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On a Problem Regarding the n-Sectors of a Triangle

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Abstract. Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n-1$ lines through A which, together with the lines AB and AC, divide the angle α in $n \geq 2$ equal parts are called the n-sectors of Δ. In this paper we determine all triangles with the property that all three edges and all $3(n-1)$ n-sectors have rational lengths. We show that such triangles exist only if $n \in \{2, 3\}$.

1. Introduction

Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n-1$ lines through A which, together with the lines AB and AC, divide the angle α in $n \geq 2$ equal parts are called the n*-sectors* of Δ. A triangle has $3(n-1)$ n-sectors. The 2-sectors and 3-sectors are also called *bisectors* and *trisectors*. In this paper we study triangles with the property that all three edges and all $3(n-1)$ n-sectors have rational lengths. We show that such triangles can exist only if $n = 2$ or 3. We also determine all triangles with the property that all edges and bisectors (trisectors) have rational lengths. In each of the cases $n = 2$ and $n = 3$, there are infinitely many nonsimilar triangles having that property.

In number theory, there are some open problems of the same type as the above-mentioned problem.

(i) Does there exists a *perfect cuboid*, i.e. a cuboid in which all 12 edges, all 12 face diagonals and all 4 body diagonals are rational? ([3, Problem D18]).

(ii) Does there exist a triangle with integer edges, medians and area? ([3, Problem D21]).

2. Some properties

An elementary proof of the following lemma can also be found in [2, p. 443].

Lemma 1. *The number* $\cos\frac{\pi}{n}$, $n \geq 2$, *is rational if and only if* $n = 2$ *or* $n = 3$.

Proof. Suppose that $\cos\frac{\pi}{n}$ is rational. Put

$$\zeta_{2n} = \cos\frac{2\pi}{2n} + i\sin\frac{2\pi}{2n},$$

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then ζ_{2n} is a zero of the polynomial $X^2 - (2 \cdot \cos\frac{\pi}{n}) \cdot X + 1 \in \mathbb{Q}[X]$. So, the minimal polynomial of ζ_{2n} over $\mathbb{Q}$ is of the first or second degree. On the other hand, we know that the minimal polynomial of ζ_{2n} over $\mathbb{Q}$ is the $2n$-th cyclotomic polynomial $\Phi_{2n}(x)$, see [4, Theorem 4.17]. The degree of $\Phi_{2n}(x)$ is $\phi(2n)$, where ϕ is the *Euler phi function*. We have $\phi(2n) = 2n \cdot \frac{p_1-1}{p_1} \cdot \frac{p_2-1}{p_2} \cdot \ldots \cdot \frac{p_k-1}{p_k}$, where $p_1, \ldots, p_k$ are the different prime numbers dividing $2n$. From $\phi(2n) \in \{1, 2\}$, it easily follows $n \in \{2, 3\}$. Obviously, $\cos\frac{\pi}{2}$ and $\cos\frac{\pi}{3}$ are rational. □

Lemma 2. *For every $n \in \mathbb{N} \setminus \{0\}$, there exist polynomials $f_n(x), g_{n-1}(x) \in \mathbb{Q}[x]$ such that*
(i) $\deg(f_n) = n$, $f_n(x) = 2^{n-1}x^n + \cdots$ *and* $\cos(nx) = f_n(\cos x)$ *for every* $x \in \mathbb{R}$*;*
(ii) $\deg(g_{n-1}) = n - 1$, $g_{n-1}(x) = 2^{n-1}x^{n-1} + \cdots$ *and* $\frac{\sin(nx)}{\sin x} = g_{n-1}(\cos x)$ *for every* $x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

Proof. From $\cos x = \cos x$, $\frac{\sin x}{\sin x} = 1$,

$$\cos(k+1)x = \cos(kx)\cos x - \frac{\sin(kx)}{\sin x}(1 - \cos^2 x),$$
$$\frac{\sin(k+1)x}{\sin x} = \frac{\sin(kx)}{\sin x}\cos x + \cos(kx)$$

for $k \geq 1$, it follows that we should make the following choices for the polynomials:

$f_1(x) := x$, $g_0(x) := 1$;
$f_{k+1}(x) := f_k(x) \cdot x - g_{k-1}(x) \cdot (1 - x^2)$ for every $k \geq 1$;
$g_k(x) := g_{k-1}(x) \cdot x + f_k(x)$ for every $k \geq 1$.

One easily verifies by induction that f_n and g_{n-1} ($n \geq 1$) have the claimed properties. □

Lemma 3. *Let $n \in \mathbb{N} \setminus \{0\}$, $q \in \mathbb{Q}^+ \setminus \{0\}$ and $x_1, \ldots, x_n \in \mathbb{R}$. If*

$$\cos x_1,\ \sqrt{q} \cdot \sin x_1,\ \ldots,\ \cos x_n,\ \sqrt{q} \cdot \sin x_n$$

are rational, then so are $\cos(x_1 + \cdots + x_n)$ *and* $\sqrt{q} \cdot \sin(x_1 + \cdots + x_n)$.

Proof. This follows by induction from the following equations ($k \geq 1$).

$$\begin{aligned}\cos(x_1 + \cdots + x_{k+1}) =& \cos(x_1 + \cdots + x_k) \cdot \cos(x_{k+1}) \\ & - \frac{1}{q}\left(\sqrt{q} \cdot \sin(x_1 + \cdots + x_k)\right) \cdot \left(\sqrt{q} \cdot \sin(x_{k+1})\right); \\ \sqrt{q} \cdot \sin(x_1 + \cdots + x_{k+1}) =& \left(\sqrt{q} \cdot \sin(x_1 + \cdots + x_k)\right) \cdot \cos(x_{k+1}) \\ & + \cos(x_1 + \cdots + x_k) \cdot \left(\sqrt{q} \cdot \sin(x_{k+1})\right).\end{aligned}$$

□

Lemma 4. *Let Δ be a triangle with vertices A, B and C. Put $a = |BC|$, $b = |AC|$, $c = |AB|$, $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{BCA}$. Let $n \in \mathbb{N} \setminus \{0\}$ and suppose that* $\cos(\frac{\alpha}{n})$, $\cos(\frac{\beta}{n})$ *and* $\cos(\frac{\gamma}{n})$ *are rational. Then the following are equivalent:*

(i) $\frac{b}{a}$ *and* $\frac{c}{a}$ *are rational numbers.*
(ii) $\frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}$ *and* $\frac{\sin\frac{\gamma}{n}}{\sin\frac{\alpha}{n}}$ *are rational numbers.*

Proof. We have

$$\frac{b}{a} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin\beta}{\sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\alpha}{n}}{\sin\alpha} \cdot \frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}.$$

By Lemma 2, $\frac{\sin\beta}{\sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\alpha}{n}}{\sin\alpha} \in \mathbb{Q}^+ \setminus \{0\}$. So, $\frac{b}{a}$ is rational if and only if $\frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}$ is rational. A similar remark holds for the fraction $\frac{c}{a}$. □

3. Necessary and sufficient conditions

Theorem 5. *Let $n \geq 2$ and $0 < \alpha, \beta, \gamma < \pi$ with $\alpha + \beta + \gamma = \pi$. There exists a triangle with angles α, β and γ all whose edges and n-sectors have rational lengths if and only if the following conditions hold:*

(1) $\cos\frac{\pi}{n} \in \mathbb{Q}$,
(2) $\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$,
(3) $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$.

Proof. (a) Let Δ be a triangle with the property that all edges and all n-sectors have rational lengths. Let A, B and C be the vertices of Δ. Put $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{ACB}$. Let $A_0, \ldots, A_n$ be the vertices on the edge BC such that $A_0 = B$, $A_n = C$ and $\widehat{A_{i-1}AA_i} = \frac{\alpha}{n}$ for all $i \in \{1, \ldots, n\}$. Put $a_i = |A_{i-1}A_i|$ for every $i \in \{1, \ldots, n\}$. For every $i \in \{1, \ldots, n-1\}$, the line AA_i is a bisector of the triangle with vertices A_{i-1}, A and A_{i+1}. Hence, $\frac{a_i}{a_{i+1}} = \frac{|AA_{i-1}|}{|AA_{i+1}|} \in \mathbb{Q}$. Together with $a_1 + \ldots + a_n = |BC| \in \mathbb{Q}$, it follows that $a_i \in \mathbb{Q}$ for every $i \in \{1, \ldots, n\}$. The cosine rule in the triangle with vertices A, A_0 and A_1 gives

$$\cos\frac{\alpha}{n} = \frac{|AA_0^2| + |AA_1|^2 - a_1^2}{2 \cdot |AA_0| \cdot |AA_1|} \in \mathbb{Q}.$$

In a similar way one shows that $\cos\frac{\beta}{n}, \cos\frac{\gamma}{n} \in \mathbb{Q}$. Put $q := (1 - \cos^2\frac{\alpha}{n})^{-1}$. By Lemma 4, $\sqrt{q} \cdot \sin\frac{\alpha}{n}$, $\sqrt{q} \cdot \sin\frac{\beta}{n}$ and $\sqrt{q} \cdot \sin\frac{\gamma}{n}$ are rational. From Lemma 3, it follows that $\cos\frac{\pi}{n} \in \mathbb{Q}$ and $\sqrt{q} \cdot \sin\frac{\pi}{n} \in \mathbb{Q}$. Hence,

$$\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} = \frac{1 + \cos\frac{\pi}{n}}{\sqrt{q} \cdot \sin\frac{\pi}{n}} \cdot \frac{\sqrt{q} \cdot \sin\frac{\alpha}{n}}{1 + \cos\frac{\alpha}{n}} \in \mathbb{Q}.$$

Similarly, $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$ and $\cot\frac{\pi}{2n} \cdot \tan\frac{\gamma}{2n} \in \mathbb{Q}$.

(b) Conversely, suppose that $\cos\frac{\pi}{n} \in \mathbb{Q}$, $\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$ and $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$. Put $q := \sin^2\frac{\pi}{n} = 1 - \cos^2\frac{\pi}{n} \in \mathbb{Q}$. From $\sqrt{q} \cdot \cot\frac{\pi}{2n} = \sqrt{q} \cdot \frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} \in \mathbb{Q}$, it follows that $\sqrt{q} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$, $\sqrt{q} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$, $\cos\frac{\alpha}{n} = \frac{1-\tan^2\frac{\alpha}{2n}}{1+\tan^2\frac{\alpha}{2n}} \in \mathbb{Q}$, $\cos\frac{\beta}{n} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin\frac{\alpha}{n} = \frac{2\sqrt{q}\cdot\tan\frac{\alpha}{2n}}{1+\tan^2\frac{\alpha}{2n}} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin\frac{\beta}{n} \in \mathbb{Q}$. By Lemma 3, also $\cos\frac{\gamma}{n}$, $\sqrt{q} \cdot \sin\frac{\gamma}{n} \in \mathbb{Q}$. Now, choose a triangle Δ with angles α, β and γ such that the edge opposite the angle α has rational length. By Lemma 4, it then follows that also the edges opposite to β and γ have rational lengths. Let A, B and C be the vertices of Δ such that $\widehat{BAC} = \alpha$, $\widehat{ABC} = \beta$ and $\widehat{ACB} = \gamma$. As before, let $A_0, \dots, A_n$ be vertices on the edge BC such that the $n+1$ lines AA_i, $i \in \{0, \dots, n\}$, divide the angle α in n equal parts. By the sine rule,

$$|AA_i| = \frac{|AB| \cdot \sin\beta}{\sin(\frac{i\alpha}{n} + \beta)}.$$

Now,

$$\frac{\sin(\frac{i\alpha}{n} + \beta)}{\sin\beta} = \frac{\sin\frac{i\alpha}{n}}{\sin\frac{\alpha}{n}} \cdot \frac{\sqrt{q}\cdot\sin\frac{\alpha}{n}}{\sqrt{q}\cdot\sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\beta}{n}}{\sin\beta} \cdot \cos\beta + \cos\frac{i\alpha}{n}.$$

By Lemma 2, this number is rational. Hence $|AA_i| \in \mathbb{Q}$. By a similar reasoning it follows that the lengths of all other n-sectors are rational as well. □

By Lemma 1 and Theorem 5 (1), we know that the problem can only have a solution in the case of bisectors or trisectors.

4. The case of bisectors

The bisector case has already been solved completely, see e.g. [1] or [5]. Here we present a complete solution based on Theorem 5. Without loss of generality, we may suppose that $\alpha \le \beta \le \gamma$. These conditions are equivalent with

$$0 < \alpha \le \frac{\pi}{3}, \tag{1}$$

$$\alpha \le \beta \le \frac{\pi}{2} - \frac{\alpha}{2}. \tag{2}$$

By Theorem 5, $q_\alpha := \tan\frac{\alpha}{4}$ and $q_\beta := \tan\frac{\beta}{4}$ are rational. Equation (1) implies $0 < q_\alpha \le \tan\frac{\pi}{12}$ and equation (2) implies $q_\alpha \le q_\beta \le x$, where $x := \tan(\frac{\pi}{8} - \frac{\alpha}{8})$. Now, $\frac{2x}{1-x^2} = \tan(\frac{\pi}{4} - \frac{\alpha}{4}) = \frac{1-q_\alpha}{1+q_\alpha}$ and hence $x = \frac{\sqrt{2+2q_\alpha^2}-1-q_\alpha}{1-q_\alpha}$. Summarizing, we have the following restrictions for $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$0 < q_\alpha \le \tan\frac{\pi}{12},$$

$$q_\alpha \le q_\beta \le \frac{\sqrt{2+2q_\alpha^2} - 1 - q_\alpha}{1 - q_\alpha}.$$

In Figure 1 we depict the area G corresponding with these inequalities. Every point in G with rational coordinates in G will give rise to a triangle all whose edges and bisectors have rational lengths. Two different points in G with rational coefficients correspond with nonsimilar triangles.

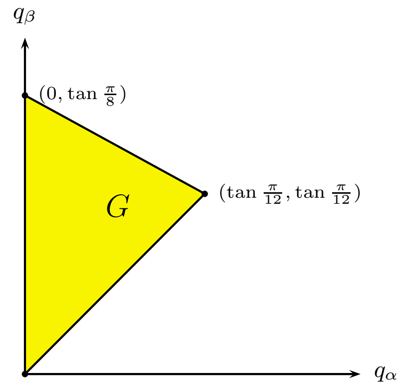

Figure 1

5. The case of trisectors

An infinite but incomplete class of solutions for the trisector case did also occur in the solution booklet of a mathematical competition in the Netherlands (universitaire wiskunde competitie, 1995). Here we present a complete solution based on Theorem 5. Again we may assume that $\alpha \leq \beta \leq \gamma$; so, equations (1) and (2) remain valid here. By Theorem 5, $q_\alpha := \sqrt{3} \cdot \tan\frac{\alpha}{6}$ and $q_\beta := \sqrt{3} \cdot \tan\frac{\beta}{6}$ are rational. As before, one can calculate the inequalities that need to be satisfied by $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$0 < q_\alpha \leq \sqrt{3} \cdot \tan\frac{\pi}{18},$$
$$q_\alpha \leq q_\beta \leq \frac{\sqrt{12 + 4q_\alpha^2} - 3 - q_\alpha}{1 - q_\alpha}.$$

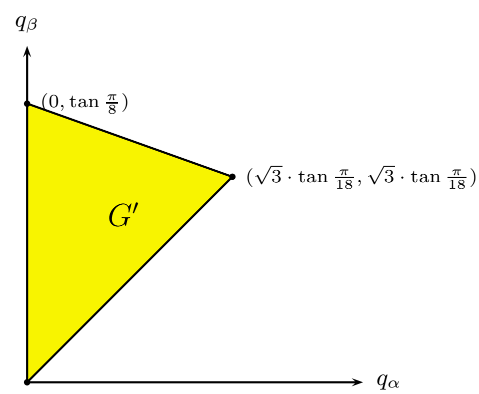

Figure 2

In Figure 2 we depict the area G' corresponding with these inequalities. Every point in G' with rational coordinates will give rise to a triangle all whose edges and trisectors have rational lengths. Two different points in G with rational coefficients correspond with nonsimilar triangles.

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