

## On Mixtilinear Incircles and Excircles

Khoa Lu Nguyen and Juan Carlos Salazar

**Abstract.** A mixtilinear incircle (respectively excircle) of a triangle is tangent to two sides and to the circumcircle internally (respectively externally). We study the configuration of the three mixtilinear incircles (respectively excircles). In particular, we give easy constructions of the circle (apart from the circumcircle) tangent the three mixtilinear incircles (respectively excircles). We also obtain a number of interesting triangle centers on the line joining the circumcenter and the incenter of a triangle.

### 1. Preliminaries

In this paper we study two triads of circles associated with a triangle, the mixtilinear incircles and the mixtilinear excircles. For an introduction to these circles, see [4] and §§2, 3 below. In this section we collect some important basic results used in this paper.

**Proposition 1** (d’Alembert’s Theorem [1]). *Let  $O_1(r_1)$ ,  $O_2(r_2)$ ,  $O_3(r_3)$  be three circles with distinct centers. According as  $\varepsilon = +1$  or  $-1$ , denote by  $A_{1\varepsilon}$ ,  $A_{2\varepsilon}$ ,  $A_{3\varepsilon}$  respectively the insimilicenters or exsimilicenters of the pairs of circles  $((O_2), (O_3))$ ,  $((O_3), (O_1))$ , and  $((O_1), (O_2))$ . For  $\varepsilon_i = \pm 1$ ,  $i = 1, 2, 3$ , the points  $A_{1\varepsilon_1}$ ,  $A_{2\varepsilon_2}$  and  $A_{3\varepsilon_3}$  are collinear if and only if  $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ . See Figure 1.*

The insimilicenter and exsimilicenter of two circles are respectively their internal and external centers of similitude. In terms of one-dimensional barycentric coordinates, these are the points

$$\text{ins}(O_1(r_1), O_2(r_2)) = \frac{r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 + r_2}, \quad (1)$$

$$\text{exs}(O_1(r_1), O_2(r_2)) = \frac{-r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 - r_2}. \quad (2)$$

**Proposition 2.** *Let  $O_1(r_1)$ ,  $O_2(r_2)$ ,  $O_3(r_3)$  be three circles with noncollinear centers. For  $\varepsilon = \pm 1$ , let  $O_\varepsilon(r_\varepsilon)$  be the Apollonian circle tangent to the three circles, all externally or internally according as  $\varepsilon = +1$  or  $-1$ . Then the Monge line containing the three exsimilicenters  $\text{exs}(O_2(r_2), O_3(r_3))$ ,  $\text{exs}(O_3(r_3), O_1(r_1))$ , and  $\text{exs}(O_1(r_1), O_2(r_2))$  is the radical axis of the Apollonian circles  $(O_+)$  and  $(O_-)$ . See Figure 1.*

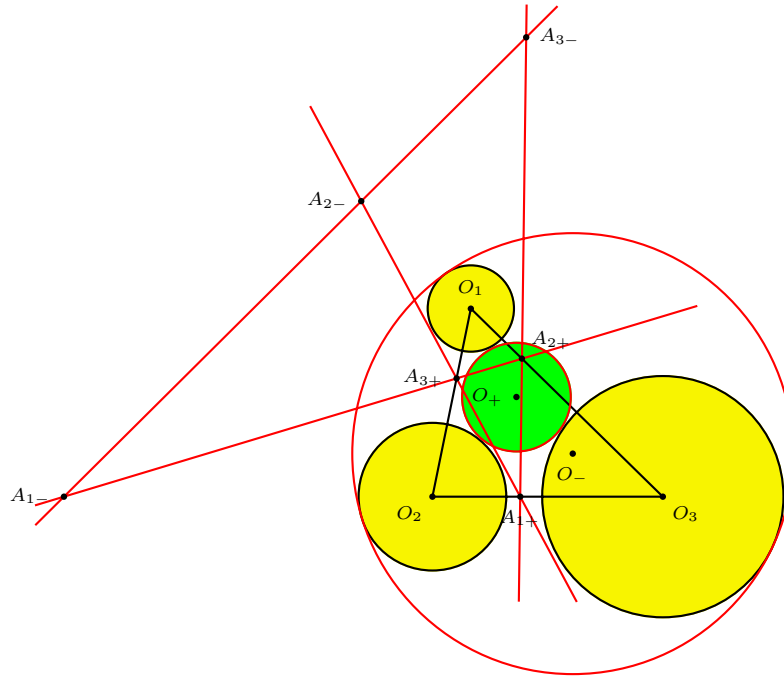


Figure 1.

**Lemma 3.** Let  $BC$  be a chord of a circle  $O(r)$ . Let  $O_1(r_1)$  be a circle that touches  $BC$  at  $E$  and intouches the circle  $(O)$  at  $D$ . The line  $DE$  passes through the midpoint  $A$  of the arc  $BC$  that does not contain the point  $D$ . Furthermore,  $AD \cdot AE = AB^2 = AC^2$ .

**Proposition 4.** The perspectrix of the circumcevian triangle of  $P$  is the polar of  $P$  with respect to the circumcircle.

Let  $ABC$  be a triangle with circumcenter  $O$  and incenter  $I$ . For the circumcircle and the incircle,

$$\begin{aligned} \text{ins}((O), (I)) &= \frac{r \cdot O + R \cdot I}{R + r} = X_{55}, \\ \text{exs}((O), (I)) &= \frac{-r \cdot O + R \cdot I}{R - r} = X_{56}. \end{aligned}$$

in the notations of [3]. We also adopt the following notations.

- $A_0$  point of tangency of incircle with  $BC$
- $A_1$  intersection of  $AI$  with the circumcircle
- $A_2$  antipode of  $A_1$  on the circumcircle

Similarly define  $B_0, B_1, B_2, C_0, C_1$  and  $C_2$ . Note that

- (i)  $A_0B_0C_0$  is the intouch triangle of  $ABC$ ,
- (ii)  $A_1B_1C_1$  is the circumcevian triangle of the incenter  $I$ ,

(iii)  $A_2B_2C_2$  is the medial triangle of the excentral triangle, *i.e.*,  $A_2$  is the midpoint between the excenters  $I_b$  and  $I_c$ . It is also the midpoint of the arc  $BAC$  of the circumcircle.

## 2. Mixtilinear incircles

The  $A$ -mixtilinear incircle is the circle  $(O_a)$  that touches the rays  $AB$  and  $AC$  at  $C_a$  and  $B_a$  and the circumcircle  $(O)$  internally at  $X$ . See Figure 2. Define the  $B$ - and  $C$ -mixtilinear incircles  $(O_b)$  and  $(O_c)$  analogously, with points of tangency  $Y$  and  $Z$  with the circumcircle. See [4]. We begin with an alternative proof of the main result of [4].

**Proposition 5.** *The lines  $AX$ ,  $BY$ ,  $CZ$  are concurrent at  $\text{exs}((O), (I))$ .*

*Proof.* Since  $A = \text{exs}((O_a), (I))$  and  $X = \text{exs}((O), (O_a))$ , the line  $AX$  passes through  $\text{exs}((O), (I))$  by d'Alembert's Theorem. For the same reason,  $BY$  and  $CZ$  also pass through the same point.  $\square$

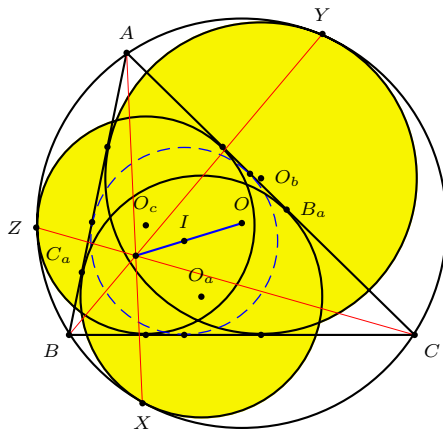


Figure 2

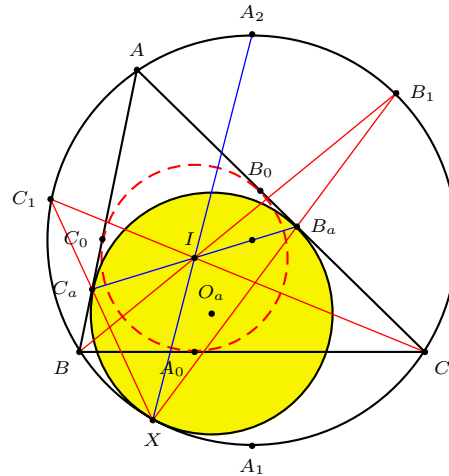


Figure 3

**Lemma 6.** (1)  $I$  is the midpoint of  $B_aC_a$ .

(2) The  $A$ -mixtilinear incircle has radius  $r_a = \frac{r}{\cos^2 \frac{A}{2}}$ .

(3)  $XI$  bisects angle  $BXC$ .

See Figure 3.

Consider the radical axis  $\ell_a$  of the mixtilinear incircles  $(O_b)$  and  $(O_c)$ .

**Proposition 7.** *The radical axis  $\ell_a$  contains*

- (1) the midpoint  $A_1$  of the arc  $BC$  of  $(O)$  not containing the vertex  $A$ ,
- (2) the midpoint  $M_a$  of  $IA_0$ , where  $A_0$  is the point of tangency of the incircle with the side  $BC$ .



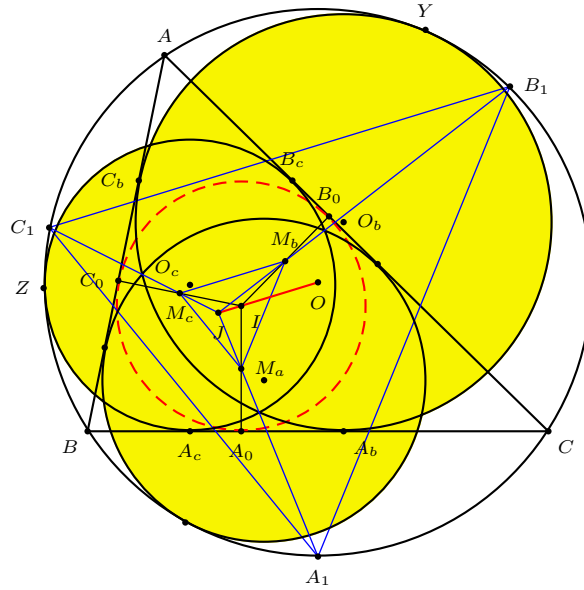


Figure 5.

the ratio

$$OJ : JI = R : -\frac{r}{2} = 2R : -r. \tag{3}$$

See Figure 5. □

*Remark.* Let  $T$  be the homothetic center of the excentral triangle  $I_aI_bI_c$  and the intouch triangle  $A_0B_0C_0$ . This is the triangle center  $X_{57}$  in [3]. Since the excentral triangle has circumcenter  $T'$ , the reflection of  $I$  in  $O$ ,

$$OT : TT' = 2R : -r.$$

Comparison with (3) shows that  $J$  is the reflection of  $T$  in  $O$ .

### 3. The mixtilinear excircles

The mixtilinear excircles are defined analogously to the mixtilinear incircles, except that the tangencies with the circumcircle are external. The  $A$ -mixtilinear excircle ( $O'_a$ ) can be easily constructed by noting that the polar of  $A$  passes through the excenter  $I_a$ ; similarly for the other two mixtilinear excircles. See Figure 6.

**Theorem 9.** *If the mixtilinear excircles touch the circumcircle at  $X'$ ,  $Y'$ ,  $Z'$  respectively, the lines  $AX'$ ,  $BY'$ ,  $CZ'$  are concurrent at  $\text{ins}((O), (I))$ .*

**Theorem 10.** *The radical center of the mixtilinear excircles is the reflection of  $J$  in  $O$ , where  $J$  is the radical center of the mixtilinear incircles.*

*Proof.* The polar of  $A$  with respect to  $(O'_a)$  passes through the excenter  $I_a$ . Similarly for the other two polars of  $B$  with respect to  $(O'_b)$  and  $C$  with respect to  $(O'_c)$ .

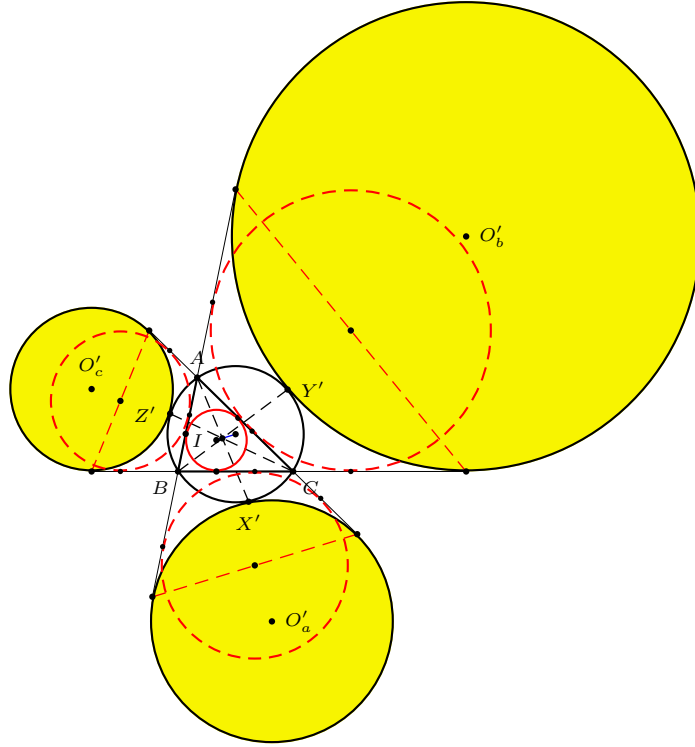


Figure 6.

Let  $A_3B_3C_3$  be the triangle bounded by these three polars. Let  $A_4, B_4, C_4$  be the midpoints of  $A_0A_3, B_0B_3, C_0C_3$  respectively. See Figure 7.

Since  $I_aI_bA_3I_c$  is a parallelogram, and  $A_2$  is the midpoint of  $I_bI_c$ , it is also the midpoint of  $A_3I_a$ . Since  $B_3C_3$  is parallel to  $I_bI_c$  (both being perpendicular to the bisector  $AA_1$ ),  $I_a$  is the midpoint of  $B_3C_3$ . Similarly,  $I_b, I_c$  are the midpoints of  $C_3A_3$  and  $A_3B_3$ , and the excentral triangle is the medial triangle of  $A_3B_3C_3$ . Note also that  $I$  is the circumcenter of  $A_3B_3C_3$  (since it lies on the perpendicular bisectors of its three sides). This is homothetic to the intouch triangle  $A_0B_0C_0$  at  $I$ , with ratio of homothety  $-\frac{r}{4R}$ .

If  $A_4$  is the midpoint of  $A_0A_3$ , similarly for  $B_4$  and  $C_4$ , then  $A_4B_4C_4$  is homothetic to  $A_3B_3C_3$  with ratio  $\frac{4R-r}{4R}$ .

We claim that  $A_4B_4C_4$  is homothetic to  $A_2B_2C_2$  at a point  $J'$ , which is the radical center of the mixtilinear excircles. The ratio of homothety is clearly  $\frac{4R-r}{R}$ .

Consider the isosceles trapezoid  $B_0C_0B_3C_3$ . Since  $B_4$  and  $C_4$  are the midpoints of the diagonals  $B_0B_3$  and  $C_0C_3$ , and  $B_3C_3$  contains the points of tangency  $B_a, C_a$  of the circle  $(O'_a)$  with  $AC$  and  $AB$ , the line  $B_4C_4$  also contains the midpoints of  $B_0B_a$  and  $C_0C_a$ , which are on the radical axis of  $(I)$  and  $(O'_a)$ . This means that the line  $B_4C_4$  is the radical axis of  $(I)$  and  $(O'_a)$ .

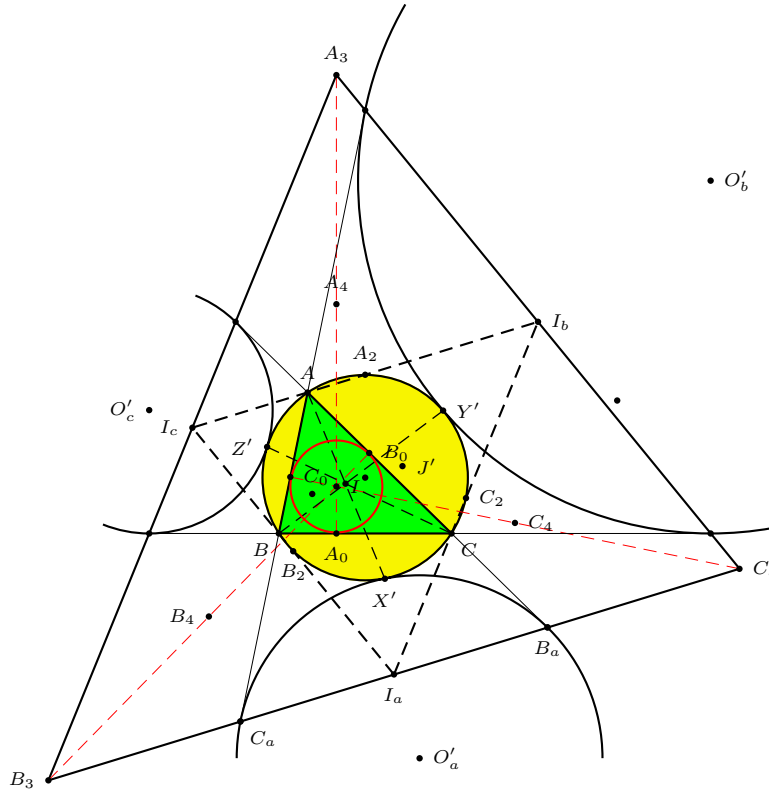


Figure 7.

It follows that  $A_4$  is on the radical axis of  $(O'_b)$  and  $(O'_c)$ . Clearly,  $A_2$  also lies on the same radical axis. This means that the radical center of the mixtilinear excircles is the homothetic center of the triangles  $A_2B_2C_2$  and  $A_4B_4C_4$ . Since these two triangles have circumcenters  $O$  and  $I$ , and circumradii  $R$  and  $\frac{4R-r}{2}$ , the homothetic center is the point  $J'$  which divides  $IO$  in the ratio

$$J'I : J'O = 4R - r : 2R. \tag{4}$$

Equivalently,  $OJ' : J'I = -2R : 4R - r$ . The reflection of  $J'$  in  $O$  divides  $OI$  in the ratio  $2R : -r$ . This is the radical center  $J$  of the mixtilinear incircles.  $\square$

#### 4. Apollonian circles

Consider the circle  $O_5(r_5)$  tangent internally to the mixtilinear incircles at  $A_5$ ,  $B_5$ ,  $C_5$  respectively. We call this the inner Apollonian circle of the mixtilinear incircles. It can be constructed from  $J$  since  $A_5$  is the second intersection of the line  $JX$  with the  $A$ -mixtilinear incircle, and similarly for  $B_5$  and  $C_5$ . See Figure 8. Theorem 11 below gives further details of this circle, and an easier construction.

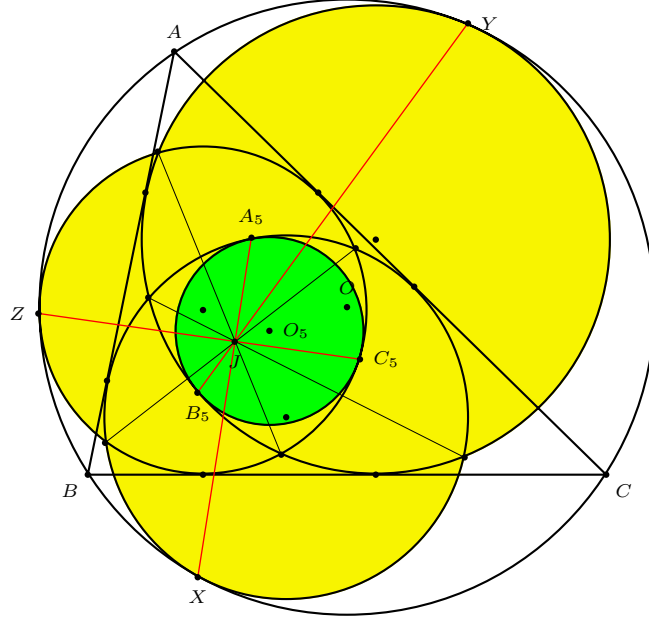


Figure 8.

**Theorem 11.** (1) Triangles  $A_5B_5C_5$  and  $ABC$  are perspective at  $\text{ins}((O), (I))$ .

(2) The inner Apollonian circle of the mixtilinear incircles has center  $O_5$  dividing the segment  $OI$  in the ratio  $4R : r$  and radius  $r_5 = \frac{3Rr}{4R+r}$ .

*Proof.* (1) Let  $P = \text{exs}((O_5), (I))$ , and  $Q_a = \text{exs}((O_5), (O'_a))$ . The following triples of points are collinear by d'Alembert's Theorem:

- (1)  $A, A_5, P$  from the circles  $(O_5), (I), (O_a)$ ;
- (2)  $A, Q_a, A_5$  from the circles  $(O_5), (O_a), (O'_a)$ ;
- (3)  $A, Q_a, P$  from the circles  $(O_5), (I), (O'_a)$ ;
- (4)  $A, X', \text{ins}((O), (I))$  from the circles  $(O), (I), (O'_a)$ .

See Figure 9. Therefore the lines  $AA_5$  contains the points  $P$  and  $\text{ins}((O), (I))$  (along with  $Q_a, X'$ ). For the same reason, the lines  $BB_5$  and  $CC_5$  contain the same two points. It follows that  $P$  and  $\text{ins}((O), (I))$  are the same point, which is common to  $AA_5, BB_5$  and  $CC_5$ .

(2) Now we compute the radius  $r_5$  of the circle  $(O_5)$ . From Theorem 8,  $OJ : JI = 2R : -r$ . As  $J = \text{exs}((O), (O_5))$ , we have  $OJ : JO_5 = R : -r_5$ . It follows that  $OJ : JI : JO_5 = 2R : -r : -2r_5$ , and

$$\frac{OO_5}{OI} = \frac{2(R - r_5)}{2R - r}. \quad (5)$$

Since  $P = \text{exs}((O_5), (I)) = \text{ins}((O), (I))$ , it is also  $\text{ins}((O), (O_5))$ . Thus,  $OP : PO_5 : PI = R : r_5 : r$ , and

$$\frac{OO_5}{OI} = \frac{R + r_5}{R + r}. \quad (6)$$

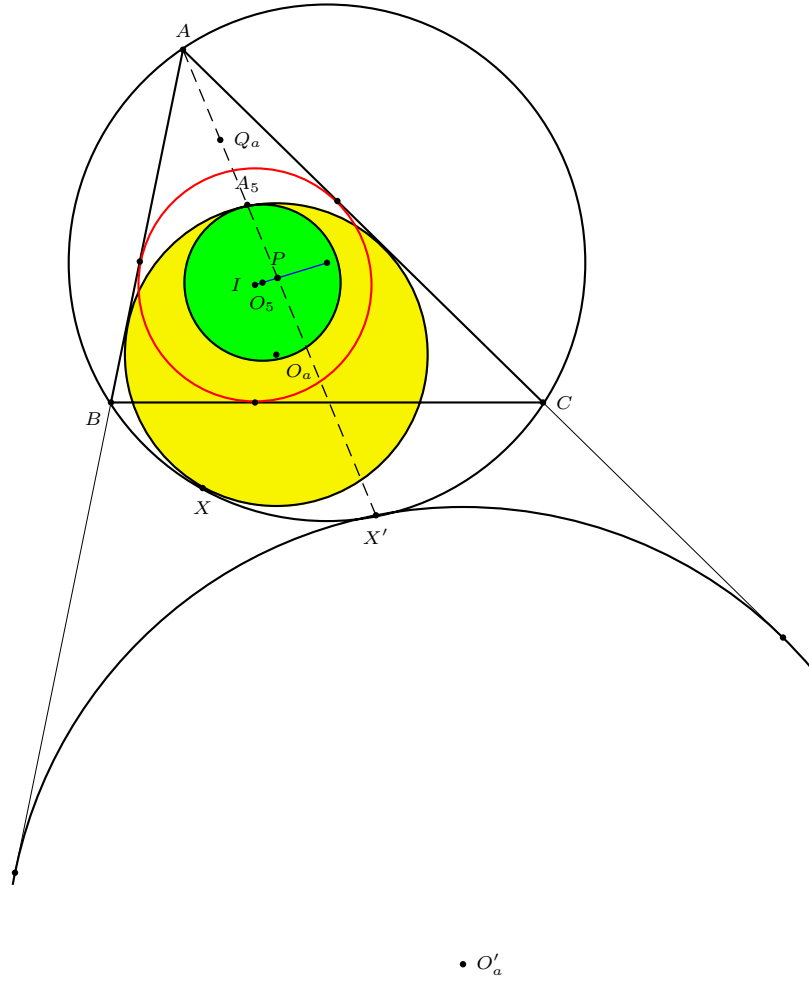


Figure 9.

Comparing (5) and (6), we easily obtain  $r_5 = \frac{3Rr}{4R+r}$ . Consequently,  $\frac{OO_5}{OI} = \frac{4R}{4R+r}$  and  $O_5$  divides  $OI$  in the ratio  $OO_5 : O_5I = 4R : r$ .  $\square$

The outer Apollonian circle of the mixtilinear excircles can also be constructed easily. If the lines  $J'X', J'Y', J'Z'$  intersect the mixtilinear excircles again at  $A_6, B_6, C_6$  respectively, then the circle  $A_6B_6C_6$  is tangent internally to each of the mixtilinear excircles. Theorem 12 below gives an easier construction without locating the radical center.

**Theorem 12.** (1) *Triangles  $A_6B_6C_6$  and  $ABC$  are perspective at  $\text{exs}((O), (I))$ .*  
 (2) *The outer Apollonian circle of the mixtilinear excircles has center  $O_6$  dividing the segment  $OI$  in the ratio  $-4R : 4R + r$  and radius  $r_6 = \frac{R(4R-3r)}{r}$ .*

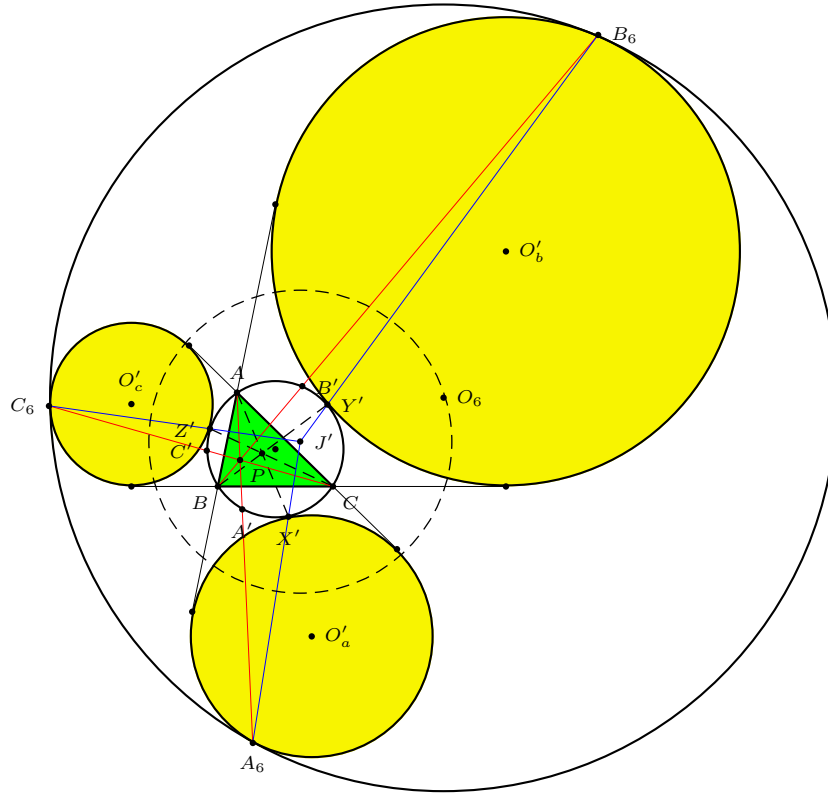


Figure 10.

*Proof.* (1) Since  $A_6 = \text{exs}((O'_a), (O_6))$  and  $A = \text{exs}((I), (O'_a))$ , by d'Alembert's Theorem, the line  $AA_6$  passes through  $P = \text{exs}((O_6), (I))$ . For the same reason  $BB_6$  and  $CC_6$  pass through the same point, the triangles  $A_6B_6C_6$  and  $ABC$  are perspective at  $P = \text{exs}((I), (O_6))$ . See Figure 10.

By Proposition 2,  $AB, X'Y', A_6B_6$ , and  $O'_aO'_b$  concur at  $S = \text{exs}((O'_a), (O'_b))$  on the radical axis of  $(O)$  and  $(O_6)$ . Now:  $SA \cdot SB = SX' \cdot SY' = SA_6 \cdot SB_6$ . Let  $AA_6, BB_6, CC_6$  intersect the circumcircle  $(O)$  at  $A', B', C'$  respectively. Since  $SA \cdot SB = SA_6 \cdot SB_6$ ,  $ABA_6B_6$  is cyclic. Since  $\angle BAA_6 = \angle BB'A' = \angle BB_6A_6$ ,  $A'B'$  is parallel to  $A_6B_6$ . Similarly,  $B'C'$  and  $C'A'$  are parallel to  $B_6C_6$  and  $C_6A_6$  respectively. Therefore the triangles  $A'B'C'$  and  $A_6B_6C_6$  are directly homothetic, and the center of homothety is  $P = \text{exs}((O), (O_6))$ .

Since  $P = \text{exs}((O), (O_6)) = \text{exs}((I), (O_6))$ , it is also  $\text{exs}((O), (I))$ , and  $PI : PO = r : R$ .

(2) Since  $A_6 = \text{exs}((O_6), (O'_a))$  and  $X' = \text{ins}((O'_a), (O))$ , by d'Alembert's Theorem, the line  $A_6X'$  passes through  $K = \text{ins}((O), (O_6))$ . For the same reason,  $B_6Y'$  and  $C_6Z'$  pass through the same point  $K$ .

We claim that  $K$  is the radical center  $J'$  of the mixtilinear excircles. Since  $SX' \cdot SY' = SA_6 \cdot SB_6$ , we conclude that  $X'A_6Y'B_6$  is cyclic, and  $KX' \cdot KA_6 = KY' \cdot KB_6$ . Also,  $Y'B_6C_6Z'$  is cyclic, and  $KY' \cdot KB_6 = KZ' \cdot KC_6$ . It follows

that

$$KX' \cdot KA_6 = KY' \cdot KB_6 = KZ' \cdot KC_6,$$

showing that  $K = \text{ins}((O), (O_6))$  is the radical center  $J'$  of the mixtilinear excircles. Hence,  $J'O : J'O_6 = R : -r_6$ . Note also  $PO : PO_6 = R : r_6$ . Then, we have the following relations.

$$OJ' : IO = 2R : 2R - r,$$

$$J'O_6 : IO = 2r_6 : 2R - r,$$

$$OO_6 : IO = r_6 - R : R - r.$$

Since  $OJ' + J'O_6 = OO_6$ , we have

$$\frac{2R}{2R - r} + \frac{2r_6}{2R - r} = \frac{r_6 - R}{R - r}.$$

This gives:  $r_6 = \frac{R(4R-3r)}{r}$ . Since  $K = \text{ins}((O), (O_6)) = \frac{r_6 \cdot O + R \cdot O_6}{R+r_6}$  and  $J' = \frac{(4R-r)O - 2R \cdot I}{2R-r}$  are the same point, we obtain  $O_6 = \frac{(4R+r)O - 4R \cdot I}{r}$ .  $\square$

*Remark.* The radical circle of the mixtilinear excircles has center  $J$  and radius  $\frac{R}{2R-r} \sqrt{(4R+r)(4R-3r)}$ .

**Corollary 13.**  $IO_5 \cdot IO_6 = IO^2$ .

## 5. The cyclocevian conjugate

Let  $P$  be a point in the plane of triangle  $ABC$ , with traces  $X, Y, Z$  on the sidelines  $BC, CA, AB$  respectively. Construct the circle through  $X, Y, Z$ . This circle intersects the sidelines  $BC, CA, AB$  again at points  $X', Y', Z'$ . A simple application of Ceva's Theorem shows that the  $AX', BY', CZ'$  are concurrent. The intersection point of these three lines is called the cyclocevian conjugate of  $P$ . See, for example, [2, p.226]. We denote this point by  $P^\circ$ . Clearly,  $(P^\circ)^\circ = P$ . For example, the centroid and the orthocenters are cyclocevian conjugates, and Gergonne point is the cyclocevian conjugate of itself.

We prove two interesting locus theorems.

**Theorem 14.** *The locus of  $Q$  whose circumcevian triangle with respect to  $XYZ$  is perspective to  $X'Y'Z'$  is the line  $PP^\circ$ . For  $Q$  on  $PP^\circ$ , the perspector is also on the same line.*

*Proof.* Let  $Q$  be a point on the line  $PP^\circ$ . By Pascal's Theorem for the six points,  $X', B', X, Y', A', Y$ , the intersections of the lines  $X'A'$  and  $Y'B'$  lies on the line connecting  $Q$  to the intersection of  $XY'$  and  $X'Y$ , which according to Pappus' theorem (for  $Y, Z, Z'$  and  $C, Y, Y'$ ), lies on  $PP^\circ$ . Since  $Q$  lies on  $PP^\circ$ , it follows that  $X'A', Y'B'$ , and  $PP^\circ$  are concurrent. Similarly,  $A'B'C'$  and  $X'Y'Z'$  are perspective at a point on  $PP^\circ$ . The same reasoning shows that if  $A'B'C'$  and  $X'Y'Z'$  are perspective at a point  $S$ , then both  $Q$  and  $S$  lie on the line connecting the intersections  $XY' \cap X'Y$  and  $YZ' \cap Y'Z$ , which is the line  $PP^\circ$ .  $\square$

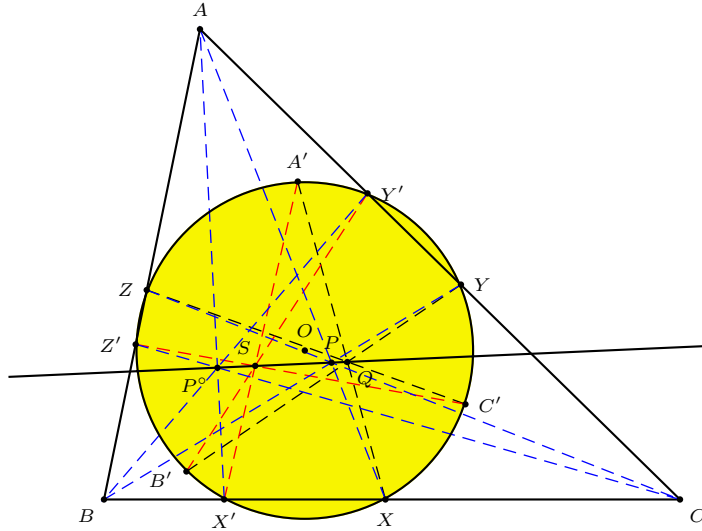


Figure 11.

For example, if  $P = G$ , then  $P^\circ = H$ . The line  $PP^\circ$  is the Euler line. If  $Q = O$ , the circumcenter, then the circumcevian triangle of  $O$  (with respect to the medial triangle) is perspective with the orthic triangle at the nine-point center  $N$ .

**Theorem 15.** *The locus of  $Q$  whose circumcevian triangle with respect to  $XYZ$  is perspective to  $ABC$  is the line  $PP^\circ$ .*

*Proof.* First note that

$$\begin{aligned} \frac{\sin Z'X'A'}{\sin A'X'Y'} &= \frac{\sin Z'XA'}{\sin A'YY'} = \frac{\sin Z'ZA'}{\sin ZAA'} \cdot \frac{\sin A'AY}{\sin A'YY'} \cdot \frac{\sin ZAA'}{\sin A'AY} \\ &= \frac{AA'}{ZA'} \cdot \frac{A'Y}{AA'} \cdot \frac{\sin BAX}{\sin XAC} = \frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin BAA'}{\sin A'AC}. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{\sin Z'X'A'}{\sin A'X'Y'} \cdot \frac{\sin X'Y'B'}{\sin B'Y'Z'} \cdot \frac{\sin Y'Z'C'}{\sin C'Z'X'} \\ &= \left( \frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin ZYB'}{\sin B'YX} \cdot \frac{\sin XZC'}{\sin C'ZY} \right) \left( \frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA} \right) \\ &= \frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA}. \end{aligned}$$

Therefore,  $A'B'C'$  is perspective with  $ABC$  if and only if it is perspective with  $X'Y'Z'$ . By Theorem 14, the locus of  $Q$  is the line  $PP^\circ$ .  $\square$

## 6. Some further results

We establish some further results on the mixtilinear incircles and excircles relating to points on the line  $OI$ .

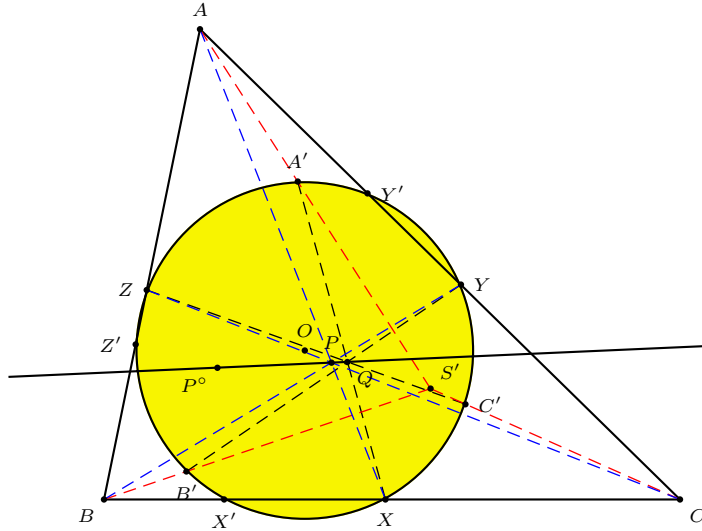


Figure 12.

**Theorem 16.** *The line  $OI$  is the locus of  $P$  whose circumcevian triangle with respect to  $A_1B_1C_1$  is perspective with  $XYZ$ .*

*Proof.* We first show that  $D = B_1Y \cap C_1Z$  lies on the line  $AA_1$ . Applying Pascal's Theorem to the six points  $Z, B_1, B_2, Y, C_1, C_2$  on the circumcircle, the points  $D = B_1Y \cap C_1Z, I = B_2Y \cap C_2Z$ , and  $B_1C_2 \cap B_2C_1$  are collinear. Since  $B_1C_2$  and  $B_2C_1$  are parallel to the bisector  $AA_1$ , it follows that  $D$  lies on  $AA_1$ . See Figure 13.

Now, if  $E = C_1Z \cap A_1X$  and  $F = A_1X \cap B_1Y$ , the triangle  $DEF$  is perspective with  $A_1B_1C_1$  at  $I$ . Equivalently,  $A_1B_1C_1$  is the circumcevian triangle of  $I$  with respect to triangle  $DEF$ . Triangle  $XYZ$  is formed by the second intersections of the circumcircle of  $A_1B_1C_1$  with the side lines of  $DEF$ . By Theorem 14, the locus of  $P$  whose circumcevian triangle with respect to  $A_1B_1C_1$  is perspective with  $XYZ$  is a line through  $I$ . This is indeed the line  $IO$ , since  $O$  is one such point. (The circumcevian triangle of  $O$  with respect to  $A_1B_1C_1$  is perspective with  $XYZ$  at  $I$ ).  $\square$

*Remark.* If  $P$  divides  $OI$  in the ratio  $OP : PI = t : 1 - t$ , then the perspector  $Q$  divides the same segment in the ratio  $OQ : QI = (1 + t)R : -2tr$ . In particular, if  $P = \text{ins}((O), (I))$ , this perspector is  $T$ , the homothetic center of the excentral and intouch triangles.

**Corollary 17.** *The line  $OI$  is the locus of  $Q$  whose circumcevian triangle with respect to  $A_1B_1C_1$  (or  $XYZ$ ) is perspective with  $DEF$ .*

**Proposition 18.** *The triangle  $A_2B_2C_2$  is perspective*  
 (1) *with  $XYZ$  at the incenter  $I$ ,*  
 (2) *with  $X'Y'Z'$  at the centroid of the excentral triangle.*

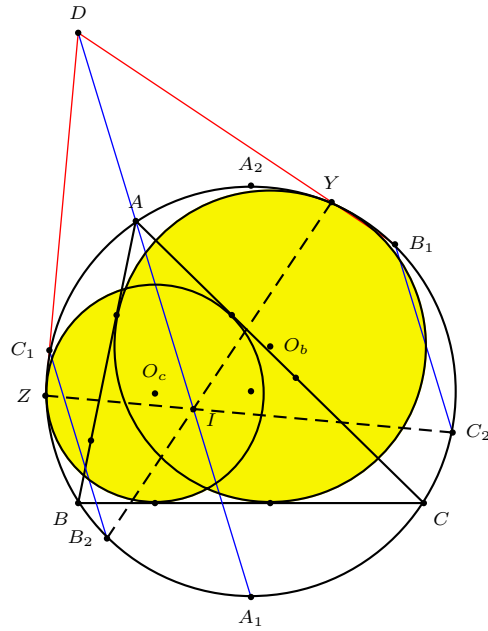


Figure 13.

*Proof.* (1) follows from Lemma 6(b).

(2) Referring to Figure 7, the excenter  $I_a$  is the midpoint  $B_aC_a$ . Therefore  $X'I_a$  is a median of triangle  $X'B_aC_a$ , and it intersects  $B_2C_2$  at its midpoint  $X''$ . Since  $A_2B_2I_aC_2$  is a parallelogram,  $A_2, X', X''$  and  $I_a$  are collinear. In other words, the line  $A_2X'$  contains a median, hence the centroid, of the excentral triangle. So do  $B_2Y'$  and  $C_2Z$ .  $\square$

Let  $A_7$  be the second intersection of the circumcircle with the line  $\ell_a$ , the radical axis of the mixtilinear incircles  $(O_b)$  and  $(O_c)$ . Similarly define  $B_7$  and  $C_7$ . See Figure 14.

**Theorem 19.** *The triangles  $A_7B_7C_7$  and  $XYZ$  are perspective at a point on the line  $OI$ .*

*Remark.* This point divides  $OI$  in the ratio  $4R - r : -4r$  and has homogeneous barycentric coordinates

$$\left( \frac{a(b+c-5a)}{b+c-a} : \frac{b(c+a-5b)}{c+a-b} : \frac{c(a+b-5c)}{a+b-c} \right).$$

## 7. Summary

We summarize the triangle centers on the  $OI$ -line associated with mixtilinear incircles and excircles by listing, for various values of  $t$ , the points which divide  $OI$  in the ratio  $R : tr$ . The last column gives the indexing of the triangle centers in [2, 3].

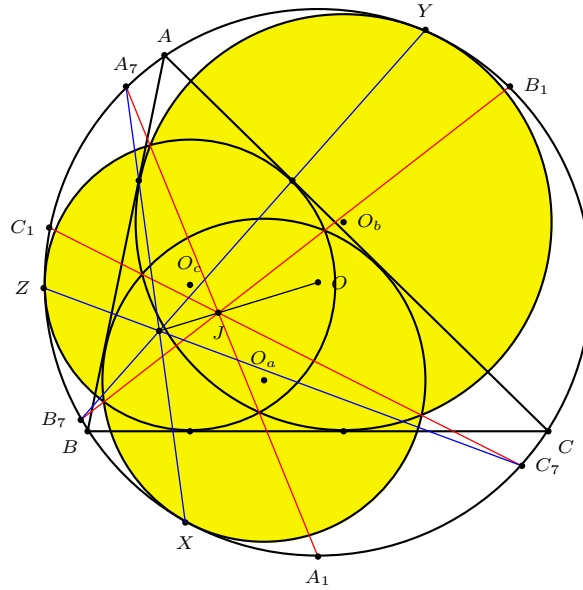


Figure 14.

$t$	first barycentric coordinate		$X_n$
1	$a^2(s - a)$	ins( $(O)$ , $(I)$ ) perspector of $ABC$ and $X'Y'Z'$ perspector of $ABC$ and $A_5B_5C_5$	$X_{55}$
-1	$\frac{a^2}{s-a}$	exs( $(O)$ , $(I)$ ) perspector of $ABC$ and $XYZ$ perspector of $ABC$ and $A_6B_6C_6$	$X_{56}$
$-\frac{2R}{2R+r}$	$\frac{a}{s-a}$	homothetic center of excentral and intouch triangles	$X_{57}$
$-\frac{1}{2}$	$a^2(b^2 + c^2 - a^2 - 4bc)$	radical center of mixtilinear incircles	$X_{999}$
$\frac{1}{4}$	$a^2(b^2 + c^2 - a^2 + 8bc)$	center of Apollonian circle of mixtilinear incircles	
$-\frac{4R-r}{2r}$	$a^2 f(a, b, c)$	radical center of mixtilinear excircles	
$-\frac{4R+r}{4r}$	$a^2 g(a, b, c)$	center of Apollonian circle of mixtilinear excircles	
$-\frac{4R}{r}$	$a(3a^2 - 2a(b + c) - (b - c)^2)$	centroid of excentral triangle	$X_{165}$
$-\frac{4R}{4R-r}$	$\frac{a(b+c-5a)}{b+c-a}$	perspector of $A_7B_7C_7$ and $XYZ$	

The functions  $f$  and  $g$  are given by

$$\begin{aligned}
 f(a, b, c) &= a^4 - 2a^3(b+c) + 10a^2bc + 2a(b+c)(b^2 - 4bc + c^2) \\
 &\quad - (b-c)^2(b^2 + 4bc + c^2), \\
 g(a, b, c) &= a^5 - a^4(b+c) - 2a^3(b^2 - bc + c^2) + 2a^2(b+c)(b^2 - 5bc + c^2) \\
 &\quad + a(b^4 - 2b^3c + 18b^2c^2 - 2bc^3 + c^4) - (b-c)^2(b+c)(b^2 - 8bc + c^2).
 \end{aligned}$$

## References

- [1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.
- [2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
- [3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at <http://faculty.evansville.edu/ck6/encyclopedia/ETC.html>.
- [4] P. Yiu, Mixtilinear incircles, *Amer. Math. Monthly*, 106 (1999) 952–955.
- [5] P. Yiu, Mixtilinear incircles II, unpublished manuscript, 1998.

Khoa Lu Nguyen: 806 Candler Dr., Houston, Texas, 77037-4122, USA  
*E-mail address:* treegoner@yahoo.com

Juan Carlos Salazar: Calle Maturín N°C 19, Urb. Mendoza, Puerto Ordaz 8015, Estado Bolívar,  
 Venezuela  
*E-mail address:* caisersal@yahoo.com