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Some Brocard-like points of a triangle

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Abstract. In this note, we prove that for every triangle ABC, there exists a unique interior point M the cevians AA', BB', and CC' through which have the property that $\angle AC'B' = \angle BA'C' = \angle CB'A'$, and a unique interior point M' the cevians AA', BB', and CC' through which have the property that $\angle AB'C' = \angle BC'A' = \angle CA'B'$. We study some properties of these Brocard-like points, and characterize those centers for which the angles $AC'B'$, $BA'C'$, and $CB'A'$ are linear forms in the angles A, B, and C of ABC.

1. Notations

Let ABC be a non-degenerate triangle, with angles A, B, and C. To every point P inside ABC, we associate, as shown in Figure 1, the following angles and lengths.

$$
\begin{array}{lll}
\xi = \angle BAA', & \eta = \angle CBB', & \zeta = \angle ACC'; \\
\xi' = \angle CAA', & \eta' = \angle ABB', & \zeta' = \angle BCC'; \\
\alpha = \angle AC'B', & \beta = \angle BA'C', & \gamma = \angle CB'A'; \\
\alpha' = \angle AB'C', & \beta' = \angle BC'A', & \gamma' = \angle CA'B'; \\
x = BA', & y = CB', & z = AC'; \\
x' = A'C, & y' = B'A, & z' = C'B.
\end{array}
$$

The well-known Brocard or Crelle-Brocard points are defined by the requirements $\xi = \eta = \zeta$ and $\xi' = \eta' = \zeta'$; see [11]. The angles ω and ω' that satisfy $\xi = \eta = \zeta = \omega$ and $\xi' = \eta' = \zeta' = \omega'$ are equal, and their common value is called the Brocard angle. The points known as Yff's analogues of the Brocard points are defined by the similar requirements $x = y = z$ and $x' = y' = z'$. These were introduced by Peter Yff in [12], and were so named by Clark Kimberling in a talk that later appeared as [8]. For simplicity, we shall refer to these points as *the Yff-Brocard points*.

2. The cevian Brocard points

In this note, we show that each of the requirements $\alpha = \beta = \gamma$ and $\alpha' = \beta' = \gamma'$ defines a unique interior point, and that the angles Ω and Ω' that satisfy $\alpha = \beta = \gamma = \Omega$ and $\alpha' = \beta' = \gamma' = \Omega'$ are equal. We shall call the resulting two points the first and second cevian Brocard points respectively, and the common value of Ω and Ω', the cevian Brocard angle of ABC.

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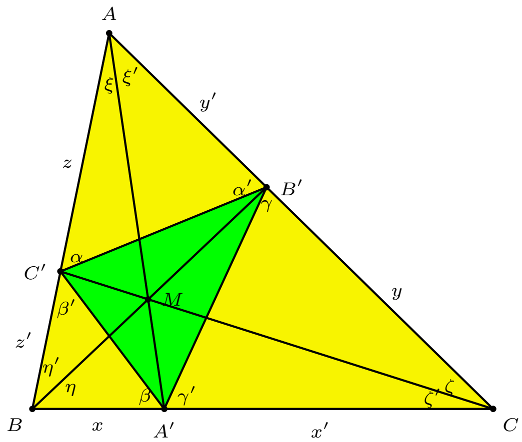

Figure 1.

We shall freely use the trigonometric forms

$$\sin\xi\sin\eta\sin\zeta = \sin\xi'\sin\eta'\sin\zeta' = \sin(A-\xi)\sin(B-\eta)\sin(C-\zeta)$$
$$\sin\alpha\sin\beta\sin\gamma = \sin\alpha'\ \sin\beta'\sin\gamma' = \sin(A+\alpha)\sin(B+\beta)\ \sin(C+\gamma)$$

of the cevian concurrence condition. We shall also freely use a theorem of Seebach stating that for any triangles ABC and UVW, there exists inside ABC a unique point P the cevians AA', BB', and CC' through which have the property that $(A', B', C') = (U, V, W)$, where A', B', and C' are the angles of $A'B'C'$ and U, V, and W are the angles of UVW; see [10] and [7].

Theorem 1. *For every triangle ABC, there exists a unique interior point M the cevians AA', BB', and CC' through which have the property that*

$$\angle AC'B' = \angle BA'C' = \angle CB'A' \ (= \Omega,\ \textit{say}), \tag{1}$$

and a unique interior point M' the cevians AA', BB', and CC' through which have the property that

$$\angle AB'C' = \angle BC'A' = \angle CA'B' \ (= \Omega',\ \textit{say}). \tag{2}$$

Also, the angles Ω and Ω' are equal and acute. See Figures 2A and 2B.

Proof. It is obvious that (1) is equivalent to the condition $(A', B', C') = (C, A, B)$, where A', B', and C' are the angles of the cevian triangle $A'B'C'$. Similarly, (2) is equivalent to the condition $(A', B', C') = (B, C, A)$. According to Seebach's theorem, the existence and uniqueness of M and M' follow by taking $(U, V, W) = (C, A, B)$ and $(U, V, W) = (B, C, A)$.

To prove that Ω is acute, observe that if Ω is obtuse, then the angles Ω, $A+\Omega$, $B+\Omega$, and $C+\Omega$ would all lie in the interval $[\pi/2, \pi]$ where the sine function is positive and decreasing. This would imply that

$$\sin^3\Omega \quad > \quad \sin(A+\Omega)\ \sin(B+\Omega)\ \sin(C+\Omega),$$

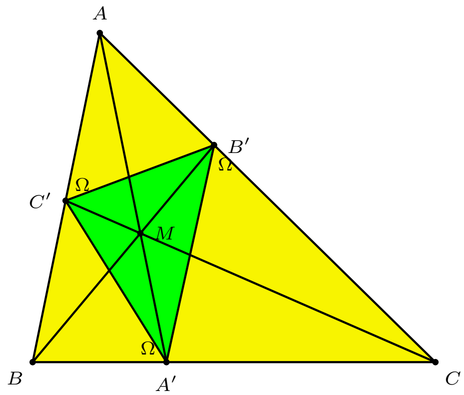

Figure 2A

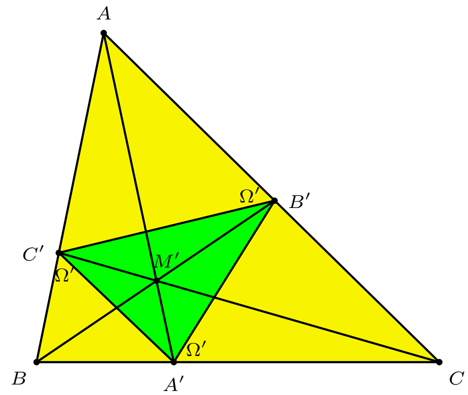

Figure 2B

contradicting the cevian concurrence condition

$$\sin^3 \Omega \quad = \quad \sin(A + \Omega)\ \sin(B + \Omega)\ \sin(C + \Omega). \tag{3}$$

Thus Ω, and similarly Ω', are acute.

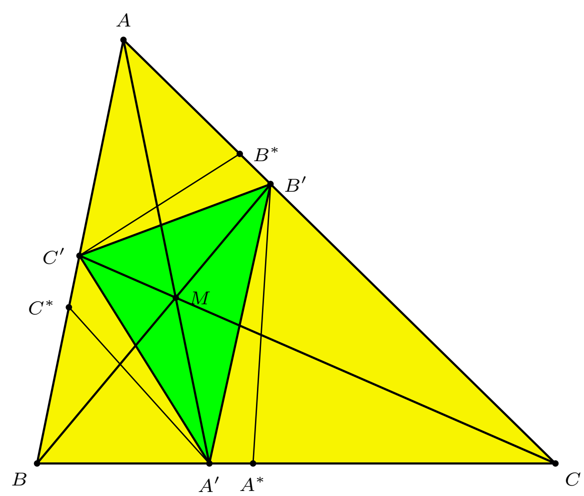

Figure 3.

It remains to prove that $\Omega' = \Omega$. Let $A'B'C'$ be the cevian triangle of M, and suppose that $\Omega' < \Omega$. Then there exist, as shown in Figure 3, points B^*, C^*, and A^* on the line segments $A'C$, $B'A$, and $C'B$, respectively, such that

$$\angle AC'B^* = \angle BA'C^* = \angle CB'A^* = \Omega'.$$

Then

$$\begin{aligned} 1 &= \frac{AB'}{B'C} \cdot \frac{CA'}{A'B} \cdot \frac{BC'}{C'A} > \frac{AB^*}{B'C} \cdot \frac{CA^*}{A'B} \cdot \frac{BC^*}{C'A} = \frac{AB^*}{AC'} \cdot \frac{CA^*}{CB'} \cdot \frac{BC^*}{BA'} \\ &= \frac{\sin \Omega'}{\sin(A+\Omega')} \cdot \frac{\sin \Omega'}{\sin(C+\Omega')} \cdot \frac{\sin \Omega'}{\sin(B+\Omega')}. \end{aligned}$$

This contradicts the cevian concurrence condition

$$\sin^3 \Omega' \quad = \quad \sin(A+\Omega')\sin(B+\Omega')\sin(C+\Omega')$$

for M'. □

The points M and M' in Theorem 1 will be called the *first* and *second cevian Brocard points* and the common value of Ω and Ω' the *cevian Brocard angle*.

3. An alternative proof of Theorem 1

An alternative proof of Theorem 1 can be obtained by noting that the existence and uniqueness of M are equivalent to the existence and uniqueness of a positive solution $\Omega < \min\{\pi - A, \pi - B, \pi - C\}$ of (3). Letting $u = \sin \Omega$, $U = \cos \Omega$, and $T = U/u = \cot \Omega$, and setting

$$\begin{aligned} c_0 &= \sin A \sin B \sin C, \\ c_1 &= \cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C, \\ c_2 &= \cos A \cos B \sin C + \cos A \sin B \cos C + \sin A \cos B \cos C, \\ c_3 &= \cos A \cos B \cos C, \end{aligned}$$

(3) simplifies into

$$u^3 = c_0 U^3 + c_1 U^2 u + c_2 U u^2 + c_3 u^3. \tag{4}$$

Using the formulas

$$c_2 = c_0 \quad \text{and} \quad c_1 = c_3 + 1 \tag{5}$$

taken from [5, Formulas 674 and 675, page 165], this further simplifies into

$$\begin{aligned} u^3 &= c_0 U^3 + (c_3+1) U^2 u + c_0 U u^2 + c_3 u^3 \\ &= c_0 U (U^2 + u^2) + c_3 u (U^2 + u^2) + U^2 u \\ &= c_0 U + c_3 u + U^2 u \\ &= u(c_0 T + c_3 + U^2). \end{aligned}$$

Since $u^2 = \frac{1}{1+T^2}$ and $U^2 = \frac{T^2}{1+T^2}$, this in turn reduces to $f(T) = 0$, where

$$f(X) = c_0 X^3 + (c_3+1) X^2 + c_0 X + (c_3 - 1). \tag{6}$$

Arguing as in the proof of Theorem 1 that Ω must be acute, we restrict our search to the interval $\Omega \in [0, \pi/2]$, i.e., to $T \in [0, \infty)$. On this interval, f is clearly increasing. Also, $f(0) < 0$ and $f(\infty) > 0$. Therefore f has a unique zero in $[0, \infty)$. This proves the existence and uniqueness of M. A similar treatment of M' leads to the same f, proving that M' exists and is unique, and that $\Omega = \Omega'$.

This alternative proof of Theorem 1 has the advantage of exhibiting the defining polynomial of $\cot\Omega$, which is needed in proving Theorems 2 and 3.

4. The cevian Brocard angle

Theorem 2. *Let Ω be the cevian Brocard angle of triangle ABC.*
(i) $\cot\Omega$ *satisfies the polynomial f given in* (6)*, where $c_0 = \sin A\sin B\sin C$ and $c_3 = \cos A\cos B\cos C$.*
(ii) $\Omega \le \pi/3$ *for all triangles.*
(iii) Ω *takes all values in* $(0, \pi/3]$.

Proof. (i) follows from the alternative proof of Theorem 1 given in the preceding section.

To prove (ii), it suffices to prove that $f(1/\sqrt{3}) \le 0$ for all triangles ABC. Let

$$G = f\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}\sin A\sin B\sin C + \frac{4}{3}\cos A\cos B\cos C - \frac{2}{3}.$$

Then $G = 0$ if ABC is equilateral, and hence it is enough to prove that G attains its maximum at such a triangle. To see this, take a non-equilateral triangle ABC. Then we may assume that $A > B$ and $C < \pi/2$. If we replace ABC by the triangle whose angles are $(A+B)/2$, $(A+B)/2$, and C, then G increases. This follows from

$$2\sin A\sin B = \cos(A-B) - \cos(A+B) < 1 - \cos(A+B) = 2\sin^2\frac{A+B}{2},$$
$$2\cos A\cos B = \cos(A-B) + \cos(A+B) < 1 + \cos(A+B) = 2\cos^2\frac{A+B}{2}.$$

Thus G attains its maximal value, 0, at equilateral triangles, and hence $G \le 0$ for all triangles, as desired.

To prove (iii), we let $S = \tan\Omega = 1/T$ and we see that S is a zero of the polynomial $F(X) = c_0 + (c_3+1)X + c_0X^2 + (c_3-1)X^3$. The non-negative zero of F when ABC is degenerate, i.e., when $c_0 = 0$, is 0. By continuity of the zeros of polynomials, we conclude that $\tan\Omega$ can be made arbitrarily close to 0 by taking a triangle whose c_0 is close enough to 0. Note that $c_3 - 1$ is bounded away from zero since $c_3 \le 3\sqrt{3}/8$ for all triangles. □

Remarks. (1) Unlike the Brocard angle ω, the cevian Brocard angle Ω is not necessarily Euclidean constructible. To see this, take the triangle ABC with $A = \pi/2$, and $B = C = \pi/4$. Then $c_3 = 0$, $c_0 = 1/2$, and $2f(T) = T^3 + 2T^2 + T - 2$. This is irreducible over $\mathbb{Z}$ since none of ±1 and ±2 is a zero of f, and therefore it is the minimal polynomial of $\cot\ \Omega$. Since it is of degree 3, it follows that $\cot\ \Omega$, and hence the angle Ω, is not constructible.

(2) By the cevian concurrence condition, the Brocard angle ω is defined by

$$\sin^3\omega \quad = \quad \sin(A-\omega)\ \sin(B-\omega)\ \sin(C-\omega). \tag{7}$$

Letting $v = \sin\omega$, $V = \cos\omega$ and $t = \cot\omega$ as before, we obtain

$$v^3 \quad = \quad c_0V^3 - c_1V^2v + c_2Vv^2 - c_3v^3. \tag{8}$$

This reduces to the very simple form $g(t) = 0$, where

$$g(X) \quad = \quad c_0 X - c_3 - 1, \tag{9}$$

showing that

$$t = \cot\omega = \frac{1 + c_3}{c_0} = \frac{c_1}{c_0} = \cot A + \cot B + \cot C, \tag{10}$$

as is well known, and exhibiting the trivial constructibility of ω. This heavy contrast with the non-constructiblity of Ω is rather curious in view of the great formal similarity between (3) and (4) on the one hand and (7) and (8) on the other.

The next theorem shows that a triangle is completely determined, up to similarity, by its Brocard and cevian Brocard angles. This implies, in particular, that Ω and ω are independent of each other, since neither of them is sufficient for determining the shape of the triangle.

Theorem 3. *If two triangles have equal Brocard angles and equal cevian Brocard angles, then they are similar.*

Proof. Let ω and Ω be the Brocard and cevian Brocard angles of triangle ABC, and let $t = \cot\omega$ and $T = \cot\Omega$. From (10) it follows that $t = c_1/c_0$ and therefore $c_1 = tc_0$. Substituting this in (6), we see that $c_0(T + t)(T^2 + 1) = 2$, and therefore

$$c_0 = \frac{2}{(T + t)(T^2 + 1)}, \quad \text{and} \quad c_1 = \frac{2t}{(T + t)(T^2 + 1)}.$$

Letting s_1, s_2, and s_3 be the elementary symmetric polynomials in $\cot A$, $\cot B$, and $\cot C$, we see that

$$\begin{aligned} s_1 &= \cot A + \cot B + \cot C = t, \\ s_2 &= \cot A \cot B + \cot B \cot C + \cot C \cot A = \frac{c_2}{c_0} = 1, \\ s_3 &= \cot A \cot B \cot C = \frac{c_3}{c_1} = \frac{c_1 - 1}{c_1} = 1 - \frac{(T + t)(T^2 + 1)}{2t}. \end{aligned}$$

Since the angles of ABC are completely determined by their cotangents, which in turn are nothing but the zeros of $X^3 - s_1X^2 + s_2X - s_3$, it follows that the angles of ABC are determined by t and T, as claimed. □

5. Some properties of the cevian Brocard points

It is easy to see that the first and second Brocard points coincide if and only if the triangle is equilateral. The same holds for the cevian Brocard points. The next theorem deals with the cases when a Brocard point and a cevian Brocard point coincide. We use the following simple theorem.

Theorem 4. *If the cevians AA', BB', and CC' through a point P inside triangle ABC have the property that two of the quadrilaterals $AC'PB'$, $BA'PC'$, $CB'PA'$, $ABA'B'$, $BCB'C'$, and $CAC'A'$ are cyclic, then P is the orthocenter of ABC. If, in addition, P is a Brocard point, then ABC is equilateral.*

Proof. The first part is nothing but [4, Theorem 4] and is easy to prove. The second part follows from $\omega = \pi/2 - A = \pi/2 - B = \pi/2 - C$. □

Theorem 5. *If any of the Brocard points L and L' of triangle ABC coincides with any of its cevian Brocard points M and M', then ABC is equilateral.*

Proof. Let AA', BB', and CC' be the cevians through L, and let ω and Ω be the Brocard and cevian Brocard angles of ABC; see Figure 4A. By the exterior angle theorem, $\angle ALB' = \omega + (B - \omega) = B$. Similarly, $\angle BLC' = C$ and $\angle CLA' = A$.

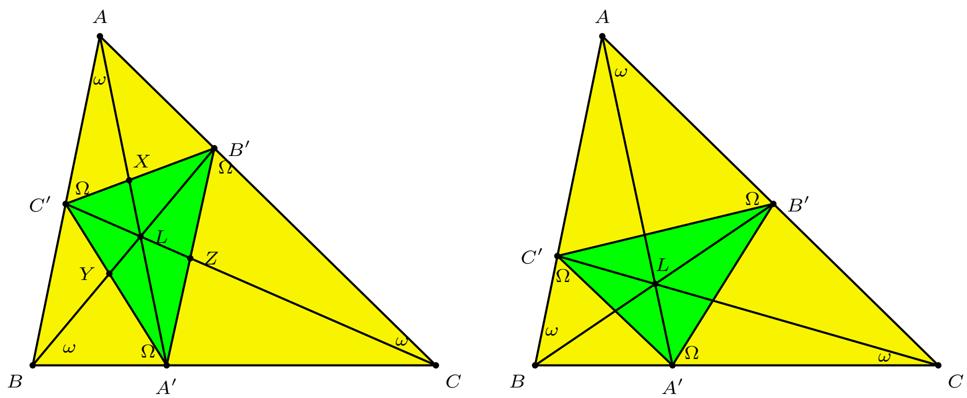

Figure 4A Figure 4B

Suppose that $L = M$. Then $(A', B', C') = (C, A, B)$. Referring to Figure 4A, let X, Y, and Z be the points where AA', BB', and CC' meet $B'C'$, $C'A'$, and $A'B'$, respectively. It follows from $\angle ALB' = B = C'$ and its iterates that the quadrilaterals $XC'YL$, $YA'ZL$, and $ZB'XL$ are cyclic. By Theorem 4, L is the orthocenter of $A'B'C'$. Therefore $\omega + \Omega = \pi/2$. Since $\omega \leq \pi/6$ and $\Omega \leq \pi/3$, it follows that $\omega = \pi/6$ and $\Omega = \pi/3$. Thus the Brocard and cevian Brocard angles of ABC coincide with those for an equilateral triangle. By Theorem 3, ABC is equilateral.

Suppose next that $L = M'$. Referring to Figure 4B, we see that $\angle AB'C' = \angle ACC' + \angle B'C'C$, and therefore $\angle B'C'C = \Omega - \omega$. Similarly $\angle C'A'A = \angle A'B'B = \Omega - \omega$. Therefore L is the second Brocard point of $A'B'C'$. Since $(A', B', C') = (B, C, A)$, it follows that ABC and $A'B'C'$ have the same Brocard angles. Therefore $\angle BAA' = \angle BB'A'$ and $ABA'B'$ is cyclic. The same holds for the quadrilaterals $BCB'C'$ and $CAC'A'$. By Theorem 4, ABC is equilateral. □

The following theorem answers questions that are raised naturally in the proof of Theorem 5. It also restates Theorem 5 in terms of the Brocard points without reference to the cevian Brocard points.

Theorem 6. *Let L be the first Brocard point of ABC, and let AA', BB', and CC' be the cevians through L. Then L coincides with one of the two Brocard points N and N' of $A'B'C'$ if and only if ABC is equilateral. The same holds for the second Brocard point L'.*

Proof. Let the angles of $A'B'C'$ be denoted by A', B', and C'. The proof of Theorem 5 shows that the condition $L = N'$ is equivalent to $L = M'$, which in turn implies that ABC is equilateral. This leaves us with the case $L = N$. In this case, let ω and μ be the Brocard angles of ABC and $A'B'C'$, respectively, as shown in Figure 5. The exterior angle theorem shows that

$$A = \pi - \angle AC'B' - \angle AB'C' = \pi - (\mu + B - \omega) - (\omega + C' - \mu) = \pi - B - C'.$$

Thus $C = C'$. Similarly, $A = A'$ and $B = B'$. Therefore $\mu = \omega$, and the quadrilaterals $AC'LB'$ and $BA'LC'$ are cyclic. By Theorem 4, ABC is equilateral. □

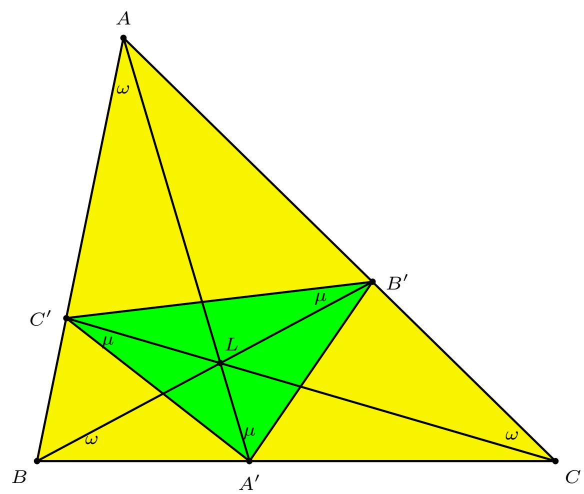

Figure 5

Remark. (3) It would be interesting to investigate whether the many inequalities involving the Brocard angle, such as Yff's inequality [1], have analogues for the cevian Brocard angles, and whether there are inequalities that involve both the Brocard and cevian Brocard angles. Similar questions can be asked about other properties of the Brocard points. For inequalities involving the Brocard angle, we refer the reader to [2] and [9, pp.329-333] and the references therein.

6. A characterization of some common triangle centers

We close with a theorem that complements Theorems 1 and 2 of [3].

Theorem 7. *The triangle centers for which the angles α, β, γ are linear forms in A, B, C are the centroid, the orthocenter, and the Gergonne point.*

Proof. Arguing as in Theorems 1 and 2 of [3], we see that α, β, γ are of the form

$$\alpha = \frac{\pi - A}{2} + t(B - C), \quad \beta = \frac{\pi - B}{2} + t(C - A), \quad \gamma = \frac{\pi - C}{2} + t(A - B).$$

In particular, $\alpha + \beta + \gamma = \pi$, and therefore

$$4 \sin \alpha \sin \beta \sin \gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma;$$

see [5, Formula 681, p. 166]. Thus the Ceva's concurrence relation takes the form

$$\begin{aligned}&\sin\left(A-2t(B-C)\right)+\sin\left(B-2t(C-A)\right)+\sin\left(C-2t(A-B)\right)\\ =&\sin\left(A+2t(B-C)\right)+\sin\left(B+2t(C-A)\right)+\sin\left(C+2t(A-B)\right),\end{aligned}$$

which reduces to

$$\cos A\sin(2t(B-C))+\cos B\sin(2t(C-A))+\cos C\sin(2t(A-B))=0.$$

Following word by word the way equation (5) of [3] was treated, we conclude that $t=-1/2$, $t=0$, or $t=1/2$.

If $t=0$, then $\alpha=(\pi-A)/2$, and therefore $\alpha=\alpha'$ and $AB'=AC'$. Thus A', B', and C' are the points of contact of the incircle, and the point of intersection of AA', BB', and CC' is the Gergonne point.

If $t=1/2$, then $(\alpha,\beta,\gamma)=(B,C,A)$, and $(A',B',C')=(A,B,C)$. This clearly corresponds to the centroid.

If $t=-1/2$, then $(\alpha,\beta,\gamma)=(C,A,B)$, and $(A',B',C')=(\pi-A,\pi-B,\pi-C)$. This clearly corresponds to the orthocenter. □

Remarks. (4) In establishing the parts pertaining to the centroid and the orthocenter in Theorem 7, we have used the uniqueness component of Seebach's theorem. Alternative proofs that do not use Seebach's theorem follow from [4, Theorems 4 and 7].

(5) In view of the proof of Theorem 7, it is worth mentioning that the proof of Theorem 2 of [3] can be simplified by noting that $\xi+\eta+\zeta=\pi/2$ and using the identity

$$1+4\sin\xi\sin\eta\sin\zeta=\cos 2\xi+\cos 2\eta+\cos 2\zeta$$

given in [5, Formula 678, p. 166].

(6) It is clear that the first and second cevian Brocard points of triangle ABC can be equivalently defined as the points whose cevian triangles $A'B'C'$ have the properties that $(A',B',C')=(C,A,B)$ and $(A',B',C')=(B,C,A)$, respectively. The point corresponding to the requirement that $(A',B',C')=(A,B,C)$ is the centroid; see [6] and [4, Theorem 7]. It would be interesting to explore the point defined by the condition $(A',B',C')=(A,C,B)$.

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