Forum Geometricorum
Volume 5 (2005) 143–148.

FORUM GEOM
ISSN 1534-1178

Applications of Homogeneous Functions to Geometric Inequalities and Identities in the Euclidean Plane

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Abstract. We study a class of geometric identities and inequalities that have a common pattern: they are generated by a homogeneous function. We show how to extend some of these homogeneous relations in the geometry of triangle. Then, we study the geometric configuration created by two intersecting lines and a pencil of n lines, where the repeated use of Menelaus's Theorem allows us to emphasize a result on homogeneous functions.

1. Introduction

The purpose of this note is to present an extension of a certain class of geometric identities or inequalities. The idea of this technique is inspired by the study of homogeneous polynomials and has the potential for additional applications besides the ones described here.

First of all, we recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous* if $f(tx_1, tx_2, ..., tx_n) = t^m f(x_1, x_2, ...x_n)$, for $t \in \mathbb{R} - \{0\}$ and $x_i \in \mathbb{R}$, $i = 1, ..., n$, $m, n \in \mathbb{N}$, $m \neq 0$, $n \geq 2$. The natural number m is called the degree of the homogeneous function f.

Remarks. 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function. If for $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we have $f(x) \geq 0$, then $f(tx) \geq 0$, for $t > 0$. Furthermore, if m is an even natural number, $f(x) \geq 0$, yields $f(tx) \geq 0$ for any real number t.

2. Any $x > 0$ can be written as $x = \frac{a}{b}$, with $a, b \in (0, 1)$.

2. Application to the geometry of triangle

Consider the homogeneous function $f_\alpha : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f_\alpha(x_1, x_2, x_3) = \alpha x_1 x_2 x_3,$$

with $\alpha \in \mathbb{R} - \{0\}$. Denote by a, b, c the lengths of the sides of a triangle ABC, by R the circumradius and by $\triangle$ the area of this triangle. By the law of sines, we get

$$f_1(a, b, c) = f_1(a, b, 2R \sin C) = 2R f_1(a, b, \sin C) = 4R\triangle.$$

Thus, we obtain $abc = 4R\triangle$.

Publication Date: October 11, 2005. Communicating Editor: Paul Yiu.

Since $f_1(a,b,c) = 8R^3 f_1(\sin A, \sin B, \sin C)$, we get also the equality

$$\triangle = 2R^2 \sin A \sin B \sin C.$$

Heron's formula can be represented by the following setting. The function $f_{\sqrt{r}}(x_1, x_2, x_3)$ for $x_1 = \sqrt{s-a}$, $x_2 = \sqrt{s-b}$, $x_3 = \sqrt{s-c}$, yields

$$f_{\sqrt{s}}(\sqrt{s-a}, \sqrt{s-b}, \sqrt{s-c}) = \triangle.$$

Furthermore, using $\cot \frac{A}{2} = \frac{s-a}{r}$ and the similar equalities in B and C, we obtain

$$f_{\sqrt{s}}(\sqrt{s-a}, \sqrt{s-b}, \sqrt{s-c}) = r\sqrt{r} f_{\sqrt{s}}\left(\sqrt{\cot \frac{A}{2}}, \sqrt{\cot \frac{B}{2}}, \sqrt{\cot \frac{C}{2}}\right),$$

which yields

$$\triangle = r^2 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

3. Homogeneous polynomials in $a^2, b^2, c^2, \triangle$ and their applications

Consider now a triangle ABC in the Euclidean plane, and denote by a, b, c the length of its sides and by $\triangle$ its area. We prove the following.

Proposition 1. *Let $p : \mathbb{R}^4 \to \mathbb{R}$ a homogeneous function with the property that $p(a^2, b^2, c^2, \triangle) \geq 0$, for any triangle in the Euclidean plane. Then for any $x > 0$ we have:*

$$p\left(xa^2, \frac{1}{x}b^2, c^2 + \left(1 - \frac{1}{x}\right)(xa^2 - b^2), \triangle\right) \geq 0. \tag{1}$$

Proof. Consider $q(x) = \left(1 - \frac{1}{x}\right)(xa^2 - b^2)$, for $x > 0$. In the triangle ABC we consider A_1 and B_1 on the sides BC and AC, respectively, such that $CA_1 = \alpha a$, $BC = a$, $CB_1 = \beta b$, $AC = b$, with $\alpha, \beta \in (0,1)$. It results that the area of triangle CA_1B_1 is $\sigma[CA_1B_1] = \alpha\beta\triangle$. By the law of cosines we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

and therefore

$$A_1B_1^2 = \alpha\beta c^2 + (\alpha - \beta)(\alpha a^2 - \beta b^2).$$

Since the given inequality $p(a^2, b^2, c^2, \triangle) \geq 0$ takes place in any triangle, then it must take place also in the triangle CA_1B_1, thus

$$p(\alpha^2 a^2, \beta^2 b^2, \alpha\beta c^2 + (\alpha - \beta)(\alpha a^2 - \beta b^2), \alpha\beta\triangle) = 0.$$

Let us take now $t = \alpha\beta$, and $x = \frac{\alpha}{\beta}$, with $\alpha, \beta \in (0,1)$. For $x \in (0, \infty)$, we have

$$p\left(xa^2, \frac{1}{x}b^2, c^2 + q(x), \triangle\right) \geq 0.$$

$\square$

Remark. In terms of identities, we state the following. Let $p : \mathbb{R}^4 \to \mathbb{R}$ a homogeneous function with the property that $p(a^2, b^2, c^2, \triangle) = 0$, for any triangle in the Euclidean plane. Then for any $x > 0$ we have

$$p\left(xa^2, \frac{1}{x}b^2, c^2 + \left(1 - \frac{1}{x}\right)(xa^2 - b^2), \triangle\right) = 0. \tag{2}$$

The proof is similar to the proof of Proposition 1.

We present now a few applications of Proposition 1.

3.1. In any triangle ABC in the Euclidean plane, for any $x \in (0, \infty)$, we have

$$4\triangle \leq \min\left[xa^2 + \frac{1}{x}b^2, xa^2 + c^2 + q(x), \frac{1}{x}b^2 + c^2 + q(x)\right].$$

To prove this inequality, it is sufficient to prove the statement for $x = 1$, then we apply Proposition 1. Let us assume, without losing any generality, that $a \geq b \geq c$. We also use $b^2 + c^2 \geq 2bc$, and $2bc \geq 2bc \sin A = 4\triangle$. Thus, $b^2 + c^2 \geq 4\triangle$, and this means

$$4\triangle \leq \min(b^2 + c^2, a^2 + c^2, a^2 + b^2).$$

Applying this result in the triangle CA_1B_1, considered as in the proof of Proposition 1, we obtain the stated inequality.

3.2. Consider $q(x) = \left(1 - \frac{1}{x}\right)(xa^2 - b^2)$, for $x > 0$. Then in any triangle we have the inequality

$$a^2b^2[c^2 + q(x)] \geq \left(\frac{4\triangle}{3\sqrt{3}}\right)^3.$$

This results as a direct consequence of Carlitz' inequality

$$a^2b^2c^2 \geq \left(\frac{4\triangle}{3\sqrt{3}}\right)^3.$$

by applying Proposition 1.

3.3. It is known that in any triangle we have Hadwiger's inequality

$$a^2 + b^2 + c^2 \geq \triangle\sqrt{3}.$$

This inequality can be generalized for any $x \in (0, \infty)$ as follows

$$(2x - 1)a^2 + \left(\frac{2}{x} + 1\right)b^2 + c^2 \geq 4\triangle\sqrt{3}.$$

(This inequality appears in *Matematika v Shkole,* No. 5, 1989.)

Hadwiger's inequality can be proven by using the law of cosines to get

$$a^2 + b^2 + c^2 = 2(b^2 + c^2) - 2bc \cos A.$$

Then, keeping in mind that $2\triangle = bc\sin A$, we get

$$\begin{aligned} a^2+b^2+c^2-4\triangle\sqrt{3} &= 2(b^2+c^2-2bc\cos A-2bc\sqrt{3}\sin A) \\ &= 2\left(b^2+c^2-4bc\cos\left(\frac{\pi}{3}-A\right)\right) \\ &\geq 2\left(b^2+c^2-4bc\cos\frac{\pi}{3}\right) \\ &= 2(b-c)^2 \\ &\geq 0. \end{aligned}$$

The equality holds when $b = c$ and $A = \frac{\pi}{3}$, i.e. when triangle ABC is equilateral.

Applying Hadwiger's inequality to the triangle CA_1B_1 constructed in Proposition 1, we get

$$\alpha^2a^2+\beta^2b^2+\alpha\beta c^2+(\alpha-\beta)(\alpha a^2-\beta b^2) \geq 4\alpha\beta\triangle\sqrt{3}.$$

Dividing by $\alpha\beta$ and denoting, as before, $x = \frac{\alpha}{\beta}$, we obtain

$$xa^2+\frac{1}{x}b^2+c^2+q(x) \geq 4\triangle\sqrt{3}.$$

After grouping the factors, we get the inequality that we wanted to prove in the first place. □

3.4. Consider Goldner's inequality

$$b^2c^2+c^2a^2+a^2b^2 \geq 16\triangle^2.$$

This inequality can be extended by using the technique presented here to the following relation:

$$a^2b^2+\left(xa^2+\frac{1}{x}b^2\right)\left[c^2+\left(1-\frac{1}{x}\right)(xa^2-b^2)\right] \geq 16\triangle^2.$$

To remind here the proof of Goldner's inequality, we use an argument based on a consequence of Heron's formula:

$$2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4) = 16\triangle^2,$$

and the inequality

$$a^4+b^4+c^4 \geq a^2b^2+a^2c^2+b^2c^2.$$

This proves Goldner's inequality. For its extension, we apply Goldner's inequality to triangle CA_1B_1, as in Proposition 1.

4. Menelaus' Theorem and homogeneous polynomials

In this section we prove the following result.

Proposition 2. *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree m, and consider n collinear points $A_1, A_2, ..., A_n$ lying on the line d. Let S be a point exterior to the line $\mathcal{L}$ and a secant $\mathcal{L}'$ whose intersection with each of the segments*

(SA_i) is denoted A_i', with $i = 1, ..., n$. Denote by K the intersection point of $\mathcal{L}$ and $\mathcal{L}'$. Then,

$$p(KA_1, KA_2, ..., KA_n) = 0$$

if and only if

$$p\left(\frac{A_1A_1'}{A_1'S}, \frac{A_2A_2'}{A_2'S}, ..., \frac{A_nA_n'}{A_n'S}\right) = 0.$$

Proof. Denote $a_i = \frac{A_iA_i'}{A_i'S}$, for $i = 1, ..., n$. Applying Menelaus' Theorem in each of the triangles SA_1A_2, $SA_2A_3, \ldots, SA_{n-1}A_n$ we have, for all $i = 1, ..., n-1$,

$$\frac{1}{a_i} \cdot \frac{A_iK}{A_{i+1}K} \cdot a_{i+1} = 1.$$

This yields

$$\frac{A_1K}{a_1} = \frac{A_2K}{a_2} = ... = \frac{A_nK}{a_n} = t,$$

where $t > 0$. The fact that $p(KA_1, KA_2, ..., KA_n) = 0$ is equivalent, by Remark 1, with

$$p(ta_1, ta_2, ..., ta_n) = 0,$$

or, furthermore

$$t^m p(a_1, a_2, ..., a_n) = 0.$$

Since $t > 0$, the conclusion follows immediately. □

Remark. 3. As in the case of Proposition 1, we can discuss this result in terms of inequalities. For example, the Proposition 2 is still true if we claim that

$$p(KA_1, KA_2, ..., KA_n) \geq 0$$

if and only if

$$p\left(\frac{A_1A_1'}{A_1'S}, \frac{A_2A_2'}{A_2'S}, ..., \frac{A_nA_n'}{A_n'S}\right) \geq 0.$$

We present now an application.

4.1. A line intersects the sides AC and BC and the median CM_0 of an arbitrary triangle in the points B_1, A_1, and M_3, respectively. Then,

$$\frac{1}{2}\left(\frac{AB_1}{B_1C} + \frac{BA_1}{A_1C}\right) = \frac{M_3M_0}{M_3C}, \tag{3}$$

$$\frac{M_3B_1}{M_3A_1} = \frac{KB_1}{KA_1} \cdot \frac{KB}{KA}. \tag{4}$$

Furthermore, (3) is still true if we apply to this configuration a projective transformation that maps K into ∞.

We use Proposition 2 to prove (3). Let $\{K\} = AB \cap A_1B_1$. Then, the relation we need to prove is equivalent to $KA + KB = 2KM_0$, which is obvious, since M_0 is the midpoint of (AB).

To prove (4), remark that the anharmonic ratios $[KM_3B_1A_1]$ and $[KM_0AB]$ are equal, since they are obtained by intersecting the pencil of lines CK, CA, CM_0, CB with the lines KA and KB. Therefore, we have

$$\frac{M_3B_1}{M_3A_1} : \frac{KB_1}{KA_1} = \frac{M_0A}{M_0B} : \frac{KA}{KB}.$$

Since $M_0A = M_0B$, we have

$$\frac{M_3B_1}{M_3A_1} = \frac{KB_1}{KA_1} \cdot \frac{KB}{KA}.$$

Finally, by mapping M into the point at infinity, the lines B_1A_1 and BA become parallel. By Thales Theorem, we have

$$\frac{B_1A}{B_1C} = \frac{BA_1}{A_1C} = \frac{M_3M_0}{M_3C},$$

therefore the relation is still true.

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