Another Proof of van Lamoen’s Theorem and Its Converse

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Abstract. We give a proof of Floor van Lamoen’s theorem and its converse on the circumcenters of the cevasix configuration of a triangle using the notion of directed angle of two lines.

1. Introduction

Let \( P \) be a point in the plane of triangle \( ABC \) with traces \( A', B', C' \) on the sidelines \( BC, CA, AB \) respectively. We assume that \( P \) does not lie on any of the sidelines. According to Clark Kimberling [1], triangles \( PCB', PC'B, PAC', PA'C, PBA', PB'A \) form the cevasix configuration of \( P \). Several years ago, Floor van Lamoen discovered that when \( P \) is the centroid of triangle \( ABC \), the six circumcenters of the cevasix configuration are concyclic. This was posed as a problem in the American Mathematical Monthly and was solved in [2, 3]. In 2003, Alexei Myakishev and Peter Y. Woo [4] gave a proof for the converse, that is, if the six circumcenters of the cevasix configuration are concyclic, then \( P \) is either the centroid or the orthocenter of the triangle.

In this note we give a new proof, which is quite different from those in [2, 3], of Floor van Lamoen’s theorem and its converse, using the directed angle of two lines. Remarkably, both necessity part and sufficiency part in our proof are basically the same. The main results of van Lamoen, Myakishev and Woo are summarized in the following theorem.

Theorem. Given a triangle \( ABC \) and a point \( P \), the six circumcenters of the cevasix configuration of \( P \) are concyclic if and only if \( P \) is the centroid or the orthocenter of \( ABC \).

We shall assume the given triangle non-equilateral, and omit the easy case when \( ABC \) is equilateral. For convenience, we adopt the following notations used in [4].

<table>
<thead>
<tr>
<th>Triangle</th>
<th>( PCB' )</th>
<th>( PC'B )</th>
<th>( PAC' )</th>
<th>( PA'C )</th>
<th>( PBA' )</th>
<th>( PB'A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notation</td>
<td>( \Delta(A_+) )</td>
<td>( \Delta(A_-) )</td>
<td>( \Delta(B_+) )</td>
<td>( \Delta(B_-) )</td>
<td>( \Delta(C_+) )</td>
<td>( \Delta(C_-) )</td>
</tr>
<tr>
<td>Circumcenter</td>
<td>( A_+ )</td>
<td>( A_- )</td>
<td>( B_+ )</td>
<td>( B_- )</td>
<td>( C_+ )</td>
<td>( C_- )</td>
</tr>
</tbody>
</table>

It is easy to see that two of these triangles may possibly share a common circumcenter only when they share a common vertex of triangle \( ABC \).

Publication Date: August 24, 2005. Communicating Editor: Floor van Lamoen.

The author thanks Le Chi Quang of Hanoi, Vietnam for his help in translation and preparation of the article.
2. Preliminary Results

**Lemma 1.** Let $P$ be a point not on the sidelines of triangle $ABC$, with traces $B', C'$ on $AC, AB$ respectively. The circumcenters of triangles $APB'$ and $APC'$ coincide if and only if $P$ lies on the reflection of the circumcircle $ABC$ in the line $BC$.

The Proof of Lemma 1 is simple and can be found in [4]. We also omit the proof of the following easy lemma.

**Lemma 2.** Given a triangle $ABC$ and $M, N$ on the line $BC$, we have
\[
\frac{\overline{BC}}{\overline{MN}} = \frac{S[ABC]}{S[AMN]},
\]
where $\overline{BC}$ and $\overline{MN}$ denote the signed lengths of the line segments $BC$ and $MN$, and $S[ABC], S[AMN]$ the signed areas of triangle $ABC$, and $AMN$ respectively.

**Lemma 3.** Let $P$ be a point not on the sidelines of triangle $ABC$, with traces $A', B', C'$ on $BC, AC, AB$ respectively, and $K$ the second intersection of the circumcircles of triangles $PCB'$ and $PC'B$. The line $PK$ is a symmedian of triangle $PBC$ if and only if $A'$ is the midpoint of $BC$.

**Proof.** Triangles $KB'B$ and $KCC'$ are directly similar (see Figure 1). Therefore,
\[
\frac{S[KB'B]}{S[KCC']} = \left(\frac{B'B}{CC'}\right)^2.
\]
On the other hand, by Lemma 2 we have
\[
\frac{S[KPB]}{S[KPC]} = \frac{PB}{PC} \cdot \frac{S[KB]}{S[KCC']}.
\]
Thus,
\[
\frac{S[KPB]}{S[KPC]} = \frac{PB}{PC} \cdot \frac{B'B}{CC'}.
\]
It follows that $PK$ is a symmedian line of triangle $PBC$, which is equivalent to the following
\[
\frac{S[KPB]}{S[KPC]} = -\left(\frac{PB}{PC}\right)^2, \quad \frac{PB \cdot B'B}{PC \cdot CC'} = -\left(\frac{PB}{PC}\right)^2, \quad \frac{B'B}{CC'} = \frac{PB}{PC}.
\]
The last equality is equivalent to $BC \parallel B'C'$, by Thales’ theorem, or $A'$ is the midpoint of $BC$, by Ceva’s theorem. □

**Remark.** Since the lines $BC'$ and $CB'$ intersect at $A$, the circumcircles of triangles $PCB'$ and $PC'B$ must intersect at two distinct points. This remark confirms the existence of the point $K$ in Lemma 3.
Lemma 4. Given a triangle $XYZ$ and pairs of points $M, N$ on $YZ$, $P, Q$ on $ZX$, and $R, S$ on $XY$ respectively. If the points in each of the quadruples $P, Q, R, S; R, S, M, N; M, N, P, Q$ are concyclic, then all six points $M, N, P, Q, R, S$ are concyclic.

Proof. Suppose that $(O_1), (O_2), (O_3)$ are the circles passing through the quadruples $(P, Q, R, S), (R, S, M, N)$, and $(M, N, P, Q)$ respectively. If $O_1, O_2, O_3$ are distinct points, then $YZ, ZX, XY$ are respectively the radical axis of pairs of circles $(O_2), (O_3); (O_3), (O_1); (O_1), (O_2)$. Hence, $YZ, ZX, XY$ are concurrent, or parallel, or coincident, which is a contradiction. Therefore, two of the three points $O_1, O_2, O_3$ coincide. It follows that six points $M, N, P, Q, R, S$ are concyclic.

Remark. In Lemma 4, if $M = N$ and the circumcircles of triangles $RSM, MPQ$ touch $YZ$ at $M$, then the five points $M, P, Q, R, S$ lie on the same circle that touches $YZ$ at the same point $M$.

3. Proof of the main theorem

Suppose that perpendicular bisectors of $AP, BP, CP$ bound a triangle $XYZ$. Evidently, the following pairs of points $B_+, C_-, C_+; A_-, A_-, A_+; A_+; B_-$ lie on the lines $YZ, ZX, XY$ respectively. Let $H$ and $K$ respectively be the feet of the perpendiculars from $P$ on $A_+A_+, B_+B_-$ (see Figure 2).

Sufficiency part. If $P$ is the orthocenter of triangle $ABC$, then $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$. Obviously, the six points $B_+, C_-, C_+, A_-, A_+, B_-$ lie on the same circle. If $P$ is the centroid of triangle $ABC$, then no more than one of the three following possibilities happen: $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$. by Lemma 1. Hence, we need to consider two cases.
Case 1. Only one of three following possibilities occurs: $B_+ = C_-, C_+ = A_-$, $A_+ = B_-$. Without loss of generality, we may assume that $B_+ = C_-, C_+ \neq A_-$ and $A_+ \neq B_-$ (see Figure 2). Since $P$ is the centroid of triangle $ABC$, $A'$ is the midpoint of the segment $BC$. By Lemma 3, we have 

$$(PH, PB) = (PC, PA') \pmod{\pi}.$$  

In addition, since $A_- A_+, A_- C_+, B_- A_+, B_- C_+$ are respectively perpendicular to $PH, PB, PC, PA'$, we have 

$$(A_- A_+, A_- C_+) \equiv (PH, PB) \pmod{\pi}.$$  

$$(B_- A_+, B_- C_+) \equiv (PC, PA') \pmod{\pi}.$$  

Thus, $(A_- A_+, A_- C_+) \equiv (B_- A_+, B_- C_+) \pmod{\pi}$, which implies that four points $C_+, A_-, A_+, B_-$ are concyclic. Similarly, we have 

$$(PK, PC) = (PA, PB') \pmod{\pi}.$$  

Moreover, since $B_- B_+, B_- A_+, YZ, B_+ A_+$ are respectively perpendicular to $PK, PC, PA, PB'$, we have 

$$(B_- B_+, B_- A_+) \equiv (PK, PC) \pmod{\pi}.$$  

$$(YZ, B_+ A_+) \equiv (PA, PB') \pmod{\pi}.$$  

Thus, $(B_- B_+, B_- A_+) \equiv (YZ, B_+ A_+) \pmod{\pi}$, which implies that the circumcircle of triangle $B_+ B_- A_+$ touches $YZ$ at $B_+$. 

Figure 2
The same reasoning also shows that the circumcircle of triangle \(B_+C_+A_-\) touches \(YZ\) at \(B_+\).

Therefore, the six points \(B_+, C_-, C_+, A_-, A_+, B_-\) lie on the same circle and this circle touches \(YZ\) at \(B_+ = C_-\) by the remark following Lemma 4.

**Case 2.** None of the three following possibilities occurs: \(B_+ = C_-; C_+ = A_-; A_+ = B_-\).

Similarly to case 1, each quadruple of points \((C_+, A_-, A_+, B_-), (A_+, B_-, B_+, C_-), (B_+, C_-, C_+, A_-)\) are concyclic. Hence, by Lemma 4, the six points \(B_+, C_-, C_+, A_-, A_+, B_-\) are concyclic.

**Necessity part.** There are three cases.

**Case 1.** No less than two of the following possibilities occur: \(B_+ = C_-; C_+ = A_-; A_+ = B_-\).

By Lemma 1, \(P\) is the orthocenter of triangle \(ABC\).

**Case 2.** Only one of the following possibilities occurs: \(B_+ = C_-; C_+ = A_-; A_+ = B_-\). We assume without loss of generality that \(B_+ = C_-; C_+ \neq A_-; A_+ \neq B_-\).

Since the six points \(B_+, C_-, C_+, A_-, A_+, B_-\) are on the same circle, so are the four points \(C_+, A_-, A_+, B_-\). It follows that

\[
(A_- A_+, A_- C_+) \equiv (B_- A_+, B_- C_+) \pmod{\pi}.
\]

Note that lines \(PH, PB, PC, PA'\) are respectively perpendicular to \(A_- A_+, A_- C_+, B_- A_+, B_- C_+\). It follows that

\[
(PH, PB) \equiv (A_- A_+, A_- C_+) \pmod{\pi},
\]

\[
(PC, PA') \equiv (B_- A_+, B_- C_+) \pmod{\pi}.
\]

Therefore, \((PH, PB) \equiv (PC, PA') \pmod{\pi}\). Consequently, \(A'\) is the midpoint of \(BC\) by Lemma 3.

On the other hand, it is evident that \(B_+ A_- \parallel B_- A_+; B_+ A_+ \parallel C_+ A_-\), and we note that each quadruple of points \((B_+, A_-, B_-, A_+), (B_+, A_+, C_+, A_-)\) are concyclic. Therefore, we have \(B_+ B_- = A_+ A_- = B_+ C_+.\) It follows that triangle \(B_+ B_- C_+\) is isosceles with \(C_+ B_+ = B_+ B_-\). Note that \(YZ\) passes \(B_+\) and is parallel to \(C_+ B_-\), so that we have \(YZ\) touches the circle passing six points \(B_+ = C_-; C_+, A_-, A_+, B_-\) at \(B_+ = C_-\). It follows that

\[
(B_- B_+, B_- A_+) \equiv (YZ, B_+ A_+) \pmod{\pi}.
\]

In addition, since \(PK, PC, PA, PB'\) are respectively perpendicular to \(B_- B_+, B_- A_+, YZ, B_+ A_+\), we have

\[
(PK, PC) \equiv (B_- B_+, B_- A_+) \pmod{\pi},
\]

\[
(PA, PB') \equiv (YZ, B_+ A_+) \pmod{\pi}.
\]

Thus, \((PK, PC) \equiv (PA, PB') \pmod{\pi}\). By Lemma 3, \(B'\) is the midpoint of \(CA\). We conclude that \(P\) is the centroid of triangle \(ABC\).
Case 3. None of the three following possibilities occur: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$. Similarly to case 2, we can conclude that $A', B'$ are respectively the midpoints of $BC$, $CA$. Thus, $P$ is the centroid of triangle $ABC$.

This completes the proof of the main theorem.

References


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