1}}\right|, & \text{if } a \neq b, \\ \\ \left|\dfrac{na}{n-2i}\right|, & \text{if } a = b. \end{cases}$$

Proof. Suppose $a \neq b$. Since $\dfrac{b}{a} = \mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n) = \dfrac{a}{b}$ is a geometric sequence with common ratio $\left(\dfrac{a}{b}\right)^{\frac{2}{n}}$, we have $\mu(\alpha_i) = \left(\dfrac{a}{b}\right)^{\frac{2i}{n}}\left(\dfrac{b}{a}\right) = \left(\dfrac{a}{b}\right)^{\frac{2i}{n}-1}$. If n is even and $i = \dfrac{n}{2}$, then $\mu(\alpha_i) = 1$ and α_i is the line. Otherwise, $\mu(\alpha_i) \neq 1$ and α_i is a semicircle. Let a_i be the x-coordinate of its center. The radius of α_i is $|a_i|$ and $\dfrac{a_i - a + b}{a_i} = \left(\dfrac{a}{b}\right)^{\frac{2i}{n}-1}$. From this, $a_i = \dfrac{b^{\frac{2i}{n}-1}(a-b)}{b^{\frac{2i}{n}-1} - a^{\frac{2i}{n}-1}}$.

The proof for the case $a = b$ is similar. □

3. Embedded patterns of the arbelos

Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. There exists a semicircle γ' which is tangent to all Archimedean circles externally. It is clearly concentric to γ. (If $n = 1$ we will take for γ' the semicircle concentric to γ and tangent to the Archimedean circle externally). Let α', β' be two semicircles in $y \geq 0$, tangent to $\alpha_i s$ at O and also tangent to γ'. We take α' in the region $x \geq 0$ and β' in the region $x \leq 0$. Let a' and b' be the radii of α' and β' respectively. Clearly α', β' are tangent externally at O, and γ' intersects the x-axis at $-2b'$ and $2a'$, and $\Phi(\alpha, \beta, \gamma) \subseteq \Phi(\alpha', \beta', \gamma')$. Moreover, for any $\alpha_i \in \Phi(\alpha, \beta, \gamma)$, $\mu(\alpha_i)$ considered in $\Phi(\alpha, \beta, \gamma)$ is equal to $\mu(\alpha_i)$ considered in $\Phi(\alpha', \beta', \gamma')$ since the centers of γ and γ' coincide.

Lemma 8. (a) *If* $a \neq b$, $\left(\dfrac{a'}{b'}\right)^n = \left(\dfrac{a}{b}\right)^{n+2}$.

(b) *If* $a = b$, $\dfrac{a'}{n} = \dfrac{a}{n+2}$.

Proof. If $a \neq b$ we have

$$a' = a - \frac{ab\left(a^{\frac{2}{n}} - b^{\frac{2}{n}}\right)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}} = \frac{a^{\frac{2}{n}+1}(a-b)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}},$$

$$b' = b - \frac{ab\left(a^{\frac{2}{n}} - b^{\frac{2}{n}}\right)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}} = \frac{b^{\frac{2}{n}+1}(a-b)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}},$$

by the definitions of a' and b'. Then the the first assertion follows. The second assertion follows similarly. □

Theorem 9. $\{\alpha', \alpha_0, \alpha_1, \dots, \alpha_n, \beta', \gamma'\}$ *is an arbelos in* $(n+2)$*-aliquot parts.*

Proof. Let us assume $a \neq b$. By Lemma 8 and the proof of Theorem 6, $\mu(\alpha_0)$, $\mu(\alpha_1), \ldots, \mu(\alpha_n)$ is a geometric sequence with common ratio $\left(\dfrac{a'}{b'}\right)^{\frac{2}{n+2}}$. Also by Lemma 8 we have

$$\frac{\mu(\alpha_0)}{\mu(\alpha')} = \frac{b}{a}\frac{a'}{b'} = \left(\frac{b'}{a'}\right)^{\frac{n}{n+2}} \frac{a'}{b'} = \left(\frac{a'}{b'}\right)^{\frac{2}{n+2}},$$

and

$$\frac{\mu(\beta')}{\mu(\alpha_n)} = \frac{a'}{b'}\frac{b}{a} = \frac{a'}{b'}\left(\frac{b'}{a'}\right)^{\frac{n}{n+2}} = \left(\frac{a'}{b'}\right)^{\frac{2}{n+2}}.$$

The case $a = b$ follows similarly. □

Let $\{\alpha, \beta, \gamma\}$ be an arbelos and all the semicircles be constructed in $y \geq 0$ such that the diameters lie on the x-axis. Let $\alpha_{-1} = \alpha$, $\alpha_1 = \beta$ and $\gamma_1 = \gamma$. If there exists an arbelos in $(2n-1)$-aliquot parts $\{\alpha_{-n}, \alpha_{-(n-1)}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_n, \gamma_{2n-1}\}$ with $\alpha_{-n} < \alpha_{-(n-1)} < \cdots < \alpha_{-1} < \alpha_1 < \cdots < \alpha_n$, we shall construct an arbelos in $(2n+1)$-aliquot parts as follows.

Let γ_{2n+1} be the semicircle concentric to γ and tangent externally to all Archimedean circles of the above arbelos. This meets the x-axis at two points one of which is in the region $x > 0$ and the other in $x < 0$. We write the semicircle passing through O and the former point as $\alpha_{-(n+1)}$ and the semicircle passing through O and the latter point as α_{n+1}. Then $\{\alpha_{-(n+1)}, \alpha_{-n}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_{n+1}, \gamma_{2n+1}\}$ is an arbelos in $(2n+1)$-aliquot parts by Theorem 9. Now we get the set of semicircles

$$\{\ldots, \alpha_{-(n+1)}, \alpha_{-n}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots \gamma_1, \gamma_3, \ldots, \gamma_{2n-1} \ldots\},$$

where $\{\alpha_{-n}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_n, \gamma_{2n-1}\}$ form the arbelos in $(2n-1)$-aliquot parts for any positive integer n. We shall call the above configuration the *odd* pattern.

Theorem 10. *Let δ_{2n-1} be one of the Archimedean circles in*

$$\{\alpha_{-n}, \alpha_{-(n-1)}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_n, \gamma_{2n-1}\}.$$

Then the radii of α_{-n} and α_n are

$$\frac{a^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}} \quad \text{and} \quad \frac{b^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}},$$

and the radii of γ_{2n-1} and δ_{2n-1} are respectively

$$\frac{(a^{2n-1} + b^{2n-1})(a-b)}{a^{2n-1} - b^{2n-1}} \quad \text{and} \quad \frac{a^{2n-1}b^{2n-1}(a-b)(a^2-b^2)}{(a^{2n-1} - b^{2n-1})(a^{2n+1} - b^{2n+1})}.$$

Proof. Let $\overline{a_{-n}}$ and $\overline{a_n}$ be the radii of α_{-n} and α_n respectively. By Lemma 8 we have

$$\left(\frac{\overline{a_{-n}}}{\overline{a_n}}\right)^{\frac{1}{2n-1}} = \left(\frac{\overline{a_{-(n-1)}}}{\overline{a_{n-1}}}\right)^{\frac{1}{2n-3}} = \cdots = \frac{\overline{a_{-1}}}{\overline{a_1}} = \frac{a}{b}. \qquad (3)$$

Since γ_{2n-1} and γ are concentric, we have

$$\overline{a_{-n}} - \overline{a_n} = a - b\,. \tag{4}$$

By (3) and (4) we have

$$\overline{a_{-n}} = \frac{a^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}}\,,$$
$$\overline{a_n} = \frac{b^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}}.$$

It follows that the radius of γ_{2n-1} is

$$\overline{a_{-n}} + \overline{a_n} = \frac{(a^{2n-1} + b^{2n-1})(a-b)}{a^{2n-1} - b^{2n-1}},$$

and that of δ_{2n-1} is

$$\frac{(a^{2n-1} + b^{2n-1})(a-b)}{a^{2n-1} - b^{2n-1}} - \frac{(a^{2n+1} + b^{2n+1})(a-b)}{a^{2n+1} - b^{2n+1}}$$
$$= \frac{a^{2n-1}b^{2n-1}(a-b)(a^2-b^2)}{(a^{2n-1} - b^{2n-1})(a^{2n+1} - b^{2n+1})}\,.$$

□

As in the odd case, we can construct the *even* pattern of arbelos $\{\dots\beta_{-(n+1)},\ \beta_{-n},\ \dots,\ \beta_{-1},\ \beta_0,\ \beta_1,\ \dots,\ \beta_n,\ \beta_{n+1},\ \dots,\ \gamma_2,\ \gamma_4,\ \dots,\ \gamma_{2n}\dots\}$ inductively by starting with an arbelos in 2-aliquot parts $\{\beta_{-1},\ \beta_0,\ \beta_1, \gamma_2\}$, where $\beta_{-1} = \alpha$, $\beta_1 = \beta$ and $\gamma_2 = \gamma$. By Theorem 9, $\{\beta_{-n}, \dots, \beta_{-1},\ \beta_0,\ \beta_1, \dots, \beta_n, \gamma_{2n}\}$ forms an arbelos in $2n$-aliquot parts for any positive integer n, and β_0 is the line by Theorem 7. Analogous to Theorem 10 we have

Theorem 11. *Let δ_{2n} be one of the Archimedean circles in*

$$\{\beta_{-n},\ \beta_{-(n-1)}, \dots, \beta_{-1},\ \beta_0,\ \beta_1, \dots, \beta_n\ , \gamma_{2n}\}.$$

The radii of β_{-n} and β_n are

$$\frac{a^n(a-b)}{a^n - b^n} \quad \textit{and} \quad \frac{b^n(a-b)}{a^n - b^n},$$

and the radii of γ_{2n} and δ_{2n} are respectively

$$\frac{(a^n + b^n)(a-b)}{a^n - b^n} \quad \text{and} \quad \frac{a^n b^n (a-b)^2}{(a^n - b^n)(a^{n+1} - b^{n+1})}.$$

Corollary 12. *Let c_n and d_n be the radii of γ_n and δ_n respecgtively.*

$$a_n = b_{2n-1},$$
$$a_{-n} = b_{-(2n-1)},$$
$$c_{2n-1} = c_{2(2n-1)},$$
$$d_{2n-1} = d_{4n-2} + d_{4n}.$$

Figure 3 shows the even pattern together with the odd pattern reflected in the x-axis. The trivial case of these patterns can be found in [2].

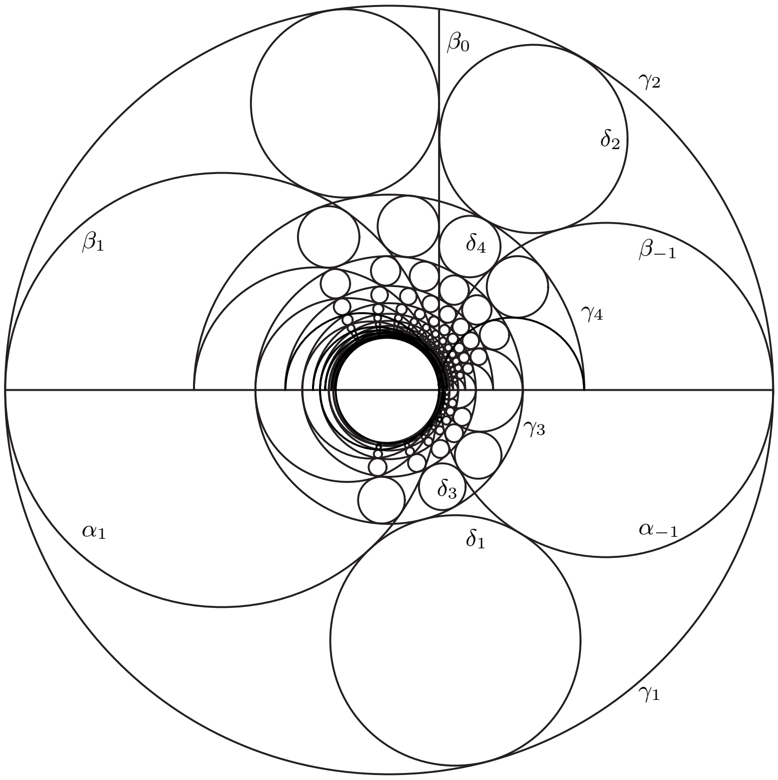

Figure 3

4. Some Applications

We give two applications here, with the same notations as in §3.

Theorem 13. *The external common tangent of β_n and β_{-n} touches γ_{4n} for any positive integer n.*

Proof. The distance between the external common tangents of β_n and β_{-n} and the center of γ_{2n} is $\dfrac{\overline{b_n}^2 + \overline{b_{-n}}^2}{\overline{b_n} + \overline{b_{-n}}}$ where $\overline{b_n}$ and $\overline{b_{-n}}$ are the radii of β_n and β_{-n}. By Theorem 11 this is equal to $\dfrac{(a-b)(a^{2n}+b^{2n})}{a^{2n}-b^{2n}}$, the radius of γ_{4n}. □

Theorem 14. *Let BK_n be the circle orthogonal to α, β and δ_{2n-1}, and let AR_n be the inscribed circle of the curvilinear triangle bounded by β_n, β_0 and γ_{2n}. The circles BK_n and AR_n are congruent for every natural number n.*

Proof. Assume $a \neq b$. Since AR_n is the Archimedean circle of the arbelos in 2-aliquot parts $\{\beta_{-n},\ \beta_0,\ \beta_n,\ \gamma_{2n}\}$, the radius of AR_n is

$$\frac{\overline{b_n}\,\overline{b_{-n}}(\overline{b_n} - \overline{b_{-n}})}{\overline{b_n}^2 - \overline{b_{-n}}^2} = \frac{a^n b^n (a-b)}{a^{2n}-b^{2n}},$$

by Theorem 6 and Theorem 11.

On the other hand BK_n is the inscribed circle of the triangle bounded by the three centers of α, β, δ_{2n-1}. Since the length of three sides of the triangle are $a + d_{2n-1}$, $b + d_{2n-1}$, $a + b$, the radius of BK_n is

$$\sqrt{\frac{abd_{2n-1}}{a+b+d_{2n-1}}} = \frac{a^n b^n (a-b)}{a^{2n} - b^{2n}},$$

by Theorem 10. □

This theorem is a generalization of Bankoff circle [1]. Bankoff's third circle corresponds to the case $n = 1$ in this theorem.

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[2] H. Okumura, Circles patterns arising from results in Japanese geometry, *Symmetry: Culture and Science*, 8 (1997) 4–23.

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