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A Simple Construction of a Triangle from its Centroid, Incenter, and a Vertex

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Abstract. We give a simple ruler and compass construction of a triangle given its centroid, incenter, and one vertex. An analysis of the number of solutions is also given.

1. Construction

The ruler and compass construction of a triangle from its centroid, incenter, and one vertex was one of the unresolved cases in [3]. An analysis of this problem, including the number of solutions, was given in [1]. In this note we give a very simple construction of triangle ABC with given centroid G, incenter I, and vertex A. The construction depends on the following propositions. For another slightly different construction, see [2].

Proposition 1. *Given triangle ABC with Nagel point N, let D be the midpoint of BC. The lines ID and AN are parallel.*

Proof. The centroid G divides each of the segments AD and NI in the ratio $AG : GD = NG : GI = 2 : 1$. See Figure 1. □

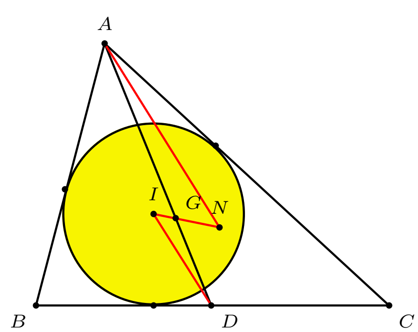

Figure 1

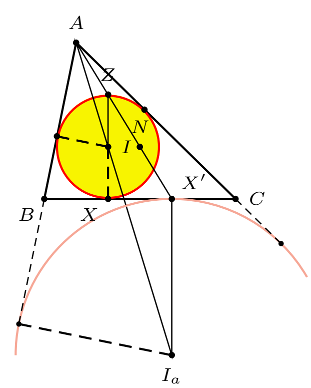

Figure 2

Proposition 2. *Let X be the point of tangency of the incircle with BC. The antipode of X on the circle with diameter ID is a point on AN.*

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Proof. This follows from the fact that the antipode of X on the incircle lies on the segment AN. See Figure 2. □

Construction. Given G, I, and A, extend AG to D such that $AG : GD = 2 : 1$. Construct the circle $\mathcal{C}$ with diameter ID, and the line $\mathcal{L}$ through A parallel parallel to ID.

Let Y be an intersection of the circle $\mathcal{C}$ and the line $\mathcal{L}$, and X the antipode of Y on $\mathcal{C}$ such that A is outside the circle $I(X)$. Construct the tangents from A to the circle $I(X)$. Their intersections with the line DX at the remaining vertices B and C of the required triangle. See Figure 3.

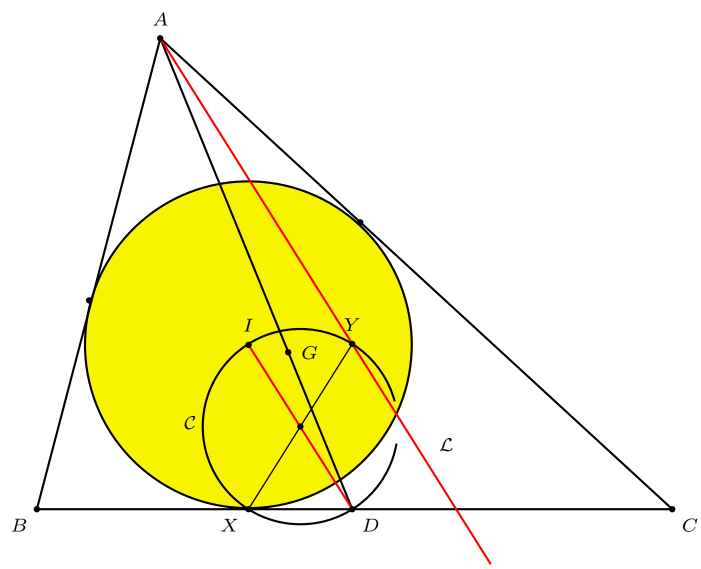

Figure 3

2. Number of solutions

We set up a Cartesian coordinate system such that $A = (0, 2k)$ and $I = (0, -k)$. If $G = (u, v)$, then $D = \frac{1}{2}(3G - A) = \left(\frac{3}{2}u,\ \frac{3}{2}v - k\right)$. The circle $\mathcal{C}$ with diameter ID has equation

$$2(x^2 + y^2) - 3ux - (3v - 4k)y + (2k^2 - 3kv) = 0$$

and the line $\mathcal{L}$ through A parallel to ID has slope $\frac{v}{u}$ and equation

$$vx - uy + 2ku = 0.$$

The line $\mathcal{L}$ and the circle $\mathcal{C}$ intersect at 0, 1, 2 real points according as

$$\Delta := (u^2 + v^2 - 4ku)(u^2 + v^2 + 4ku)$$

is negative, zero, or positive. Since $x^2 + y^2 \pm 4kx = 0$ represent the two circles of radii $2k$ tangent to each other externally and to the y-axis at $(0, 0)$, Δ is negative, zero, or positive according as G lies in the interior, on the boundary, or in the exterior of the union of the two circles.

The intersections of the circle and the line are the points

$$Y_\varepsilon = \left(\frac{3u(u^2+v^2-4kv-\varepsilon\sqrt{\Delta})}{4(u^2+v^2)},\ \frac{8k(u^2+v^2)+3v(u^2+v^2-4kv-\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}\right)$$

for $\varepsilon = \pm 1$. Their antipodes on $\mathcal{C}$ are the points

$$X_\varepsilon = \left(\frac{3u(u^2+v^2+4kv+\varepsilon\sqrt{\Delta})}{4(u^2+v^2)},\ \frac{-16k(u^2+v^2)+3v(u^2+v^2+4kv+\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}\right).$$

There is a triangle ABC tritangent to the circle $I(X_\varepsilon)$ and with DX_ε as a sideline if and only if the point A lies outside the circle $I(X_\varepsilon)$. Note that $IA = 3k$ and

$$IX_+^2 = \frac{9}{8}(u^2+v^2+\sqrt{\Delta}), \quad IX_-^2 = \frac{9}{8}(u^2+v^2-\sqrt{\Delta}).$$

From these, we make the following conclusions.

(i) If $u^2+v^2-8k^2 \geq \sqrt{\Delta}$, then A lies inside or on $I(X_-)$. In this case, there is no triangle.

(ii) If $-\sqrt{\Delta} \leq u^2+v^2-8k^2 < \sqrt{\Delta}$, then A lies outside $I(X_-)$ but not $I(X_+)$. There is exactly one triangle.

(iii) If $u^2+v^2-8k^2 < -\sqrt{\Delta}$, then A lies outside $I(X_+)$ (and also $I(X_-)$). There are in general two triangles.

It is easy to see that the condition $-\sqrt{\Delta} < u^2+v^2-8k^2 < \sqrt{\Delta}$ is equivalent to $(v-2k)(v+2k) > 0$, *i.e.*, $|v| > 2k$. We also note the following.

(i) When the line D_ε passes through A, the corresponding triangle degenerates. The condition for collinearity leads to

$$u(3u^2+3v^2-4kv \pm \sqrt{\Delta}) = 0.$$

Clearly, $u = 0$ gives the y-axis. The corresponding triangle is isosceles. On the other hand, the condition $3u^2+3v^2-4kv \pm \sqrt{\Delta} = 0$ leads to

$$(u^2+v^2)(u^2+v^2-3kv+2k^2) = 0,$$

i.e., (u,v) lying on the circle tangent to the circles $x^2+y^2 \pm 4kx = 0$ at $\left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$ and the line $y = 2k$ at A.

(ii) If $v > 0$, the circle $I(X_\varepsilon)$, instead of being the incircle, is an excircle of the triangle. If G lies inside the region $ATOT'A$ bounded by the circular segments, one of the excircles is the A-excircle. Outside this region, the excircle is always a B/C-excircle.

From these we obtain the distribution of the position of G, summarized in Table 1 and depicted in Figure 4, for the various numbers of solutions of the construction problem. In Figure 4, the number of triangles is

0 if G in an unshaded region, on a dotted line, or at a solid point other than I,

1 if G is in a yellow region or on a solid red line,

2 if G is in a green region.

Table 1. Number N of non-degenerate triangles according to the location of G relative to A and I

N	Location of centroid $G(u, v)$
0	$(0,0),\ (\pm 2k, 2k)$; $\left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$; $v = 2k$; $\lvert u\rvert > 2k - \sqrt{4k^2 - v^2},\ -2k \le v < 2k.$
1	$u = 0,\ 0 < \lvert v\rvert < 2k$; $-2k < u < 2k,\ v = -2k$; $u = 2k - \sqrt{4k^2 - v^2},\ 0 < \lvert v\rvert < 2k$; $\lvert v\rvert > 2k$; $u^2 + v^2 - 3kv + 2k^2 = 0$ except $(0, 2k),\ \left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$.
2	$\lvert u\rvert < 2k - \sqrt{4k^2 - v^2},\ 0 < \lvert v\rvert < 2k$, but $u^2 + v^2 - 3kv + 2k^2 \neq 0$.

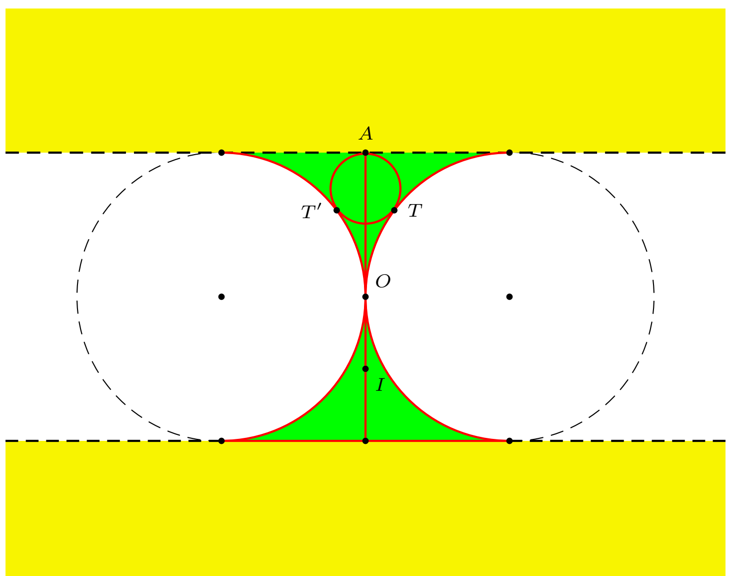

Figure 4

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