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On the Existence of Triangles with Given Circumcircle, Incircle, and One Additional Element

Victor Oxman

Abstract. We give necesssary and sufficient conditions for the existence of poristic triangles with two given circles as circumcircle and incircle, and (1) a side length, (2) the semiperimeter (area), (3) an altitude, and (4) an angle bisector. We also consider the question of construction of such triangles.

1. Introduction

It is well known that the distance d between the circumcenter and incenter of a triangle is given by the formula:

$$d^2 = R^2 - 2Rr, \tag{1}$$

where R and r are respectively the circumradius and inradius of the triangle ([3, p.29]). Therefore, if we are given two circles on the plane, with radii R and r, $(R \geq 2r)$, a necessary condition for an existence of a triangle, for which the two circles will be the circumcircle and the incircle, is that the distance d between their centers satifies (1). From Poncelet's closure theorem it follows that this condition is also sufficient. Furthermore, each point on the circle with radius R may be one of the triangle vertex, *i.e.*, in general there are infinitely many such triangles. A natural question is on the existence and uniqueness of such a triangle if we specify one additional element. We shall consider this question when this additional element is one of the following: (1) a side length, (2) the semiperimeter (area), (3) an altitude, and (4) an angle bisector.

2. Main results

Throughout this paper, we consider two given circles $O(R)$ and $I(r)$ with distance d between their centers satisfying (1). Following [2], we shall call a triangle with circumcircle $O(R)$ and incircle $I(r)$ a poristic triangle.

Theorem 1. *Let a be a given positive number.* (1). *If $d \leq r$, i.e. $R \leq (\sqrt{2}+1)r$, then there is a unique poristic triangle ABC with $BC = a$ if and only if*

$$4r(2R - r - 2d) \leq a^2 \leq 4r(2R - r + 2d). \tag{2}$$

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(2). *If* $d > r$, *i.e.* $R > (\sqrt{2}+1)r$, *then there is a unique poristic triangle* ABC *with* $BC = a$ *if and only if*

$$4r(2R-r-2d) \le a^2 < 4r(2R-r+2d) \quad \textit{or} \quad a = 2R, \tag{3}$$

and there are two such triangles if and only if

$$4r(2R-r+2d) \le a^2 < 4R^2. \tag{4}$$

Theorem 2. *Given* $s > 0$, *there is a unique poristic triangle with semiperimeter* s *if and only if*

$$\sqrt{R+r-d}(\sqrt{2R}+\sqrt{R-r+d}) \le s \le \sqrt{R+r+d}(\sqrt{2R}+\sqrt{R-r-d}). \tag{5}$$

Theorem 3. *Given* $h > 0$, *there is a unique poristic triangle with an altitude* h *if and only if*

$$R+r-d \le h \le R+r+d. \tag{6}$$

Theorem 4. *Given* $\ell > 0$, *there is a unique poristic triangle with an angle bisector* ℓ *if and only if*

$$R+r-d \le \ell \le R+r+d. \tag{7}$$

3. Proof of Theorem 1

3.1. *Case 1.* $d \le r$. The length of $BC = a$ attains its minimal value when the distance from O to BC is maximal, which is $d+r$. See Figure 1. Therefore,

$$a^2_{\min} = 4r(2R-r-2d).$$

Similarly, a attains its maximum when the distance from O to BC is minimal, *i.e.*, $r-d$. See Figure 2.

$$a^2_{\max} = 4r(2R-r+2d).$$

This shows that (2) is a necessary condition a to be a side of a poristic triangle.

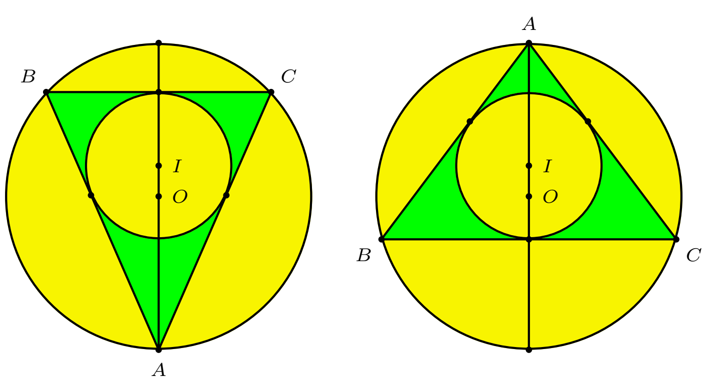

Figure 1 Figure2

We prove the sufficiency part by an explicit construction. If a satisfies (2), we construct the circle $O(R_1)$ with $R_1^2 = R^2 - \frac{a^2}{4}$, and a common tangent of this circle and $I(r)$. The segment of this tangent inside the circle $O(R)$ is a side of a poristic triangle with a side of length a. The third vertex is, by Poncelet's closure theorem, the intersection of the tangents from these endpoints to $I(r)$, and it lies on $O(R)$.

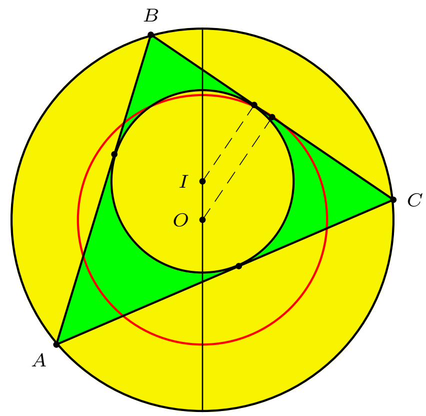

Figure 3

Remark. If $a \neq a_{\max}, a_{\min}$, we can construct two common tangents to the circles $O(R)$ and $I(r)$. These are both external common tangents and are symmetric with respect to the line OI. The resulting triangles are congruent.

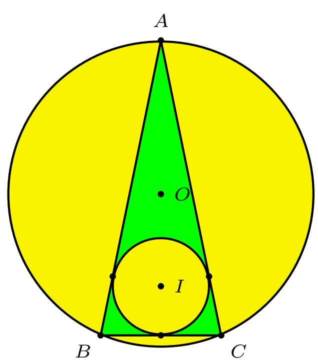

Figure 4

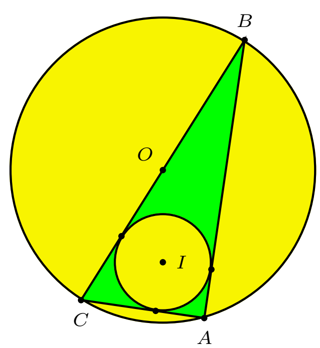

Figure 5

3.2. *Case 2.* $d > r$. In this case by the same way we have

$$a_{\min}^2 = 4r(2R - r - 2d).$$

See Figure 4. On the other hand, the maximum occurs when BC passes through the center O, *i.e.*, $a_{\max} = 2R$. See Figure 5.

For a given $a > 0$, we again construct the circle $O(R_1)$ with $R_1^2 = R^2 - \frac{a^2}{4}$. Chords of the circle (O) which are tangent to $O(R_1)$ have length a. If $R_1 > d - r$, the construction in §3.1 gives a poristic triangle with a side a. Therefore for

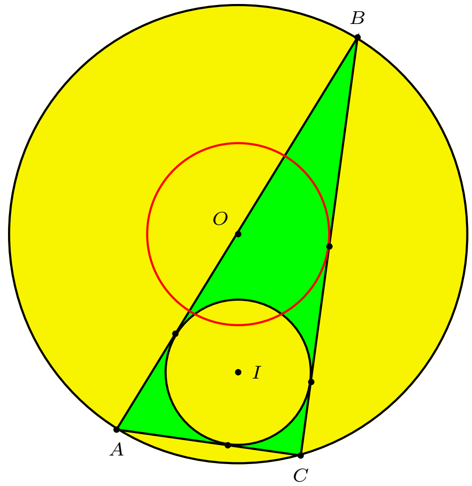

Figure 6

$4r(2R - r - 2d) \leq a^2 < 4r(2R - r + 2d)$, there is a unique poristic triangle with side a. See Figure 6. It is clear that this is also the case if $a = 2R$.

However, if $R_1 \leq d - r$, there are also internal common tangents of the circles $O(R_1)$ and $I(r)$. The internal common tangents give rise to an obtuse angled triangle. See Figures 7 and 8.

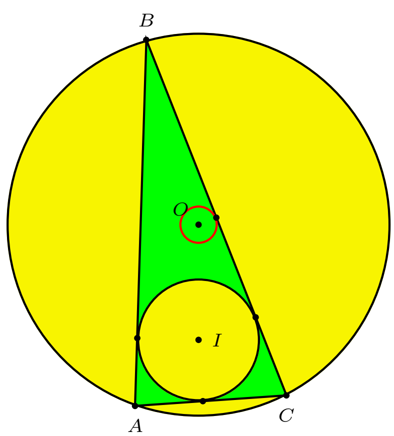

Figure 7

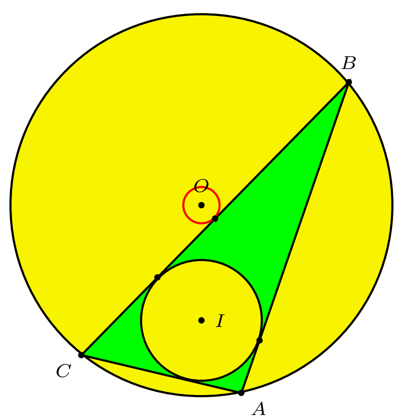

Figure 8

4. Proof of Theorem 2

Let $A_1B_1C_1$ and $A_2B_2C_2$ be the poristic triangles with A_1 and A_2 on the line OI. We assume $\angle A_1 \leq \angle A_2$. If $\angle A_1 = \angle A_2$, the triangle is equilateral and the statement of the theorem is trivial. We shall therefore assume $\angle A_1 < \angle A_2$. Consider an arbitrary poristic triangle ABC with semiperimeter s. According to

[4], s attains its maximum when the triangle coincides with $A_1B_1C_1$ and minimum when it coincides with $A_2B_2C_2$. Therefore,

$$\begin{aligned} s_{\max} =& \sqrt{R^2-(r+d)^2}+\sqrt{R^2-(r+d)^2+(R+r+d)^2} \\ =& \sqrt{R+r+d}(\sqrt{2R}+\sqrt{R-r-d}), \\ s_{\min} =& \sqrt{R^2-(r-d)^2}+\sqrt{R^2-(r-d)^2+(R+r-d)^2} \\ =& \sqrt{R+r-d}(\sqrt{2R}+\sqrt{R-r+d}). \end{aligned}$$

This proves (5).

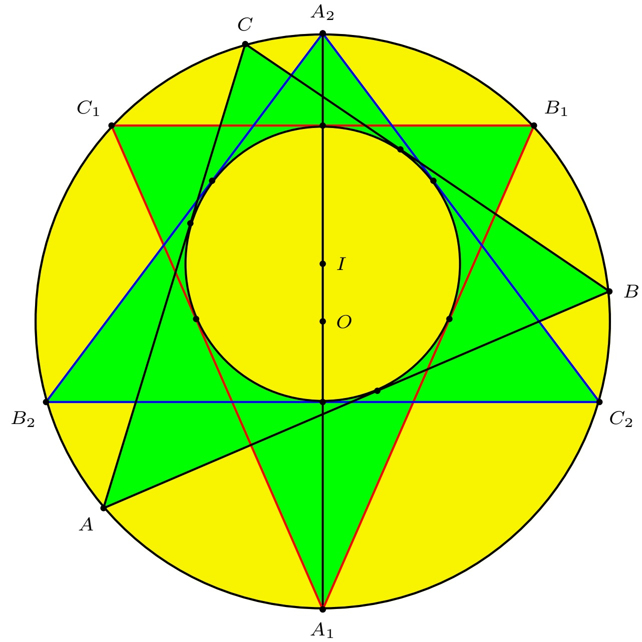

Figure 9

As A traverses a semicircle from position A_1 to A_2, the measure α of angle A is monotonically increasing from $\alpha_{\min} = \angle A_1$ to $\alpha_{\max} = \angle A_2$. For each $\alpha \in [\alpha_{\min},\ \alpha_{\max}]$,

$$s = s(\alpha) = \frac{r}{\tan\frac{\alpha}{2}} + 2R\sin\alpha.$$

Differentiating with respect to α, we have

$$s'(\alpha) = -\frac{r}{2\sin^2\frac{\alpha}{2}} + 2R\cos\alpha.$$

Clearly, $s'(\alpha) = 0$ if and only if $\sin^2\frac{\alpha}{2} = \frac{R\pm d}{4R}$. Since $\sin\frac{\alpha}{2} > 0$, there are two values of $\alpha \in (\alpha_{\min},\ \alpha_{\max})$ for which $s'(\alpha) = 0$. One of these is $\alpha_1 = \angle B_1$ for which $s(\alpha_1) = s_{\max}$ and the other is $\alpha_2 = \angle C_2$ for which $s(\alpha_2) = s_{\min}$.

Therefore for given real number $s > 0$ satisfying (5), there are three values of α (or two values if $s = s_{\min}$ or $s_{\max}$) for which $s(\alpha) = s$. These values are the values of the three angles of the same triangle that has semiperimeter s. So for such s the triangle is unique up to congruence.

Remark. Generally the ruler and compass construction of the triangle with given R, r and s is impossible. In fact, if $t = \tan\frac{\alpha}{2}$, then from $s = \frac{r}{\tan\frac{\alpha}{2}} + 2R\sin\alpha$ we have

$$st^3 - (4R + r)t^2 + st - r = 0.$$

The triangle is constructible if and only if t is constructible. It is known that the roots of a cubic equation with rational coefficients are constructible if and only if the equation has a rational root [1, p.16]. For $R = 4$, $r = 1$, $s = 8$ (such a triangle exists by Theorem 2) we have

$$8t^3 - 17t^2 + 8t - 1 = 0. \tag{8}$$

It is easy to see that it does not have rational roots. Therefore the roots of (8) are not constructible, and the triangle with given R, r, s is also not constructible.

5. Proof of Theorem 3

Let α be the measure of angle A.

$$h = \frac{2rs}{a} = \frac{\frac{2r^2}{\tan\frac{\alpha}{2}} + 4Rr\sin\alpha}{2R\sin\alpha} = \frac{r^2}{2R\sin^2\frac{\alpha}{2}} + 2r.$$

Since α is monotonically increasing (from $\alpha_{\min}$ to $\alpha_{\max}$ while vertex A moves from A_1 to A_2 along the arc A_1A_2, $h = h(\alpha)$ monotonically decreases from $h_{\max} = h(\alpha_{\min})$ to $h_{\min} = h(\alpha_{\max})$. Furthermore,

$$\begin{aligned} h_{\min} =& R + r - d, \\ h_{\max} =& R + r + d. \end{aligned}$$

This completes the proof of Theorem 3.

Remark. It is easy to construct the triangle by given R, r and h with the help of ruler and compass. Indeed, for a triangle ABC with given altitude $AH = h$ we have

$$AI^2 = \frac{r^2}{\sin^2\frac{\alpha}{2}} = 2R(h - 2r).$$

6. Proof of Theorem 4

The length of the bisector of angle A is given by

$$\ell = \frac{2bc\cos\frac{\alpha}{2}}{b + c}.$$

Since $R = \frac{abc}{4\triangle} = \frac{abc}{4rs}$, we have

$$\ell = \frac{\frac{8Rrs}{a}\cdot\cos\frac{\alpha}{2}}{2s - a} = \frac{r}{\sin\frac{\alpha}{2}} + \frac{2Rr\sin\frac{\alpha}{2}}{r + 2R\sin^2\frac{\alpha}{2}}.$$

Differentiating with respect to α, we have

$$\begin{aligned}\frac{\ell'(\alpha)}{r} &= -\frac{\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}} + \frac{R\cos\frac{\alpha}{2}(r - 2R\sin^2\frac{\alpha}{2})}{(r + 2R\sin^2\frac{\alpha}{2})^2} \\ &= -\frac{\cos\frac{\alpha}{2}(r^2 + 2Rr\sin^2\frac{\alpha}{2} + 8R^2\sin^4\frac{\alpha}{2})}{2\sin^2\frac{\alpha}{2}(r + 2R\sin^2\frac{\alpha}{2})^2} \\ &< 0.\end{aligned}$$

Therefore, $\ell(\alpha)$ monotonically decreases on $[\alpha_{\min}, \alpha_{\max}]$ from $\ell_{\max} = R + r + d$ to $\ell_{\min} = R + r - d$.

Remark. Generally the ruler and compass construction of the triangle with given R, r and ℓ is impossible. Indeed, if $t = \sin\frac{\alpha}{2}$, then

$$2R\ell t^3 - 4Rrt^2 + r\ell t - r^2 = 0.$$

For $R = 3$, $r = 1$ and $\ell = 5$ (such a triangle exists by Theorem 4), we have

$$30t^3 - 12t^2 + 5t - 1 = 0.$$

It can be easily checked that this equation doe not have a rational root. This shows that the ruler and compass construction of the triangle is not possible.

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Victor Oxman (Western Galilee College): Derech HaYam 191A, Haifa 34890, Israel

E-mail address: `voxman@013.net`