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On Mixtilinear Incircles and Excircles

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Abstract. A mixtilinear incircle (respectively excircle) of a triangle is tangent to two sides and to the circumcircle internally (respectively externally). We study the configuration of the three mixtilinear incircles (respectively excircles). In particular, we give easy constructions of the circle (apart from the circumcircle) tangent the three mixtilinear incircles (respectively excircles). We also obtain a number of interesting triangle centers on the line joining the circumcenter and the incenter of a triangle.

1. Preliminaries

In this paper we study two triads of circles associated with a triangle, the mixtilinear incircles and the mixtilinear excircles. For an introduction to these circles, see [4] and §§2, 3 below. In this section we collect some important basic results used in this paper.

Proposition 1 (d'Alembert's Theorem [1]). *Let $O_1(r_1)$, $O_2(r_2)$, $O_3(r_3)$ be three circles with distinct centers. According as $\varepsilon = +1$ or -1, denote by $A_{1\varepsilon}$, $A_{2\varepsilon}$, $A_{3\varepsilon}$ respectively the insimilicenters or exsimilicenters of the pairs of circles $((O_2),\ (O_3))$, $((O_3),\ (O_1))$, and $((O_1),\ (O_2))$. For $\varepsilon_i = \pm 1$, $i = 1, 2, 3$, the points $A_{1\varepsilon_1}$, $A_{2\varepsilon_2}$ and $A_{3\varepsilon_3}$ are collinear if and only if $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.* See Figure 1.

The insimilicenter and exsimilicenter of two circles are respectively their internal and external centers of similitude. In terms of one-dimensional barycentric coordinates, these are the points

$$\text{ins}(O_1(r_1),\ O_2(r_2)) = \frac{r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 + r_2}, \tag{1}$$

$$\text{exs}(O_1(r_1),\ O_2(r_2)) = \frac{-r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 - r_2}. \tag{2}$$

Proposition 2. *Let $O_1(r_1)$, $O_2(r_2)$, $O_3(r_3)$ be three circles with noncollinears centers. For $\varepsilon = \pm 1$, let $O_\varepsilon(r_\varepsilon)$ be the Apollonian circle tangent to the three circles, all externally or internally according as $\varepsilon = +1$ or -1. Then the Monge line containing the three exsimilicenters $\text{exs}(O_2(r_2), O_3(r_3))$, $\text{exs}(O_3(r_3), O_1(r_1))$, and $\text{exs}(O_1(r_1), O_2(r_2))$ is the radical axis of the Apollonian circles (O_+) and (O_-).* See Figure 1.

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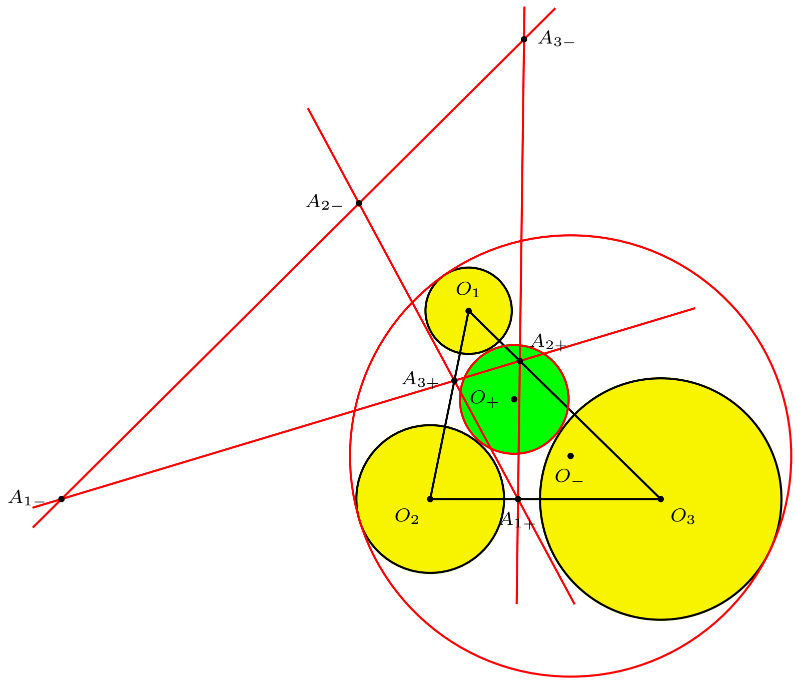

Figure 1.

Lemma 3. *Let BC be a chord of a circle $O(r)$. Let $O_1(r_1)$ be a circle that touches BC at E and intouches the circle (O) at D. The line DE passes through the midpoint A of the arc BC that does not contain the point D. Furthermore, $AD \cdot AE = AB^2 = AC^2$.*

Proposition 4. *The perspectrix of the circumcevian triangle of P is the polar of P with respect to the circumcircle.*

Let ABC be a triangle with circumcenter O and incenter I. For the circumcircle and the incircle,

$$\text{ins}((O),\ (I)) = \frac{r \cdot O + R \cdot I}{R + r} = X_{55},$$
$$\text{exs}((O),\ (I)) = \frac{-r \cdot O + R \cdot I}{R - r} = X_{56}.$$

in the notations of [3]. We also adopt the following notations.

A_0	point of tangency of incircle with BC
A_1	intersection of AI with the circumcircle
A_2	antipode of A_1 on the circumcircle

Similarly define B_0, B_1, B_2, C_0, C_1 and C_2. Note that
(i) $A_0B_0C_0$ is the intouch triangle of ABC,
(ii) $A_1B_1C_1$ is the circumcevian triangle of the incenter I,

(iii) $A_2B_2C_2$ is the medial triangle of the excentral triangle, *i.e.*, A_2 is the midpoint between the excenters I_b and I_c. It is also the midpoint of the arc BAC of the circumcircle.

2. Mixtilinear incircles

The A-mixtilinear incircle is the circle (O_a) that touches the rays AB and AC at C_a and B_a and the circumcircle (O) internally at X. See Figure 2. Define the B- and C-mixtilinear incircles (O_b) and (O_c) analogously, with points of tangency Y and Z with the circumcircle. See [4]. We begin with an alternative proof of the main result of [4].

Proposition 5. *The lines AX, BY, CZ are concurrent at* $\mathrm{exs}((O),\ (I))$.

Proof. Since $A = \mathrm{exs}((O_a),\ (I))$ and $X = \mathrm{exs}((O),\ (O_a))$, the line AX passes through $\mathrm{exs}((O),\ (I))$ by d'Alembert's Theorem. For the same reason, BY and CZ also pass through the same point. □

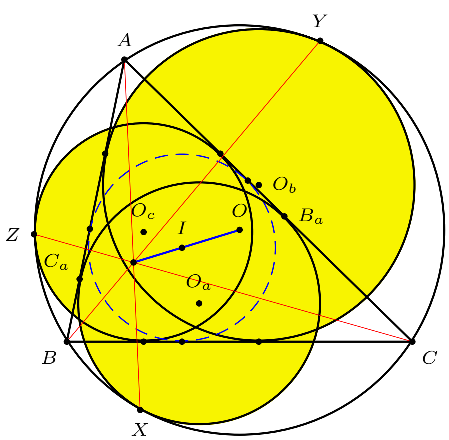

Figure 2

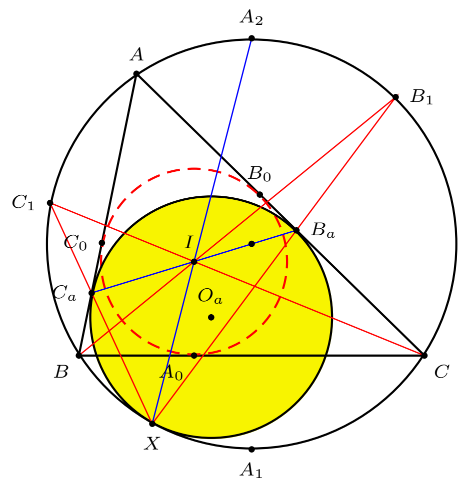

Figure 3

Lemma 6. (1) *I is the midpoint of B_aC_a.*
(2) *The A-mixtilinear incircle has radius $r_a = \frac{r}{\cos^2 \frac{A}{2}}$.*
(3) *XI bisects angle BXC.*
See Figure 3.

Consider the radical axis ℓ_a of the mixtilinear incircles (O_b) and (O_c).

Proposition 7. *The radical axis ℓ_a contains*
(1) *the midpoint A_1 of the arc BC of (O) not containing the vertex A,*
(2) *the midpoint M_a of IA_0, where A_0 is the point of tangency of the incircle with the side BC.*

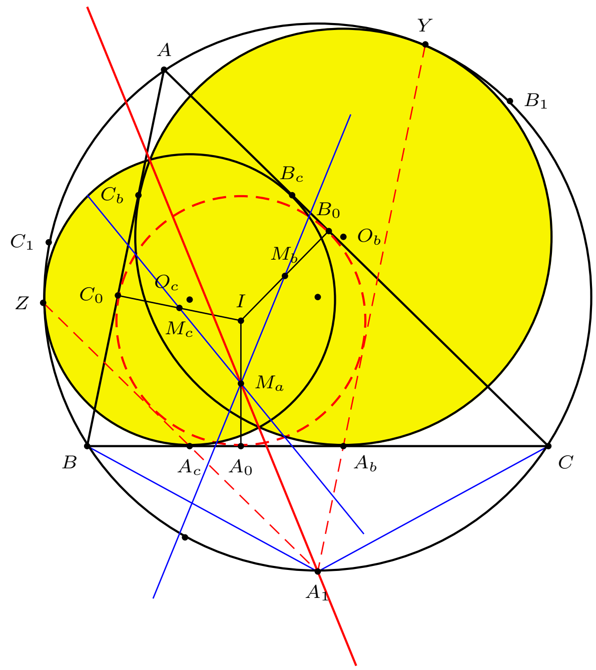

Figure 4.

Proof. (1) By Lemma 3, Z, A_c and A_1 are collinear, so are Y, A_b, A_1. Also, $A_1A_c \cdot A_1Z = A_1B^2 = A_1C^2 = A_1A_b \cdot A_1Y$. This shows that A_1 is on the radical axis of (O_b) and (O_c).

(2) Consider the incircle (I) and the B-mixtilinear incircle (O_b) with common ex-tangents BA and BC. Since the circle (I) touches BA and BC at C_0 and A_0, and the circle (O_b) touches the same two lines at C_b and A_b, the radical axis of these two circles is the line joining the midpoints of C_bC_0 and A_bA_0. Since A_b, I, C_b are collinear, the radical axis of (I) and (O_b) passes through the midpoints of IA_0 and IC_0. Similarly, the radical axis of (I) and (O_c) passes through the midpoints of IA_0 and IB_0. It follows that the midpoint of IA_0 is the common point of these two radical axes, and is therefore a point on the radical axis of (O_b) and (O_c). □

Theorem 8. *The radical center of (O_a), (O_b), (O_c) is the point J which divides OI in the ratio*

$$OJ : JI = 2R : -r.$$

Proof. By Proposition 7, the radical axis of (O_b) and (O_c) is the line A_1M_a. Let M_b and M_c be the midpoints of IB_0 and IC_0 respectively. Then the radical axes of (O_c) and (O_a) is the line B_1M_b, and that of (O_a) and (O_b) is the line C_1M_c. Note that the triangles $A_1B_1C_1$ and $M_aM_bM_c$ are directly homothetic. Since $A_1B_1C_1$ is inscribed in the circle $O(R)$ and $M_aM_bM_c$ in inscribed in the circle $I(\frac{r}{2})$, the homothetic center of the triangles is the point J which divides the segment OI in

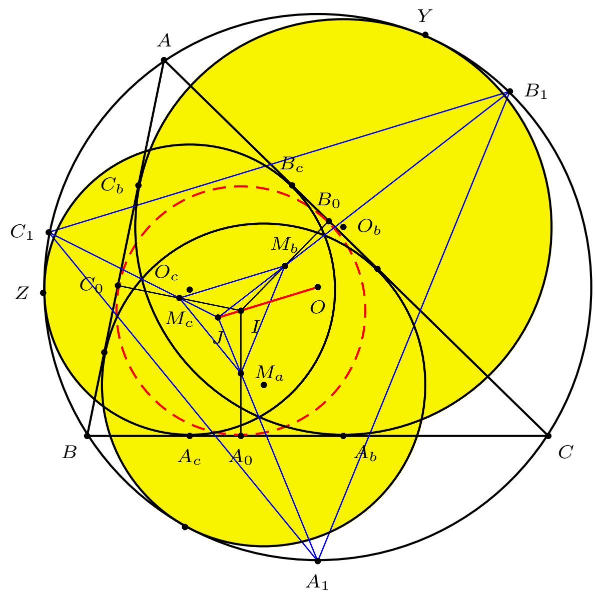

Figure 5.

the ratio

$$OJ : JI = R : -\frac{r}{2} = 2R : -r. \tag{3}$$

See Figure 5. □

Remark. Let T be the homothetic center of the excentral triangle $I_aI_bI_c$ and the intouch triangle $A_0B_0C_0$. This is the triangle center X_{57} in [3]. Since the excentral triangle has circumcenter I', the reflection of I in O,

$$OT : TI' = 2R : -r.$$

Comparison with (3) shows that J is the reflection of T in O.

3. The mixtilinear excircles

The mixtilinear excircles are defined analogously to the mixtilinear incircles, except that the tangencies with the circumcircle are external. The A-mixtilinear excircle (O'_a) can be easily constructed by noting that the polar of A passes through the excenter I_a; similarly for the other two mixtilinear excircles. See Figure 6.

Theorem 9. *If the mixtilinear excircles touch the circumcircle at X', Y', Z' respectively, the lines AX', BY', CZ' are concurrent at $\text{ins}((O), (I))$.*

Theorem 10. *The radical center of the mixtilinear excircles is the reflection of J in O, where J is the radical center of the mixtilinear incircles.*

Proof. The polar of A with respect to (O'_a) passes through the excenter I_a. Similarly for the other two polars of B with respect to (O'_b) and C with respect to (O'_c).

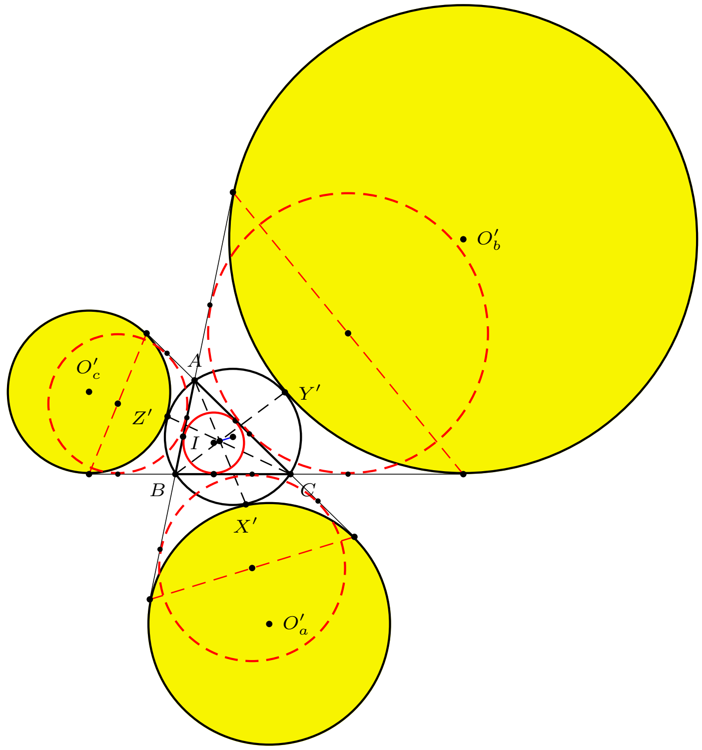

Figure 6.

Let $A_3B_3C_3$ be the triangle bounded by these three polars. Let A_4, B_4, C_4 be the midpoints of A_0A_3, B_0B_3, C_0C_3 respectively. See Figure 7.

Since $I_aI_bA_3I_c$ is a parallelogram, and A_2 is the midpoint of I_bI_c, it is also the midpoint of A_3I_a. Since B_3C_3 is parallel to I_bI_c (both being perpendicular to the bisector AA_1), I_a is the midpoint of B_3C_3. Similarly, I_b, I_c are the midpoints of C_3A_3 and A_3B_3, and the excentral triangle is the medial triangle of $A_3B_3C_3$. Note also that I is the circumcenter of $A_3B_3C_3$ (since it lies on the perpendicular bisectors of its three sides). This is homothetic to the intouch triangle $A_0B_0C_0$ at I, with ratio of homothety $-\frac{r}{4R}$.

If A_4 is the midpoint of A_0A_3, similarly for B_4 and C_4, then $A_4B_4C_4$ is homothetic to $A_3B_3C_3$ with ratio $\frac{4R-r}{4R}$.

We claim that $A_4B_4C_4$ is homothetic to $A_2B_2C_2$ at a point J', which is the radical center of the mixtilinear excircles. The ratio of homothety is clearly $\frac{4R-r}{R}$.

Consider the isosceles trapezoid $B_0C_0B_3C_3$. Since B_4 and C_4 are the midpoints of the diagonals B_0B_3 and C_0C_3, and B_3C_3 contains the points of tangency B_a, C_a of the circle (O'_a) with AC and AB, the line B_4C_4 also contains the midpoints of B_0B_a and C_0C_a, which are on the radical axis of (I) and (O'_a). This means that the line B_4C_4 is the radical axis of (I) and (O'_a).

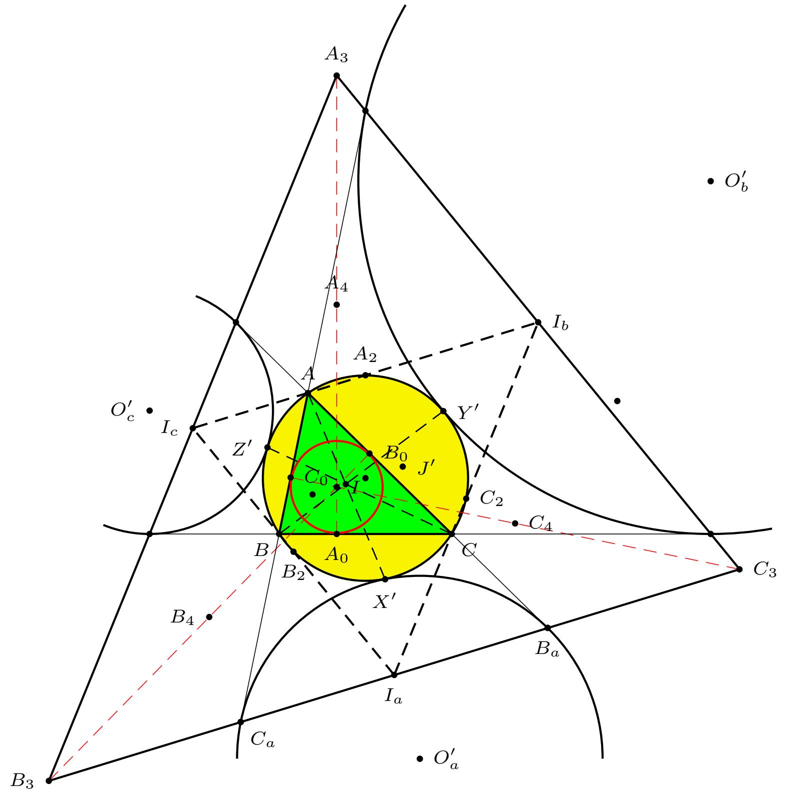

Figure 7.

It follows that A_4 is on the radical axis of (O'_b) and (O'_c). Clearly, A_2 also lies on the same radical axis. This means that the radical center of the mixtilinear excircles is the homothetic center of the triangles $A_2B_2C_2$ and $A_4B_4C_4$. Since these two triangles have circumcenters O and I, and circumradii R and $\frac{4R-r}{2}$, the homothetic center is the point J' which divides IO in the ratio

$$J'I : J'O = 4R - r : 2R. \tag{4}$$

Equivalently, $OJ' : J'I = -2R : 4R - r$. The reflection of J' in O divides OI in the ratio $2R : -r$. This is the radical center J of the mixtilinear incircles. □

4. Apollonian circles

Consider the circle $O_5(r_5)$ tangent internally to the mixtilinear incircles at A_5, B_5, C_5 respectively. We call this the inner Apollonian circle of the mixtilinear incircles. It can be constructed from J since A_5 is the second intersection of the line JX with the A-mixtilinear incircle, and similarly for B_5 and C_5. See Figure 8. Theorem 11 below gives further details of this circle, and an easier construction.

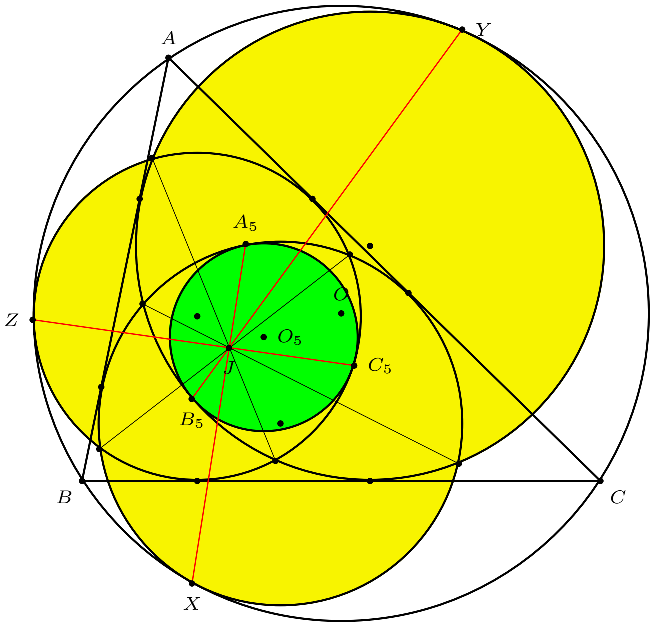

Figure 8.

Theorem 11. (1) *Triangles* $A_5B_5C_5$ *and* ABC *are perspective at* $\text{ins}((O),\ (I))$.

(2) *The inner Apollonian circle of the mixtilinear incircles has center* O_5 *dividing the segment* OI *in the ratio* $4R : r$ *and radius* $r_5 = \frac{3Rr}{4R+r}$.

Proof. (1) Let $P = \text{exs}((O_5),(I))$, and $Q_a = \text{exs}((O_5),(O'_a))$. The following triples of points are collinear by d'Alembert's Theorem:
(1) A, A_5, P from the circles (O_5), (I), (O_a);
(2) A, Q_a, A_5 from the circles (O_5), (O_a), (O'_a);
(3) A, Q_a, P from the circles (O_5), (I), (O'_a);
(4) A, X', $\text{ins}((O),(I))$ from the circles (O), (I), (O'_a).

See Figure 9. Therefore the lines AA_5 contains the points P and $\text{ins}((O),(I))$ (along with Q_a, X'). For the same reason, the lines BB_5 and CC_5 contain the same two points. It follows that P and $\text{ins}((O),(I))$ are the same point, which is common to AA_5, BB_5 and CC_5.

(2) Now we compute the radius r_5 of the circle (O_5). From Theorem 8, $OJ : JI = 2R : -r$. As $J = \text{exs}((O),(O_5))$, we have $OJ : JO_5 = R : -r_5$. It follows that $OJ : JI : JO_5 = 2R : -r : -2r_5$, and

$$\frac{OO_5}{OI} = \frac{2(R - r_5)}{2R - r}. \tag{5}$$

Since $P = \text{exs}((O_5),(I)) = \text{ins}((O),(I))$, it is also $\text{ins}((O),(O_5))$. Thus, $OP : PO_5 : PI = R : r_5 : r$, and

$$\frac{OO_5}{OI} = \frac{R + r_5}{R + r}. \tag{6}$$

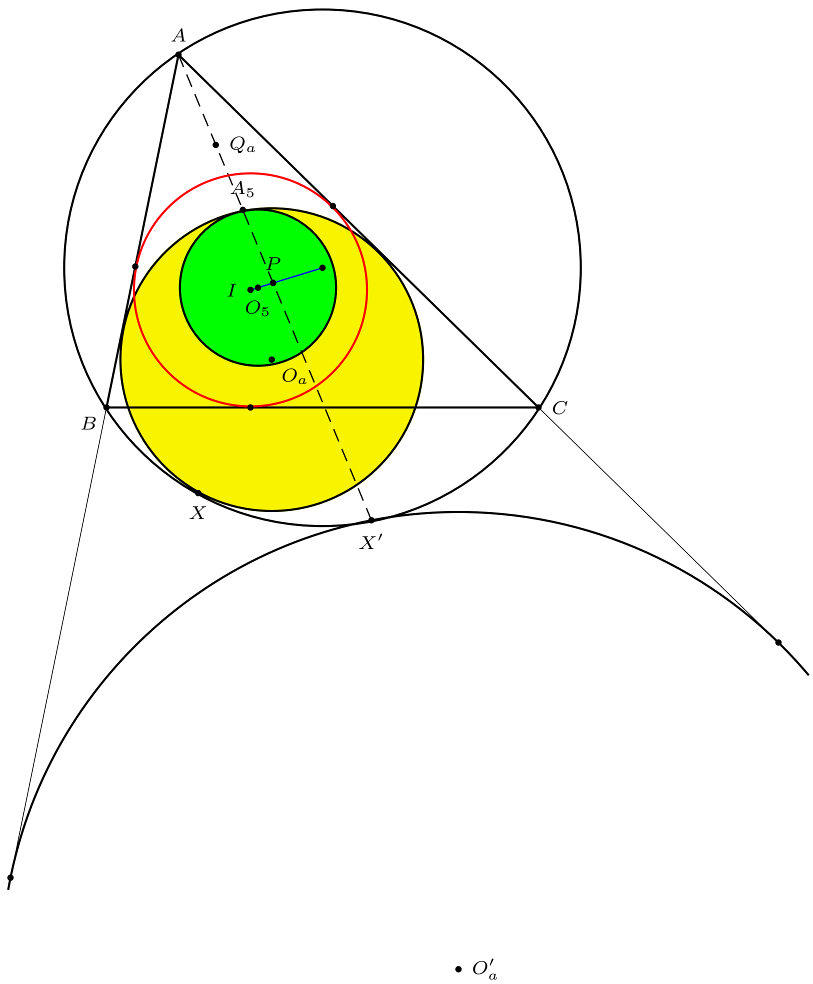

Figure 9.

Comparing (5) and (6), we easily obtain $r_5 = \frac{3Rr}{4R+r}$. Consequently, $\frac{OO_5}{OI} = \frac{4R}{4R+r}$ and O_5 divides OI in the ratio $OO_5 : O_5I = 4R : r$. □

The outer Apollonian circle of the mixtilinear excircles can also be constructed easily. If the lines $J'X'$, $J'Y'$, $J'Z'$ intersect the mixtilinear excircles again at A_6, B_6, C_6 respectively, then the circle $A_6B_6C_6$ is tangent internally to each of the mixtilinear excircles. Theorem 12 below gives an easier construction without locating the radical center.

Theorem 12. (1) *Triangles* $A_6B_6C_6$ *and* ABC *are perspective at* $\text{exs}((O),(I))$.

(2) *The outer Apollonian circle of the mixtilinear excircles has center* O_6 *dividing the segment* OI *in the ratio* $-4R : 4R + r$ *and radius* $r_6 = \frac{R(4R-3r)}{r}$.

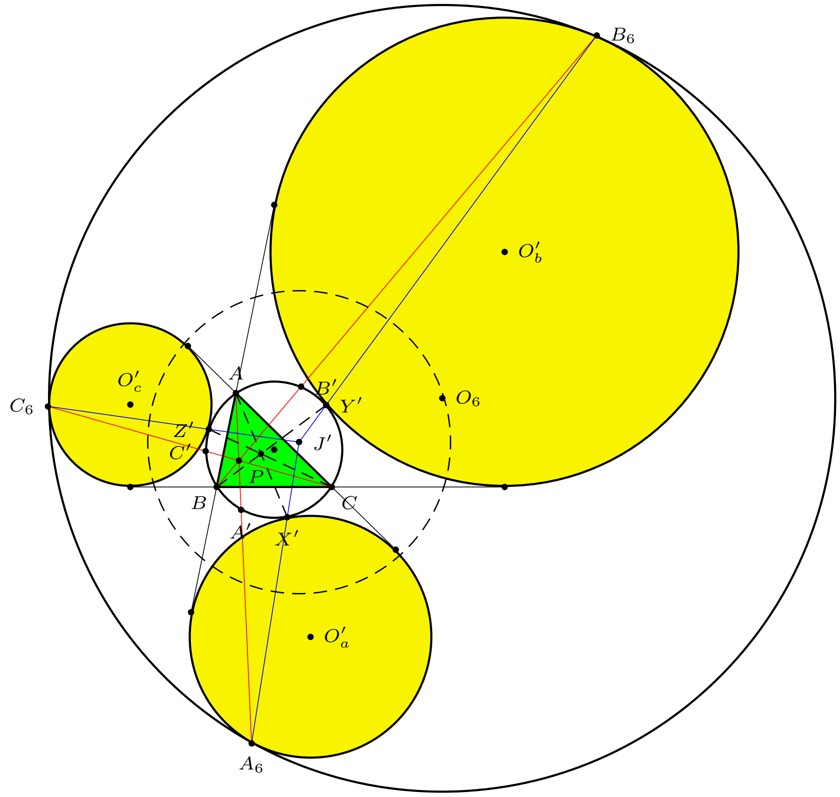

Figure 10.

Proof. (1) Since $A_6 = \text{exs}((O'_a), (O_6))$ and $A = \text{exs}((I), (O'_a))$, by d'Alembert's Theorem, the line AA_6 passes through $P = \text{exs}((O_6), (I))$. For the same reason BB_6 and CC_6 pass through the same point, the triangles $A_6B_6C_6$ and ABC are perspective at $P = \text{exs}((I), (O_6))$. See Figure 10.

By Proposition 2, AB, $X'Y'$, A_6B_6, and $O'_aO'_b$ concur at $S = \text{exs}((O'_a), (O'_b))$ on the radical axis of (O) and (O_6). Now: $SA \cdot SB = SX' \cdot SY' = SA_6 \cdot SB_6$. Let AA_6, BB_6, CC_6 intersect the circumcircle (O) at A', B', C' respectively. Since $SA \cdot SB = SA_6 \cdot SB_6$, ABA_6B_6 is cyclic. Since $\angle BAA_6 = \angle BB'A' = \angle BB_6A_6$, $A'B'$ is parallel to A_6B_6. Similarly, $B'C'$ and $C'A'$ are parallel to B_6C_6 and C_6A_6 respectively. Therefore the triangles $A'B'C'$ and $A_6B_6C_6$ are directly homothetic, and the center of homothety is $P = \text{exs}((O), (O_6))$.

Since $P = \text{exs}((O), (O_6)) = \text{exs}((I), (O_6))$, it is also $\text{exs}((O), (I))$, and $PI : PO = r : R$.

(2) Since $A_6 = \text{exs}((O_6), (O'_a))$ and $X' = \text{ins}((O'_a), (O))$, by d'Alembert's Theorem, the line A_6X' passes through $K = \text{ins}((O), (O_6))$. For the same reason, B_6Y' and C_6Z' pass through the same point K.

We claim that K is the radical center J' of the mixtilinear excircles. Since $SX' \cdot SY' = SA_6 \cdot SB_6$, we conclude that $X'A_6Y'B_6$ is cyclic, and $KX' \cdot KA_6 = KY' \cdot KB_6$. Also, $Y'B_6C_6Z'$ is cyclic, and $KY' \cdot KB_6 = KZ' \cdot KC_6$. It follows that

$$KX' \cdot KA_6 = KY' \cdot KB_6 = KZ' \cdot KC_6,$$

showing that $K = \text{ins}((O), (O_6))$ is the radical center J' of the mixtilinear excircles. Hence, $J'O : J'O_6 = R : -r_6$. Note also $PO : PO_6 = R : r_6$. Then, we have the following relations.

$$\begin{aligned} OJ' : IO =& 2R : 2R - r, \\ J'O_6 : IO =& 2r_6 : 2R - r, \\ OO_6 : IO =& r_6 - R : R - r. \end{aligned}$$

Since $OJ' + J'O_6 = OO_6$, we have

$$\frac{2R}{2R - r} + \frac{2r_6}{2R - r} = \frac{r_6 - R}{R - r}.$$

This gives: $r_6 = \frac{R(4R-3r)}{r}$. Since $K = \text{ins}((O), (O_6)) = \frac{r_6 \cdot O + R \cdot O_6}{R + r_6}$ and $J' = \frac{(4R-r)O - 2R \cdot I}{2R-r}$ are the same point, we obtain $O_6 = \frac{(4R+r)O - 4R \cdot I}{r}$. □

Remark. The radical circle of the mixtilinear excircles has center J' and radius $\frac{R}{2R-r}\sqrt{(4R+r)(4R-3r)}$.

Corollary 13. $IO_5 \cdot IO_6 = IO^2$.

5. The cyclocevian conjugate

Let P be a point in the plane of triangle ABC, with traces X, Y, Z on the sidelines BC, CA, AB respectively. Construct the circle through X, Y, Z. This circle intersects the sidelines BC, CA, AB again at points X', Y', Z'. A simple application of Ceva's Theorem shows that the AX', BY', CZ' are concurrent. The intersection point of these three lines is called the cyclocevian conjugate of P. See, for example, [2, p.226]. We denote this point by P°. Clearly, $(P^\circ)^\circ = P$. For example, the centroid and the orthocenters are cyclocevian conjugates, and Gergonne point is the cyclocevian conjugate of itself.

We prove two interesting locus theorems.

Theorem 14. *The locus of Q whose circumcevian triangle with respect to XYZ is perspective to $X'Y'Z'$ is the line PP°. For Q on PP°, the perspector is also on the same line.*

Proof. Let Q be a point on the line PP°. By Pascal's Theorem for the six points, X', B', X, Y', A', Y, the intersections of the lines $X'A'$ and $Y'B'$ lies on the line connecting Q to the intersection of XY' and $X'Y$, which according to Pappus' theorem (for Y, Z, Z' and C, Y, Y'), lies on PP°. Since Q lies on PP°, it follows that $X'A'$, $Y'B'$, and PP° are concurrent. Similarly, $A'B'C'$ and $X'Y'Z'$ are perspective at a point on PP°. The same reasoning shows that if $A'B'C'$ and $X'Y'Z'$ are perspective at a point S, then both Q and S lie on the line connecting the intersections $XY' \cap X'Y$ and $YZ' \cap Y'Z$, which is the line PP°. □

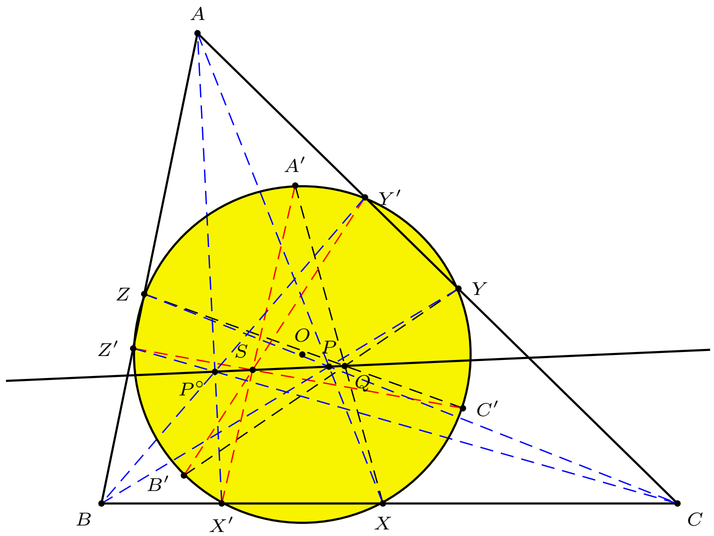

Figure 11.

For example, if $P = G$, then $P^\circ = H$. The line PP° is the Euler line. If $Q = O$, the circumcenter, then the circumcevian triangle of O (with respect to the medial triangle) is perspective with the orthic triangle at the nine-point center N.

Theorem 15. *The locus of Q whose circumcevian triangle with respect to XYZ is perspective to ABC is the line PP°.*

Proof. First note that

$$\begin{aligned}\frac{\sin Z'X'A'}{\sin A'X'Y'} &= \frac{\sin Z'XA'}{\sin A'YY'} = \frac{\sin Z'ZA'}{\sin ZAA'} \cdot \frac{\sin A'AY}{\sin A'YY'} \cdot \frac{\sin ZAA'}{\sin A'AY} \\ &= \frac{AA'}{ZA'} \cdot \frac{A'Y}{AA'} \cdot \frac{\sin BAX}{\sin XAC} = \frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin BAA'}{\sin A'AC}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\sin Z'X'A'}{\sin A'X'Y'} \cdot \frac{\sin X'Y'B'}{\sin B'Y'Z'} \cdot \frac{\sin Y'Z'C'}{\sin C'Z'X'} \\ =&\left(\frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin ZYB'}{\sin B'YX} \cdot \frac{\sin XZC'}{\sin C'ZY}\right)\left(\frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA}\right) \\ =&\frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA}.\end{aligned}$$

Therefore, $A'B'C'$ is perspective with ABC if and only if it is perspective with $X'Y'Z'$. By Theorem 14, the locus of Q is the line PP°. □

6. Some further results

We establish some further results on the mixtilinear incircles and excircles relating to points on the line OI.

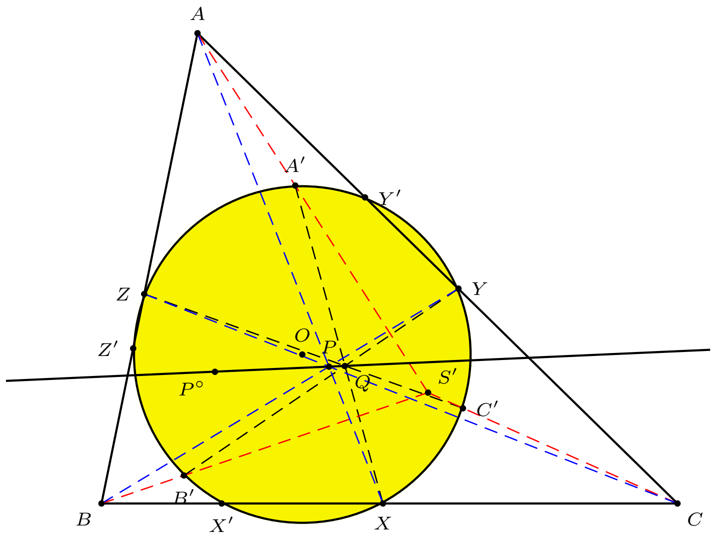

Figure 12.

Theorem 16. *The line OI is the locus of P whose circumcevian triangle with respect to $A_1B_1C_1$ is perspective with XYZ.*

Proof. We first show that $D = B_1Y \cap C_1Z$ lies on the line AA_1. Applying Pascal's Theorem to the six points Z, B_1, B_2, Y, C_1, C_2 on the circumcircle, the points $D = B_1Y \cap C_1Z$, $I = B_2Y \cap C_2Z$, and $B_1C_2 \cap B_2C_1$ are collinear. Since B_1C_2 and B_2C_1 are parallel to the bisector AA_1, it follows that D lies on AA_1. See Figure 13.

Now, if $E = C_1Z \cap A_1X$ and $F = A_1X \cap B_1Y$, the triangle DEF is perspective with $A_1B_1C_1$ at I. Equivalently, $A_1B_1C_1$ is the circumcevian triangle of I with respect to triangle DEF. Triangle XYZ is formed by the second intersections of the circumcircle of $A_1B_1C_1$ with the side lines of DEF. By Theorem 14, the locus of P whose circumcevian triangle with respect to $A_1B_1C_1$ is perspective with XYZ is a line through I. This is indeed the line IO, since O is one such point. (The circumcevian triangle of O with respect to $A_1B_1C_1$ is perspective with XYZ at I). □

Remark. If P divides OI in the ratio $OP : PI = t : 1 - t$, then the perspector Q divides the same segment in the ratio $OQ : QI = (1 + t)R : -2tr$. In particular, if $P = \text{ins}((O),\ (I))$, this perspector is T, the homothetic center of the excentral and intouch triangles.

Corollary 17. *The line OI is the locus of Q whose circumcevian triangle with respect to $A_1B_1C_1$ (or XYZ) is perspective with DEF.*

Proposition 18. *The triangle $A_2B_2C_2$ is perspective*
(1) *with XYZ at the incenter I,*
(2) *with $X'Y'Z'$ at the centroid of the excentral triangle.*

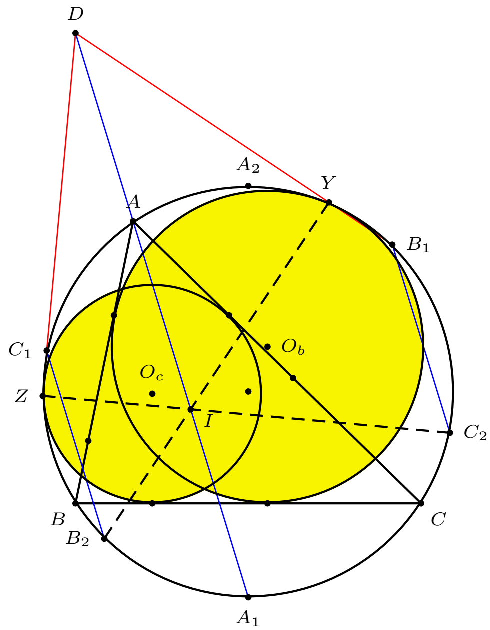

Figure 13.

Proof. (1) follows from Lemma 6(b).

(2) Referring to Figure 7, the excenter I_a is the midpoint B_aC_a. Therefore $X'I_a$ is a median of triangle $X'B_aC_a$, and it intersects B_2C_2 at its midpoint X''. Since $A_2B_2I_aC_2$ is a parallelogram, A_2, X', X'' and I_a are collinear. In other words, the line A_2X' contains a median, hence the centroid, of the excentral triangle. So do B_2Y' and C_2Z. □

Let A_7 be the second intersection of the circumcircle with the line ℓ_a, the radical axis of the mixtilinear incircles (O_b) and (O_c). Similarly define B_7 and C_7. See Figure 14.

Theorem 19. *The triangles $A_7B_7C_7$ and XYZ are perspective at a point on the line OI.*

Remark. This point divides OI in the ratio $4R - r : -4r$ and has homogeneous barycentric coordinates

$$\left(\frac{a(b+c-5a)}{b+c-a} : \frac{b(c+a-5b)}{c+a-b} : \frac{c(a+b-5c)}{a+b-c}\right).$$

7. Summary

We summarize the triangle centers on the OI-line associated with mixtilinear incircles and excircles by listing, for various values of t, the points which divide OI in the ratio $R : tr$. The last column gives the indexing of the triangle centers in [2, 3].

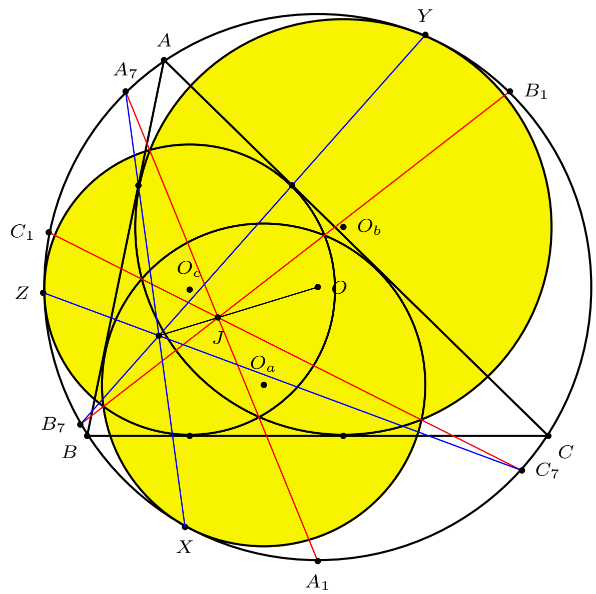

Figure 14.

t	first barycentric coordinate		X_n
1	$a^2(s-a)$	$\text{ins}((O),\ (I))$ perspector of ABC and $X'Y'Z'$ perspector of ABC and $A_5B_5C_5$	X_{55}
-1	$\frac{a^2}{s-a}$	$\text{exs}((O),\ (I))$ perspector of ABC and XYZ perspector of ABC and $A_6B_6C_6$	X_{56}
$-\frac{2R}{2R+r}$	$\frac{a}{s-a}$	homothetic center of excentral and intouch triangles	X_{57}
$-\frac{1}{2}$	$a^2(b^2+c^2-a^2-4bc)$	radical center of mixtilinear incircles	X_{999}
$\frac{1}{4}$	$a^2(b^2+c^2-a^2+8bc)$	center of Apollonian circle of mixtilinear incircles	
$-\frac{4R-r}{2r}$	$a^2f(a,b,c)$	radical center of mixtilinear excircles	
$-\frac{4R+r}{4r}$	$a^2g(a,b,c)$	center of Apollonian circle of mixtilinear excircles	
$-\frac{4R}{r}$	$a(3a^2-2a(b+c)-(b-c)^2)$	centroid of excentral triangle	X_{165}
$-\frac{4R}{4R-r}$	$\frac{a(b+c-5a)}{b+c-a}$	perspector of $A_7B_7C_7$ and XYZ	

The functions f and g are given by

$$f(a,b,c) = a^4 - 2a^3(b+c) + 10a^2bc + 2a(b+c)(b^2-4bc+c^2) - (b-c)^2(b^2+4bc+c^2),$$
$$g(a,b,c) = a^5 - a^4(b+c) - 2a^3(b^2-bc+c^2) + 2a^2(b+c)(b^2-5bc+c^2) + a(b^4-2b^3c+18b^2c^2-2bc^3+c^4) - (b-c)^2(b+c)(b^2-8bc+c^2).$$

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