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Statics and the Moduli Space of Triangles

Geoff C. Smith

Abstract. The variance of a weighted collection of points is used to prove classical theorems of geometry concerning homogeneous quadratic functions of length (Apollonius, Feuerbach, Ptolemy, Stewart) and to deduce some of the theory of major triangle centers. We also show how a formula for the distance of the incenter to the reflection of the centroid in the nine-point center enables one to simplify Euler's method for the reconstruction of a triangle from its major centers. We also exhibit a connection between Poncelet's porism and the location of the incenter in the circle on diameter GH (the orthocentroidal or critical circle). The interior of this circle is the moduli (classification) space of triangles.

1. Introduction

There are some theorems of Euclidean geometry which have elegant proofs by means of mechanical principles. For example, if ABC is an acute triangle, one can ask which point P in the plane minimizes $AP + BP + CP$? The answer is the Fermat point, the place where $\angle APB = \angle BPC = \angle CPA = 2\pi/3$. The mechanical solution is to attach three pieces of inextensible massless string to P, and to dangle the three strings over frictionless pulleys at the vertices of the triangle, and attach the same mass to each string. Now hold the triangle flat and dangle the masses in a uniform gravitational field. The forces at P must balance so the angle equality is obtained, and the potential energy of the system is minimized when $AP + BP + CP$ is minimized.

In this article we will develop a geometric technique which involves a notion analogous to the moment of inertia of a mechanical system, but because of an averaging process, this notion is actually more akin to *variance* in statistics. The main result is well known to workers in the analysis of variance. The applications we give will (in the main) not yield new results, but rather give alternative proofs of classical results (Apollonius, Feuerbach, Stewart, Ptolemy) and make possible a systematic statical development of some of the theory of triangle centers. We will conclude with some remarks concerning the problem of reconstructing a triangle from O, G and I which will, we hope, shed more light on the constructions of Euler [3] and Guinand [4].

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For geometrical background we recommend [1] and [2].

Definition. Let X and Y be non-empty finite subsets of an inner product space V. We have weight maps $m : X \longrightarrow \mathbb{R}$ and $n : Y \longrightarrow \mathbb{R}$ with the property that $M = \sum_x m(x) \neq 0 \neq \sum_{y\in Y} n(y) = N$. The *mean square distance* between these weighted sets is

$$d^2(X, m, Y, n) = \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - y||^2.$$

Let

$$\overline{x} = \frac{1}{M} \sum_{x\in X} m(x)x$$

be the centroid of X. Ignoring the distinction between $\overline{x}$ and $\{\overline{x}\}$, and assigning the weight 1 to $\overline{x}$, we put

$$\sigma^2(X, m) = d^2(X, m, \overline{x}, 1)$$

and call this the *variance* of X, m. In fact the non-zero weight assigned to $\overline{x}$ is immaterial since it cancels. When the weighting is clear in a particular context, mention of it may be suppressed. We will also be cavalier with the arguments of these functions for economy.

We call the main result the generalized parallel axis theorem (abbreviated to GPAT) because of its relationship to the corresponding result in mechanics.

Theorem 1 (GPAT)**.**

$$d^2(X, m, Y, n) = \sigma^2(X, m) + ||\overline{x} - \overline{y}||^2 + \sigma^2(Y, m).$$

Proof.

$$\frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - y||^2$$

$$= \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - \overline{x} + \overline{x} - \overline{y} + \overline{y} - y||^2$$

$$= \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - \overline{x}||^2 + ||\overline{x} - \overline{y}||^2 + \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||y - \overline{y}||^2$$

since the averaging process makes the cross terms vanish. We are done. □

Corollary 2. $d^2(X, m, X, m) = 2\sigma^2(X, m)$.

Note that the averaging process ensures that scaling the weights of a given set does not alter mean square distances or variances.

The method of areal co-ordinates involves fixing a reference triangle ABC in the plane, and given a point P in its interior, assigning weights which are the areas of triangles: the weights $[PBC], [PCA]$ and $[PAC]$ are assigned to the points A, B and C respectively. The center of mass of $\{A, B, C\}$ with the given weights is P. With appropriate signed area conventions, this can be extended to define a co-ordinate system for the whole plane. If the weights are scaled by dividing by the area of $\triangle ABC$, then one obtains normalized areal co-ordinates; the co-ordinates of A are then $(1, 0, 0)$ for example. A similar arrangement works in Euclidean space of any dimension. The GPAT has much to say about these co-ordinate systems.

2. Applications

2.1. *Theorems of Apollonius and Stewart.* Let ABC be a triangle with corresponding sides of length a, b and c. A point D on the directed line CB is such that $CD = m$, $DB = n$ and these quantities may be negative. Let AD have length x. Weighting B with m and C with n, the center of mass of $\{B, C\}$ is at D and the variance of the weighted $\{B, C\}$ is $\sigma^2 = (mn^2 + nm^2)/(m + n) = mn$. The GPAT now asserts that

$$\frac{nb^2 + mc^2}{m + n} = 0 + x^2 + \sigma^2$$

or rather

$$nb^2 + mc^2 = (m + n)(x^2 + mn).$$

This is Stewart's theorem. If $m = n$ we deduce Apollonius's result that $b^2 + c^2 = 2(x^2 + (\frac{a}{2})^2)$.

2.2. *Ptolemy's Theorem.* Let A, B, C and D be four points in Euclidean 3-space. Consider the two sets $\{A, C\}$ and $\{B, D\}$ with weight 1 at each point. The GPAT asserts that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4t^2$$

where t is the distance between the midpoints of the line segments AC and BD. This may be familiar in the context that $t = 0$ and $ABCD$ is a parallelogram.

Recall that Ptolemy's theorem asserts that if $ABCD$ is a cyclic quadrilateral, then

$$AC \cdot BD = AB \cdot CD + BC \cdot DA.$$

We prove this as follows. Let the diagonals AC and BD meet at X. Now weight A, B, C and D so that the centers of mass of both $\{A, C\}$ and $\{B, D\}$ are at X. The GPAT now asserts that

$$\frac{XC \cdot AX^2 + AX \cdot XC^2}{AC} + \frac{XB \cdot DX^2 + DX \cdot BX^2}{BD}$$

$$= \frac{XC \cdot AB^2 \cdot XD + XC \cdot AD^2 \cdot BX + XA \cdot CB^2 \cdot XD + XA \cdot CD^2 \cdot XB}{AC \cdot BD}.$$

The left side of this equation tidies to $AX \cdot XC + BX \cdot XD$. One could regard this equation as a generalization of Ptolemy's theorem to quadrilaterals which are not necessarily cyclic.

Now we invoke cyclicity: $AX \cdot XC = BX \cdot XD = x$ by the intersecting chords theorem. Therefore $AC \cdot BD =$

$$\frac{XC \cdot AB^2 \cdot XD + XC \cdot AD^2 \cdot BX + XA \cdot CB^2 \cdot XD + XA \cdot CD^2 \cdot XB}{2x}.$$

However $AB/CD = BX/XC = AX/XD$ and $DA/BC = AX/BX = DX = CX$ (by similarity) so the right side of this equation is $AB \cdot CD + BC \cdot DA$ and Ptolemy's theorem is established.

2.3. *A geometric interpretation of* σ^2. Let ABC be a triangle with circumcenter O and incenter I and the usual side lengths a, b and c. We can arrange that the center of mass of $\{A, B, C\}$ is at I by placing weights a, b and c at A, B and C respectively. By calculating the mean square distance of this set of weighted triangle vertices to itself, we obtain the variance $\sigma_I^2 = \frac{abc}{a+b+c}$. However $abc/4R = [ABC]$, the area of the triangle, and $(a+b+c)r = 2[ABC]$ where R, r are the circumradius and inradius respectively. Therefore

$$\sigma_I^2 = 2Rr = \frac{abc}{a+b+c}. \tag{1}$$

Now calculate the mean square distance from O to the weighted triangle vertices both in the obvious way, and also by the GPAT to obtain Euler's result

$$OI^2 = R^2 - 2Rr. \tag{2}$$

Observation More generally suppose that a finite coplanar set of points Λ is concyclic, and is weighted to have center of mass at L, Let the center of the circle be at X and its radius be ρ. By the GPAT applied to X and the weighted set Λ we obtain

$$LX^2 = \rho^2 - \sigma^2(\Lambda, L)$$

so

$$\sigma^2(\Lambda, L) = \rho^2 - LX^2 = (\rho - LX)(\rho + LX).$$

Thus we conclude that $\sigma^2(\Lambda, L)$ is minus the *power of* L with respect to the circle.

2.4. *The Euler line.* Let ABC be a triangle with circumcenter O, centroid G and orthocenter H. These three points are collinear and this line is called the Euler line. It is easy to show that $OH = 3OG$. It is well known that

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2). \tag{3}$$

We derive this formula using the GPAT. Assign unit weights to the vertices of triangle ABC. The center of mass will be at G the intersection of the medians. Calculate the mean square distance of this triangle to itself to obtain the variance σ_G^2 of this triple of points. By the GPAT we have

$$2\sigma_G^2 = \frac{2a^2 + 2b^2 + 2c^2}{9}$$

so $\sigma_G^2 = \frac{a^2+b^2+c^2}{9}$. Now calculate the mean square distance from O to this triangle with unit weight the sensible way, and also by the GPAT to obtain

$$R^2 = OG^2 + \sigma_G^2.$$

Multiply through by 9 and use the fact that $OH = 3OG$ to obtain (3).

2.5. *The Nine-point Circle.* Let ABC be a triangle. The nine-point circle of ABC is the circle which passes through the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments joining the orthocenter H to each vertex. This circle has radius $R/2$ and is tangent to the inscribed circle of triangle ABC (they touch internally to the nine-point circle), and the three escribed circles (externally). We will prove this last result using the GPAT, and calculate the squares of the distances from I to important points on the Euler line.

Proposition 3. *Let p denote the perimeter of the triangle A, B, C. The distance between the incenter I and centroid G satisfies the following equation:*

$$IG^2 = \frac{p^2}{6} - \frac{5}{18}(a^2 + b^2 + c^2) - 4Rr. \tag{4}$$

Proof. Let $\triangle_G$ denote the triangle weighted 1 at each vertex and $\triangle_I$ denote the same triangle with weights attached to the vertices which are the lengths of the opposite sides. We apply the GPAT and a direct calculation:

$$d^2(\triangle_G, \triangle_I) = \sigma_G^2 + IG^2 + \sigma_I^2 = \frac{ab^2 + ba^2 + bc^2 + cb^2 + ca^2 + ac^2}{3(a+b+c)}$$

so

$$\begin{aligned} \frac{a^2 + b^2 + c^2}{9} + IG^2 + 2Rr &= \frac{(ab + bc + ca)(a + b + c) - 3abc}{3(a+b+c)} \\ &= \frac{ab + bc + ca}{3} - 2Rr. \end{aligned}$$

Therefore

$$4Rr + IG^2 + \frac{a^2 + b^2 + c^2}{9} = \frac{(a+b+c)^2}{6} - \frac{a^2 + b^2 + c^2}{6}.$$

This equation can be tidied into the required form.

Corollary Using Euler's inequality $R \geq 2r$ (which follows from $IO^2 \geq 0$) and the condition $|IG|^2 \geq 0$ we obtain that in any triangle we have

$$3p^2 \geq 5(a^2 + b^2 + c^2) + 144r^2$$

with equality exactly when $R = 2r$ and $I = G$. Thus the inequality becomes an equality if and only if the triangle is equilateral. □

Theorem 4 (Feuerbach)**.** *The nine point circle of $\triangle ABC$ is internally tangent to the incircle.*

Proof. (outline) The radius of the nine point circle is $R/2$. The result will established if we show that $|IN| = R/2 - r$. However, in $\triangle INO$ the point G is on the side NO and $NG : GO = 1 : 2$. We know $|IO|, |IG|, |NG|$ and $|GO|$, so Stewart's theorem and some algebra enable us to deduce the result.

Since $OG : GN = 2 : 1$ Stewart's theorem applies and we have

$$IG^2 + \frac{2}{9}ON^2 = \frac{2}{3}IN^2 = \frac{1}{3}IO^2.$$

Rearranging this becomes

$$IN^2 = \frac{3}{2}IG^2 + \frac{3}{4}OG^2 - \frac{1}{2}IO^2.$$

Now we aim to show that this expression is $(R/2 - r)^2$, or rather $R^2/4 - Rr + r^2$. We put in known values in terms of the side lengths, and perform algebraic manipulations, deploying Heron's formula where necessary. Feuerbach's theorem follows. □

It must be admitted that this calculation does little to illuminate Feuerbach's result. We will give a more conceptual statics proof shortly.

2.6. *The location of the incenter.*

Proposition 5. *The incenter of a non-equilateral triangle lies strictly in the interior of the circle on diameter GH.*

This was presumably known to Euler [5], and a stronger version of the result was proved in [4]. Given Feuerbach's theorem, this result almost proves itself. Let N be the nine-point center, the midpoint of the segment OH, Feuerbach's tangency result yields $IN = R/2 - r$. However $OI^2 = R^2 - 2Rr$ so $OI^2 - 4IN^2 = R^2 - 4Rr + 4r^2 - R^2 + 2Rr = 2r(R - 2r)$. However Euler's formula for OI yields $2r < R$ (with equality only for equilateral triangles). Therefore I lies in the interior of the circle of Apollonius consisting of points P such that $OP = 2NP$, which is precisely the circle on diameter GH as required.

We can verify this result by an explicit calculation. Let J be the center of the circle on diameter GH so $OG = GJ = JH$. Using Apollonius's theorem on $\triangle IHO$ we obtain

$$2IN^2 + 2(\frac{3}{2}OG)^2 = OI^2 + IH^2$$

which expands to reveal that

$$HI^2 = \frac{OH^2 - (R^2 - 4r^2)}{2}.$$

Now use Stewart's theorem on $\triangle IHO$ to calculate IJ^2. We have

$$IJ^2 + 2OG^2 = \frac{OI^2 \cdot OG + IH^2 \cdot 2OG}{OH}$$

which after simple manipulation yields that

$$IJ^2 = OG^2 - \frac{2r}{3}(R - 2r) < OG^2. \tag{5}$$

The formulas for the squares of the distances from I to important points on the Euler line can be quite unwieldy, and some care has been taken to calculate these quantities in such a way that the algebraic dependence between the triangle sides and r, R and OG is produces relatively straightforward expressions. More interesting relationship can be found; for example using Stewart's theorem on $\triangle INO$ with Cevian IG we obtain

$$6IG^2 + 3OG^2 = (3R - 2r)(R - 2r).$$

3. Areal co-ordinates and Feuerbach revisited

The use of areal or volumetric co-ordinates is a special but important case of weighted systems of points. The GPAT tells us about the change of co-ordinate frames: given two reference triangles $\triangle_1$ with vertices A, B, C and $\triangle_2$ with vertices A', B', C' and points P and Q in the plane. it is natural to consider the relationship between the areal co-ordinates of a point P in the first frame (x, y, z) and those of Q in the second (x', y', z'). We assume that co-ordinates are normalized. Now GPAT tells us that

$$d^2(\triangle_{1,P}, \triangle_{2,Q}) = \sigma^2_{1,P} + PQ^2 + \sigma^2_{1,Q}.$$

The resulting formulas can be read off. The recipe which determines the square of the distance between two points given in areal co-ordinates with respect to the same reference triangle is straightforward. Suppose that P has areal co-ordinates (p_1, p_2, p_3) and Q has co-ordinates (q_1, q_2, q_3). Let $(x, y, z) = (p_1, p_2, p_3) - (q_1, q_2, q_3)$ (subtraction of 3-tuples) and let $(u, v, w) = (yz, zx, xy)$ (the Cremona transformation) then we deduce that

$$PQ^2 = -(a^2, b^2, c^2) \cdot (u, v, w).$$

Here we are using the ordinary dot product of 3-tuples. Note that (a^2, b^2, c^2) viewed as an areal co-ordinate is the symmedian point, the isogonal conjugate of G. We do not know if this observation has any significance.

A another special situation arises when $\triangle_1$ and $\triangle_2$ have the same circumcircle (perhaps they are the same triangle) and points P and Q are both on the common circle. In this case $\sigma^2_{1,P} = 0 = \sigma^2_{2,Q}$ and

$$d^2(\triangle_{1,P}, \triangle_{2,Q}) = PQ^2.$$

In the context of areal co-ordinates, we are now in a position to revisit Feuerbach's theorem and give a more conceptual statics proof which yields an interesting corollary.

3.1. *Proof of Feuerbach's theorem.* To prove Feuerbach's theorem it suffices to show that the power of I with respect to the nine-point circle is $-r(R - r)$ or equivalently that $\widehat{\sigma}^2_I = r(R - r)$ where the hat indicates that we are using the medial triangle (with vertices the midpoints of the sides of $\triangle ABC$) as the triangle of reference. Now the medial triangle is obtained by rotating the original triangle about G through π, and scaling by 1/2. Let I' denote the incenter of the medial triangle with co-ordinates $(a/2, b/2, c/2)$. The co-ordinates of G are $(s/3, s/3, s/3)$. Now I', G, I are collinear and $I'G : GI = 1 : 2$. The co-ordinates of I are therefore $(s-a, s-b, s-c)$, Next we use cyc to indicate a sum over cyclic permutations of a, b and c, and *sym* a sum over all permutations. We calculate

$$\widehat{\sigma}_I^2 = \sum_{cyc} \frac{(s-a)(s-b)c^2}{4s^2}$$
$$= \frac{s^2 \sum_{cyc} a^2 - s \sum_{sym} a^2 b + 2abcs}{4s^2}$$
$$= \frac{a^3+b^3+c^3}{4(a+b+c)} - \frac{\sum_{sym} a^2 b}{4(a+b+c)} + 2Rr.$$

However by Heron's formula

$$r^2 = \frac{(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)}$$

so

$$rR - r^2 = \frac{2abc}{4(a+b+c)} + \frac{a^3+b^3+c^3}{4(a+b+c)} - \frac{\sum_{sym} a^2 b}{4(a+b+c)} + \frac{2abc}{4(a+b+c)} = \widehat{\sigma}_I^2$$

since $abc/(a+b+c) = 2Rr$.

Corollary 6. *The areal co-ordinates of I with respect to the medial triangle are $(s-a, s-b, s-c)$, perhaps better written $(\frac{s}{2} - \frac{a}{2}, \frac{s}{2} - \frac{b}{2}, \frac{s}{2} - \frac{c}{2})$. Therefore the incenter of the reference triangle is the Nagel point of the medial triangle.*

4. The Euler-Guinand problem

In 1765 Euler [3] recovered the sides lengths a, b and c of a non-equilateral triangle from the positions of O, G and I. At the time he did not have access to Feuerbach's formula for IN^2 nor our formula (5). This extra data enables us to make light of Euler's calculations. From (5) we have $r(2R-r)$ and combining with (2) we obtain first R/r and then both R and r. Now (3) yields $a^2+b^2+c^2$ and (4) gives $a+b+c$. Finally (1) yields abc. Thus the polynomial $\triangle(x) = (X-a)(X-b)(X-c)$ can be easily recovered from the positions of O, G and I. We call this the triangle polynomial This may be an irreducible rational cubic so the construction of a, b and c by ruler and compasses may not be possible.

The actual locations of A, B and C may be determined as follows. Note that this addresses the critical remark (3) of [5]. The circumcircle of $\triangle ABC$ is known since O and R are known. Now by the GPAT we obtain the well known formula

$$\frac{0^2+b^2+c^2}{3} = AG^2 + \frac{a^2+b^2+c^2}{9}$$

so

$$AG^2 = \frac{2b^2+2c^2-a^2}{9}$$

and similarly of BG^2 and CG^2. By intersecting circles of appropriate radii centered at G with the circumcircle, we recover at most two candidate locations for each point A, B and C. Now triangle ABC is one of at most $2^3 = 8$ triangles. These can be inspected to see which ones have correct O, G and I. Note that there is only one correct triangle since AG^2, AO^2 and AI^2 are all determined.

In fact every point in the interior of the circle on diameter GH other than the nine-point center N arises as a possible location of an incenter I [4]. We give a new derivation of this result addressing the same question as [4] and [5] but in a different way.

Given any value $k \in (0, 1)$ there is a triangle such that $2r/R = k$. Choosing such a triangle, with circumradius R we observe that

$$\left(\frac{IO}{IN}\right)^2 = \frac{R^2 - 2Rr}{\left(\frac{R}{2} - r\right)^2}$$

so

$$\frac{IO}{IN} = 2\sqrt{\frac{R}{R - 2r}}. \tag{6}$$

If O and N were fixed, this would force I to lie on a circle of Apollonius with defining ratio $2\sqrt{\frac{R}{R-2r}}$. In what follows we rescale our diagrams (when convenient) so that the distance ON is fixed, so the circle on diameter GH (the orthocentroidal or critical [4] circle) can be deemed to be of fixed diameter.

Consider the configuration of Poncelet's porism for triangle ABC. We draw the circumcircle with radius R and center O, and the incenter I internally tangent to triangle ABC at three points. Now move the point A to A' elsewhere on the circumcircle and generate a new triangle $A'B'C'$ with the same incircle. We move A to A' continuously and monotonically, and observe how the configuration changes; the quantities R and r do not change but in the scaled diagram the corresponding point I' moves continuously on the given circle of Apollonius. When A' reaches B the initial configuration is recovered. Consideration of the largest angle in the moving triangle $A'B'C'$ shows that until the initial configuration is regained, the triangles formed are pairwise dissimilar, so inside the scaled version of the circle on diameter GH, the point I' moves continuously on the circle of Apollonius in a monotonic fashion. Therefore I' makes exactly one rotation round the circle of Apollonius and A' moves to B. Thus all points on this circle of Apollonius arise as possible incenters, and since the defining constant of the circle is arbitrary, all points (other than N) in the interior of the scaled circle on diameter GH arise as possible locations for I and Guinand's result is obtained [4].

Letting the equilateral triangle correspond to N, the open disk becomes a moduli space for direct similarity types of triangle. The boundary makes sense if we allow triangles to have two sides parallel with included angle 0. Some caution should be exercised however. The angles of a triangle are not a continuous function of the side lengths when one of the side lengths approaches 0. Fix A and let B tend to C by spiraling in towards it. The point I in the moduli space will move enthusiastically round and round the disk, ever closer to the boundary.

Isosceles triangles live in the moduli space as the points on the distinguished (Euler line) diameter. If the unequal side is short, I is near H, but if it is long, I is near G.

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Geoff C. Smith: Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, England.

E-mail address: G.C.Smith@bath.ac.uk