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A Gergonne Analogue of the Steiner - Lehmus Theorem

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Abstract. In this paper we prove an analogue of the famous Steiner - Lehmus theorem from the Gergonne cevian perspective.

1. Introduction

Can a theorem be both famous and infamous simultaneously? Certainly there is one such in Euclidean Geometry if the former is an indicator of a record number of correct proofs and the latter an indicator of a record number of incorrect ones. Most school students must have found it easy to prove the following: The angle bisectors of equal angles of a triangle are equal. However, not many can prove its converse theorem correctly:

Theorem 1 (Steiner-Lehmus)**.** *If two internal angle bisectors of a triangle are equal, then the triangle is isosceles.*

According to available history, in 1840 a Berlin professor named C. L. Lehmus (1780-1863) asked his contemporary Swiss geometer Jacob Steiner for a proof of Theorem 1. Steiner himself found a proof but published it in 1844. Lehmus proved it independently in 1850. The year 1842 found the first proof in print by a French mathematician [3]. Since then mathematicians and amateurs alike have been proving and re-proving the theorem. More than 80 correct proofs of the Steiner - Lehmus theorem are known. Even larger number of incorrect proofs have been offered. References [4, 5] provide extensive bibliographies on the Steiner - Lehmus theorem.

For completeness, we include a proof by M. Descube in 1880 below, recorded in [1, p.235]. The aim of this paper is to prove an analogous theorem in which we consider the equality of two Gergonne cevians. We offer two proofs of it and then consider an extension. Recall that a Gergonne cevian of a triangle is the line segment connecting a vertex to the point of contact of the opposite side with the incircle.

2. Proof of the Steiner - Lehmus theorem

Figure 1 shows the bisectors BE and CF of $\angle ABC$ and $\angle ACB$. We assume $BE = CF$. If $AB \neq AC$, let $AB < AC$, *i.e.*, $\angle ACB < \angle ABC$ or $\frac{C}{2} < \frac{B}{2}$. A

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comparison of triangles BEC with BFC shows that

$$CE > BF. \tag{1}$$

Complete the parallelogram $BFGE$. Since $EG = BF$, $\angle FGE = \frac{B}{2}$, $FG = BE = CF$ implying that $\angle FGC = \angle FCG$. But by assumption $\angle FGE = \frac{B}{2} > \angle FCE = \frac{C}{2}$. So $\angle EGC < \angle ECG$, and $CE < GE = BF$, contradicting (1).

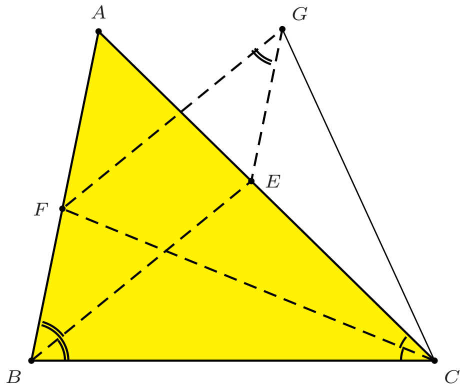

Figure 1.

Likewise, the assumption $AB > AC$ also leads to a contradiction. Hence, $AB = AC$ and $\triangle ABC$ must be isosceles.

3. The Gergonne analogue

We provide two proofs of Theorem 2 below. The first proof equates the expressions for the two Gergonne cevians to establish the result. The second one is modelled on the proof of the Steiner - Lehmus theorem in §2 above.

Theorem 2. *If two Gergonne cevians of a triangle are equal, then the triangle is isosceles.*

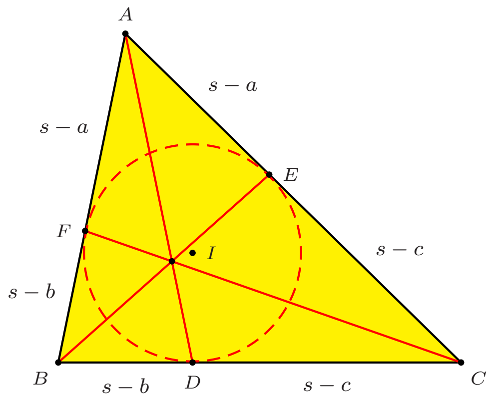

Figure 2.

3.1. *First proof.* Figure 2 shows the equal Gergonne cevians BE, CF of triangle ABC. We consider $\triangle ABE$, $\triangle ACF$ and apply the law of cosines:

$$\begin{aligned} BE^2 =& c^2 + (s-a)^2 - 2c(s-a)\cos A, \\ CF^2 =& b^2 + (s-a)^2 - 2b(s-a)\cos A. \end{aligned}$$

Equating the expressions for BE^2 and CF^2 we see that

$$2(b-c)(s-a)\cos A - (b^2-c^2) = 0$$

or

$$(b-c)\left[\frac{(-a+b+c)(b^2+c^2-a^2)}{2bc} - (b+c)\right] = 0.$$

There are two cases to consider.

(i) $b - c = 0 \Rightarrow b = c$ and triangle ABC is isosceles.

(ii) $\frac{(-a+b+c)(b^2+c^2-a^2)}{2bc} - (b+c) = 0$. This can be put, after simplification, in the form

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) = 0.$$

This clearly is an impossibility by the triangle inequality.

Therefore (i) must hold and triangle ABC is isosceles.

3.2. *Second proof.* We employ the same construction as in Figure 1 for Theorem 1. Hence we do not repeat the description here for Figure 3.

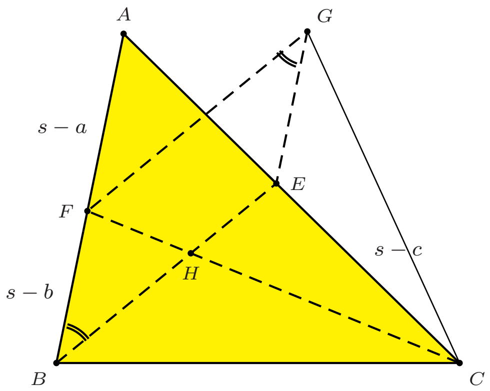

Figure 3.

If $AB \neq AC$, let $AB < AC$, *i.e.*, $c < b$, and $s - c > s - b$. As seen in the proof of Theorem 1, $\angle EBC > \angle FCB \Rightarrow CH > BH$. Since $CF = BE$, we have

$$FH < EH. \tag{2}$$

In triangles ABE and AFC, $AE = AF = s - a$, $BE = CF$ and by assumption $AB < AC$. Hence $\angle AEB < \angle AFC \Rightarrow \angle BEC > \angle BFC$ or

$$\angle HEC > \angle HFB. \tag{3}$$

Therefore, in triangles BFH and EHC, $\angle BHF = \angle EHC$ and from (3) we see that

$$\angle FBH > \angle HCE. \tag{4}$$

Triangle FGC is isosceles by construction, so $\angle FGC = \angle FCG$ or $\angle FGE + \angle EGC = \angle HCE + \angle ECG$. Because of (4) we see that $\angle EGC < \angle ECG$ or $EC < EG$, *i.e.*, $s - c < s - b \Rightarrow b < c$, contradicting the assumption.

Likewise the assumption $b > c$ would lead to a similar contradiction. Hence we must have $b = c$, and triangle ABC is isosceles.

4. An extension

Theorem 3 shows that the equality of the segments of two angle bisectors of a triangle intercepted by a Gergonne cevian itself implies that the triangle is isosceles.

Theorem 3. *The internal angle bisectors of the angles ABC and ACB of triangle ABC meet the Gergonne cevian AD at E and F respectively. If $BE = CF$, then triangle ABC is isosceles.*

Proof. We refer to Figure 4. If $AB \neq AC$, let $AB < AC$. Hence $b > c$, $s - b < s - c$ and E lies below F on AD. A simple calculation with the help of the angle bisector theorem shows that the Gergonne cevian AD lies to the left of the cevian that bisects $\angle BAC$ and hence that $\angle ADC$ is obtuse.

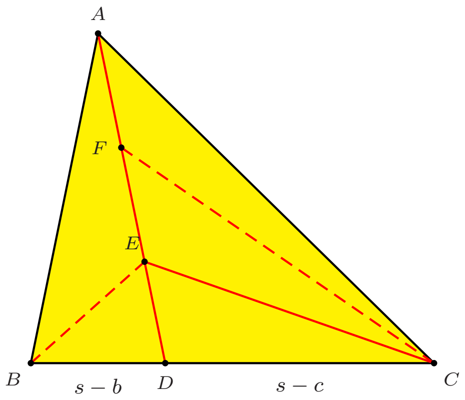

Figure 4.

By assumption, $\angle ABC > \angle ACB \Rightarrow \angle EBC > \angle FCD > \angle ECB$. Therefore,

$$CE > BE \quad \text{or} \quad CE > CF \tag{5}$$

because $BE = CF$. However, $\angle ADC = \angle EDC > \frac{\pi}{2}$ as mentioned above. Hence $\angle FEC = \angle EDC + \angle ECD > \frac{\pi}{2}$ and $\angle EFC < \frac{\pi}{2} \Rightarrow CE < CF$, contradicting (5).

Likewise, the assumption $AB > AC$ also leads to a contradiction. This means that triangle ABC must be isosceles. $\square$

5. Conclusion

The reader is invited to consider other types of analogues or extensions of the Steiner - Lehmus theorem. To conclude the discussion, we pose two problems to the reader.

(1) The external angle bisectors of $\angle ABC$ and $\angle ACB$ meet the extension of the Gergonne cevian AD at the points E and F respectively. If $BE = CF$, prove or disprove that triangle ABC is isosceles.

(2) AD is an internal cevian of triangle ABC. The internal angle bisectors of $\angle ABC$ and $\angle ACB$ meet AD at E and F respectively. Determine a necesssary and sufficient condition so that $BE = CF$ implies that triangle ABC is isosceles.

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