

On a Problem Regarding the n -Sectors of a Triangle

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Abstract. Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n - 1$ lines through A which, together with the lines AB and AC , divide the angle α in $n \geq 2$ equal parts are called the n -sectors of Δ . In this paper we determine all triangles with the property that all three edges and all $3(n - 1)$ n -sectors have rational lengths. We show that such triangles exist only if $n \in \{2, 3\}$.

1. Introduction

Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n - 1$ lines through A which, together with the lines AB and AC , divide the angle α in $n \geq 2$ equal parts are called the n -sectors of Δ . A triangle has $3(n - 1)$ n -sectors. The 2-sectors and 3-sectors are also called *bisectors* and *trisectors*. In this paper we study triangles with the property that all three edges and all $3(n - 1)$ n -sectors have rational lengths. We show that such triangles can exist only if $n = 2$ or 3 . We also determine all triangles with the property that all edges and bisectors (trisectors) have rational lengths. In each of the cases $n = 2$ and $n = 3$, there are infinitely many nonsimilar triangles having that property.

In number theory, there are some open problems of the same type as the above-mentioned problem.

(i) Does there exist a *perfect cuboid*, i.e. a cuboid in which all 12 edges, all 12 face diagonals and all 4 body diagonals are rational? ([3, Problem D18]).

(ii) Does there exist a triangle with integer edges, medians and area? ([3, Problem D21]).

2. Some properties

An elementary proof of the following lemma can also be found in [2, p. 443].

Lemma 1. *The number $\cos \frac{\pi}{n}$, $n \geq 2$, is rational if and only if $n = 2$ or $n = 3$.*

Proof. Suppose that $\cos \frac{\pi}{n}$ is rational. Put

$$\zeta_{2n} = \cos \frac{2\pi}{2n} + i \sin \frac{2\pi}{2n},$$

then ζ_{2n} is a zero of the polynomial $X^2 - (2 \cdot \cos \frac{\pi}{n}) \cdot X + 1 \in \mathbb{Q}[X]$. So, the minimal polynomial of ζ_{2n} over \mathbb{Q} is of the first or second degree. On the other hand, we know that the minimal polynomial of ζ_{2n} over \mathbb{Q} is the $2n$ -th cyclotomic polynomial $\Phi_{2n}(x)$, see [4, Theorem 4.17]. The degree of $\Phi_{2n}(x)$ is $\phi(2n)$, where ϕ is the *Euler phi function*. We have $\phi(2n) = 2n \cdot \frac{p_1-1}{p_1} \cdot \frac{p_2-1}{p_2} \cdot \dots \cdot \frac{p_k-1}{p_k}$, where p_1, \dots, p_k are the different prime numbers dividing $2n$. From $\phi(2n) \in \{1, 2\}$, it easily follows $n \in \{2, 3\}$. Obviously, $\cos \frac{\pi}{2}$ and $\cos \frac{\pi}{3}$ are rational. \square

Lemma 2. *For every $n \in \mathbb{N} \setminus \{0\}$, there exist polynomials $f_n(x), g_{n-1}(x) \in \mathbb{Q}[x]$ such that*

(i) $\deg(f_n) = n$, $f_n(x) = 2^{n-1}x^n + \dots$ and $\cos(nx) = f_n(\cos x)$ for every $x \in \mathbb{R}$;

(ii) $\deg(g_{n-1}) = n - 1$, $g_{n-1}(x) = 2^{n-1}x^{n-1} + \dots$ and $\frac{\sin(nx)}{\sin x} = g_{n-1}(\cos x)$ for every $x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

Proof. From $\cos x = \cos x$, $\frac{\sin x}{\sin x} = 1$,

$$\begin{aligned} \cos(k+1)x &= \cos(kx) \cos x - \frac{\sin(kx)}{\sin x} (1 - \cos^2 x), \\ \frac{\sin(k+1)x}{\sin x} &= \frac{\sin(kx)}{\sin x} \cos x + \cos(kx) \end{aligned}$$

for $k \geq 1$, it follows that we should make the following choices for the polynomials:

$$\begin{aligned} f_1(x) &:= x, g_0(x) := 1; \\ f_{k+1}(x) &:= f_k(x) \cdot x - g_{k-1}(x) \cdot (1 - x^2) \text{ for every } k \geq 1; \\ g_k(x) &:= g_{k-1}(x) \cdot x + f_k(x) \text{ for every } k \geq 1. \end{aligned}$$

One easily verifies by induction that f_n and g_{n-1} ($n \geq 1$) have the claimed properties. \square

Lemma 3. *Let $n \in \mathbb{N} \setminus \{0\}$, $q \in \mathbb{Q}^+ \setminus \{0\}$ and $x_1, \dots, x_n \in \mathbb{R}$. If*

$$\cos x_1, \sqrt{q} \cdot \sin x_1, \dots, \cos x_n, \sqrt{q} \cdot \sin x_n$$

are rational, then so are $\cos(x_1 + \dots + x_n)$ and $\sqrt{q} \cdot \sin(x_1 + \dots + x_n)$.

Proof. This follows by induction from the following equations ($k \geq 1$).

$$\begin{aligned} \cos(x_1 + \dots + x_{k+1}) &= \cos(x_1 + \dots + x_k) \cdot \cos(x_{k+1}) \\ &\quad - \frac{1}{q} (\sqrt{q} \cdot \sin(x_1 + \dots + x_k)) \cdot (\sqrt{q} \cdot \sin(x_{k+1})); \\ \sqrt{q} \cdot \sin(x_1 + \dots + x_{k+1}) &= (\sqrt{q} \cdot \sin(x_1 + \dots + x_k)) \cdot \cos(x_{k+1}) \\ &\quad + \cos(x_1 + \dots + x_k) \cdot (\sqrt{q} \cdot \sin(x_{k+1})). \end{aligned}$$

\square

Lemma 4. *Let Δ be a triangle with vertices A, B and C . Put $a = |BC|$, $b = |AC|$, $c = |AB|$, $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{BCA}$. Let $n \in \mathbb{N} \setminus \{0\}$ and suppose that $\cos(\frac{\alpha}{n})$, $\cos(\frac{\beta}{n})$ and $\cos(\frac{\gamma}{n})$ are rational. Then the following are equivalent:*

(i) $\frac{b}{a}$ and $\frac{c}{a}$ are rational numbers.

(ii) $\frac{\sin \frac{\beta}{n}}{\sin \frac{\alpha}{n}}$ and $\frac{\sin \frac{\gamma}{n}}{\sin \frac{\alpha}{n}}$ are rational numbers.

Proof. We have

$$\frac{b}{a} = \frac{\sin \beta}{\sin \alpha} = \frac{\sin \beta}{\sin \frac{\beta}{n}} \cdot \frac{\sin \frac{\alpha}{n}}{\sin \alpha} \cdot \frac{\sin \frac{\beta}{n}}{\sin \frac{\alpha}{n}}.$$

By Lemma 2, $\frac{\sin \beta}{\sin \frac{\beta}{n}} \cdot \frac{\sin \frac{\alpha}{n}}{\sin \alpha} \in \mathbb{Q}^+ \setminus \{0\}$. So, $\frac{b}{a}$ is rational if and only if $\frac{\sin \frac{\beta}{n}}{\sin \frac{\alpha}{n}}$ is rational. A similar remark holds for the fraction $\frac{c}{a}$. \square

3. Necessary and sufficient conditions

Theorem 5. *Let $n \geq 2$ and $0 < \alpha, \beta, \gamma < \pi$ with $\alpha + \beta + \gamma = \pi$. There exists a triangle with angles α , β and γ all whose edges and n -sectors have rational lengths if and only if the following conditions hold:*

- (1) $\cos \frac{\pi}{2n} \in \mathbb{Q}$,
- (2) $\cot \frac{\pi}{2n} \cdot \tan \frac{\alpha}{2n} \in \mathbb{Q}$,
- (3) $\cot \frac{\pi}{2n} \cdot \tan \frac{\beta}{2n} \in \mathbb{Q}$.

Proof. (a) Let Δ be a triangle with the property that all edges and all n -sectors have rational lengths. Let A , B and C be the vertices of Δ . Put $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{ACB}$. Let A_0, \dots, A_n be the vertices on the edge BC such that $A_0 = B$, $A_n = C$ and $\widehat{A_{i-1}AA_i} = \frac{\alpha}{n}$ for all $i \in \{1, \dots, n\}$. Put $a_i = |A_{i-1}A_i|$ for every $i \in \{1, \dots, n\}$. For every $i \in \{1, \dots, n-1\}$, the line AA_i is a bisector of the triangle with vertices A_{i-1} , A and A_{i+1} . Hence, $\frac{a_i}{a_{i+1}} = \frac{|AA_{i-1}|}{|AA_{i+1}|} \in \mathbb{Q}$. Together with $a_1 + \dots + a_n = |BC| \in \mathbb{Q}$, it follows that $a_i \in \mathbb{Q}$ for every $i \in \{1, \dots, n\}$. The cosine rule in the triangle with vertices A , A_0 and A_1 gives

$$\cos \frac{\alpha}{n} = \frac{|AA_0|^2 + |AA_1|^2 - a_1^2}{2 \cdot |AA_0| \cdot |AA_1|} \in \mathbb{Q}.$$

In a similar way one shows that $\cos \frac{\beta}{n}, \cos \frac{\gamma}{n} \in \mathbb{Q}$. Put $q := (1 - \cos^2 \frac{\alpha}{n})^{-1}$. By Lemma 4, $\sqrt{q} \cdot \sin \frac{\alpha}{n}$, $\sqrt{q} \cdot \sin \frac{\beta}{n}$ and $\sqrt{q} \cdot \sin \frac{\gamma}{n}$ are rational. From Lemma 3, it follows that $\cos \frac{\pi}{2n} \in \mathbb{Q}$ and $\sqrt{q} \cdot \sin \frac{\pi}{2n} \in \mathbb{Q}$. Hence,

$$\cot \frac{\pi}{2n} \cdot \tan \frac{\alpha}{2n} = \frac{1 + \cos \frac{\pi}{2n}}{\sqrt{q} \cdot \sin \frac{\pi}{2n}} \cdot \frac{\sqrt{q} \cdot \sin \frac{\alpha}{2n}}{1 + \cos \frac{\alpha}{2n}} \in \mathbb{Q}.$$

Similarly, $\cot \frac{\pi}{2n} \cdot \tan \frac{\beta}{2n} \in \mathbb{Q}$ and $\cot \frac{\pi}{2n} \cdot \tan \frac{\gamma}{2n} \in \mathbb{Q}$.

(b) Conversely, suppose that $\cos \frac{\pi}{2n} \in \mathbb{Q}$, $\cot \frac{\pi}{2n} \cdot \tan \frac{\alpha}{2n} \in \mathbb{Q}$ and $\cot \frac{\pi}{2n} \cdot \tan \frac{\beta}{2n} \in \mathbb{Q}$. Put $q := \sin^2 \frac{\pi}{2n} = 1 - \cos^2 \frac{\pi}{2n} \in \mathbb{Q}$. From $\sqrt{q} \cdot \cot \frac{\pi}{2n} = \sqrt{q} \cdot \frac{1 + \cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n}} \in \mathbb{Q}$, it follows that $\sqrt{q} \cdot \tan \frac{\alpha}{2n} \in \mathbb{Q}$, $\sqrt{q} \cdot \tan \frac{\beta}{2n} \in \mathbb{Q}$, $\cos \frac{\alpha}{n} = \frac{1 - \tan^2 \frac{\alpha}{2n}}{1 + \tan^2 \frac{\alpha}{2n}} \in \mathbb{Q}$, $\cos \frac{\beta}{n} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin \frac{\alpha}{n} = \frac{2\sqrt{q} \cdot \tan \frac{\alpha}{2n}}{1 + \tan^2 \frac{\alpha}{2n}} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin \frac{\beta}{n} \in \mathbb{Q}$. By Lemma 3, also $\cos \frac{\gamma}{n}$, $\sqrt{q} \cdot \sin \frac{\gamma}{n} \in \mathbb{Q}$. Now, choose a triangle Δ with angles α , β and γ such that the edge

opposite the angle α has rational length. By Lemma 4, it then follows that also the edges opposite to β and γ have rational lengths. Let A, B and C be the vertices of Δ such that $\widehat{BAC} = \alpha$, $\widehat{ABC} = \beta$ and $\widehat{ACB} = \gamma$. As before, let A_0, \dots, A_n be vertices on the edge BC such that the $n + 1$ lines AA_i , $i \in \{0, \dots, n\}$, divide the angle α in n equal parts. By the sine rule,

$$|AA_i| = \frac{|AB| \cdot \sin \beta}{\sin(\frac{i\alpha}{n} + \beta)}.$$

Now,

$$\frac{\sin(\frac{i\alpha}{n} + \beta)}{\sin \beta} = \frac{\sin \frac{i\alpha}{n}}{\sin \frac{\alpha}{n}} \cdot \frac{\sqrt{q} \cdot \sin \frac{\alpha}{n}}{\sqrt{q} \cdot \sin \frac{\beta}{n}} \cdot \frac{\sin \frac{\beta}{n}}{\sin \beta} \cdot \cos \beta + \cos \frac{i\alpha}{n}.$$

By Lemma 2, this number is rational. Hence $|AA_i| \in \mathbb{Q}$. By a similar reasoning it follows that the lengths of all other n -sectors are rational as well. \square

By Lemma 1 and Theorem 5 (1), we know that the problem can only have a solution in the case of bisectors or trisectors.

4. The case of bisectors

The bisector case has already been solved completely, see e.g. [1] or [5]. Here we present a complete solution based on Theorem 5. Without loss of generality, we may suppose that $\alpha \leq \beta \leq \gamma$. These conditions are equivalent with

$$0 < \alpha \leq \frac{\pi}{3}, \tag{1}$$

$$\alpha \leq \beta \leq \frac{\pi}{2} - \frac{\alpha}{2}. \tag{2}$$

By Theorem 5, $q_\alpha := \tan \frac{\alpha}{4}$ and $q_\beta := \tan \frac{\beta}{4}$ are rational. Equation (1) implies $0 < q_\alpha \leq \tan \frac{\pi}{12}$ and equation (2) implies $q_\alpha \leq q_\beta \leq x$, where $x := \tan(\frac{\pi}{8} - \frac{\alpha}{8})$.

Now, $\frac{2x}{1-x^2} = \tan(\frac{\pi}{4} - \frac{\alpha}{4}) = \frac{1-q_\alpha}{1+q_\alpha}$ and hence $x = \frac{\sqrt{2+2q_\alpha^2}-1-q_\alpha}{1-q_\alpha}$. Summarizing, we have the following restrictions for $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$0 < q_\alpha \leq \tan \frac{\pi}{12},$$

$$q_\alpha \leq q_\beta \leq \frac{\sqrt{2+2q_\alpha^2}-1-q_\alpha}{1-q_\alpha}.$$

In Figure 1 we depict the area G corresponding with these inequalities. Every point in G with rational coordinates in G will give rise to a triangle all whose edges and bisectors have rational lengths. Two different points in G with rational coefficients correspond with nonsimilar triangles.

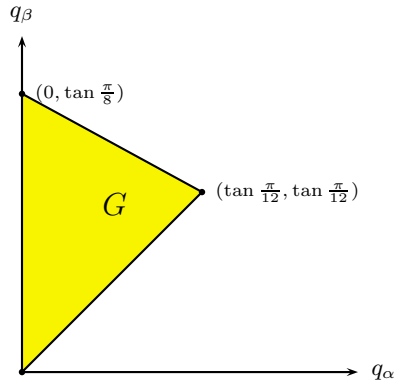


Figure 1

5. The case of trisectors

An infinite but incomplete class of solutions for the trisector case did also occur in the solution booklet of a mathematical competition in the Netherlands (universitaire wiskunde competitie, 1995). Here we present a complete solution based on Theorem 5. Again we may assume that $\alpha \leq \beta \leq \gamma$; so, equations (1) and (2) remain valid here. By Theorem 5, $q_\alpha := \sqrt{3} \cdot \tan \frac{\alpha}{6}$ and $q_\beta := \sqrt{3} \cdot \tan \frac{\beta}{6}$ are rational. As before, one can calculate the inequalities that need to be satisfied by $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$0 < q_\alpha \leq \sqrt{3} \cdot \tan \frac{\pi}{18},$$

$$q_\alpha \leq q_\beta \leq \frac{\sqrt{12 + 4q_\alpha^2} - 3 - q_\alpha}{1 - q_\alpha}.$$

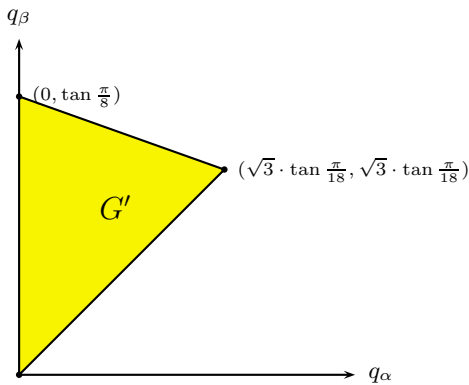


Figure 2

In Figure 2 we depict the area G' corresponding with these inequalities. Every point in G' with rational coordinates will give rise to a triangle all whose edges and

trisectors have rational lengths. Two different points in G' with rational coefficients correspond with nonsimilar triangles.

References

- [1] W. E. Buker and E. P. Starke. Problem E418, *Amer. Math. Monthly*, 47 (1940) 240; solution, 48 (1941) 67–68.
- [2] H. S. M. Coxeter. *Introduction to Geometry*. 2nd edition, John Wiley & Sons, New York, 1989.
- [3] R. K. Guy. *Unsolved problems in number theory*. Problem books in Mathematics. Springer Verlag, New York, 2004.
- [4] N. Jacobson. *Basic Algebra I*. Freeman, New York, 1985.
- [5] D. L. Mackay and E. P. Starke. Problem E331, *Amer. Math. Monthly*, 45 (1938) 249; solution, 46 (1939) 172.

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