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A Conic Associated with Euler Lines

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Abstract. We study the locus of a point C for which the Euler line of triangle ABC with given A and B has a given slope m. This is a conic through A and B, and with center (if it exists) at the midpoint of AB. The main properties of such an Euler conic are described. We also give a construction of a point C for which triangle ABC, with A and B fixed, has a prescribed Euler line.

1. The Euler conic

Given two points A and B and a real number m, we study the locus of a point C for which the Euler line of triangle ABC has slope m. We show that this locus is a conic through A and B. Without loss of generality, we assume a Cartesian coordinate system in which

$$A = (-1, 0) \quad \text{and} \quad B = (1, 0),$$

and write $C = (x, y)$. The centroid G and the orthocenter H of a triangle can be determined from the coordinates of its vertices. They are the points

$$G = \left(\frac{x}{3}, \frac{y}{3}\right) \quad \text{and} \quad H = \left(x, \frac{-x^2+1}{y}\right). \tag{1}$$

See, for example, [2]. The vector

$$\overrightarrow{GH} = \left(\frac{2x}{3}, \frac{-3x^2-y^2+3}{3y}\right), \tag{2}$$

is parallel to the Euler line. When the Euler line is not vertical, its slope is given by:

$$m = \frac{-3x^2-y^2+3}{2xy}, \quad x, y \neq 0.$$

Therefore, the coordinates of the vertex C satisfy the equation

$$3x^2 + 2mxy + y^2 = 3. \tag{3}$$

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This clearly represents a conic. We call this the Euler conic associated with A, B and slope m. It clearly has center at the origin, the midpoint M of AB. Its axes are the eigenvectors of the matrix

$$\begin{pmatrix} 3 & m \\ m & 1 \end{pmatrix}.$$

The characteristic polynomial being $\lambda^2 - 4\lambda - (m^2 - 3)$, its eigenvectors with corresponding eigenvalues are as follows.

eigenvector	eigenvalue
$(\sqrt{m^2+1}+1,\ m)$	$2+\sqrt{m^2+1}$
$(\sqrt{m^2+1}-1,\ -m)$	$2-\sqrt{m^2+1}$

Thus, equation (3) can be rewritten in the form
(1) $(mx+y)^2 = 3$, if $m = \pm\sqrt{3}$, or
(2) $(2+\sqrt{m^2+1})(x\cos\alpha - y\sin\alpha)^2 + (2-\sqrt{m^2+1})(x\sin\alpha + y\cos\alpha)^2 = 3$, if $m \neq \pm\sqrt{3}$, where

$$\cos\alpha = \sqrt{\frac{\sqrt{m^2+1}+1}{2\sqrt{m^2+1}}}, \qquad \sin\alpha = \sqrt{\frac{\sqrt{m^2+1}-1}{2\sqrt{m^2+1}}}.$$

Remarks. (1) The pairs $(\pm 1, 0)$ are always solutions of (3) and correspond to the singular cases in which the vertex C coincides, respectively, with A or B, and consequently, it is not possible to define the triangle ABC.

(2) The pairs $\left(0, \pm\sqrt{3}\right)$ are also solutions of (3) and correspond to the trivial case when the triangle ABC is equilateral. In this case, the centroid, the orthocenter, and the circumcenter coincide.

2. Classification of the Euler conic

The Euler conic is an ellipse or a hyperbola according as $m^2 < 3$ or $m^2 > 3$. It degenerates into a pair of straight lines when $m^2 = 3$.

Proposition 1. *Suppose $m^2 < 3$. The Euler conic is an ellipse with eccentricity*

$$\varepsilon = \sqrt{\frac{2\sqrt{m^2+1}}{\sqrt{m^2+1}+2}}.$$

The foci are the points

$$\pm\left(-\operatorname{sgn}(m)\cdot\sqrt{\frac{3(\sqrt{m^2+1}-1)}{3-m^2}},\ \sqrt{\frac{3(\sqrt{m^2+1}+1)}{3-m^2}}\right),$$

where $\operatorname{sgn}(m) = +1$, 0*, or* -1 *according as* $m >, =,$ *or* < 0.

Figure 1 shows the Euler ellipse for $m = \frac{3}{4}$, a triangle ABC with C on the ellipse, and its Euler line of slope m.

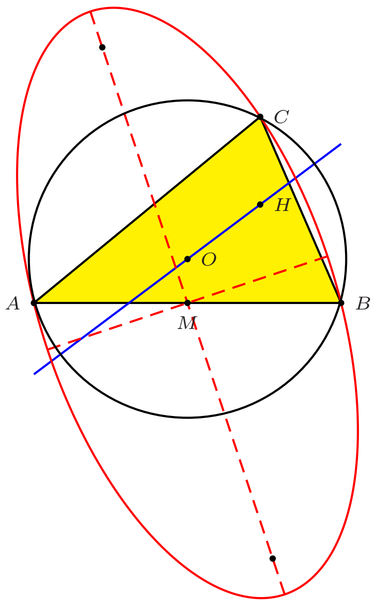

Figure 1

Proposition 2. *Suppose $m^2 > 3$. The Euler conic is a hyperbola with eccentricity*

$$\varepsilon = \sqrt{\frac{2\sqrt{m^2+1}}{\sqrt{m^2+1}-2}}.$$

The foci are the points

$$\pm\left(\operatorname{sgn}(m)\cdot\sqrt{\frac{3(\sqrt{m^2+1}+1)}{m^2-3}}, \sqrt{\frac{3(\sqrt{m^2+1}-1)}{m^2-3}}\right),$$

where $\operatorname{sgn}(m) = +1$ or -1 according as $m > 0$ or < 0. The asymptotes are the lines

$$y = (-m \pm \sqrt{m^2-3})x.$$

Figure 2 shows the Euler hyperbola for $m = \frac{12}{5}$, a triangle ABC with C on the hyperbola, and its Euler line of slope m.

When $|m| = \sqrt{3}$, the Euler conic degenerates into a pair of parallel lines, whose equations are:

$$y = -mx \pm \sqrt{3}.$$

Examples of triangles for $m = \sqrt{3}$ are shown in Figures 3A and 3B, and for $m = -\sqrt{3}$ in Figures 4A and 4B.

Corollary 3. *The slope of the Euler line of the triangle ABC is*

$$m = \pm\sqrt{3},$$

if and only if, one of angles A and B is $60°$ or $120°$.

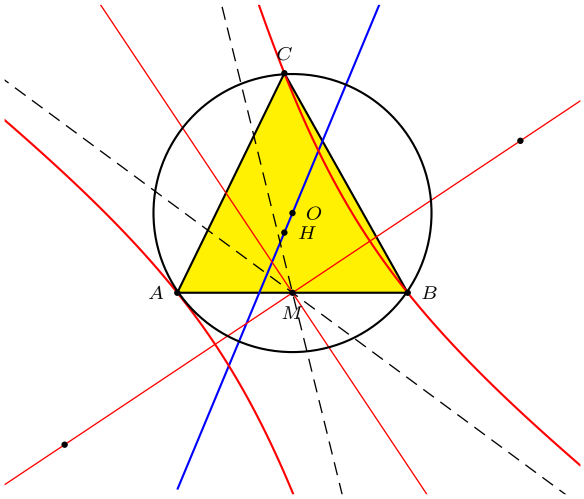

Figure 2

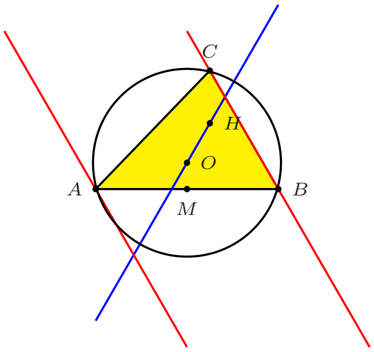

Figure 3A

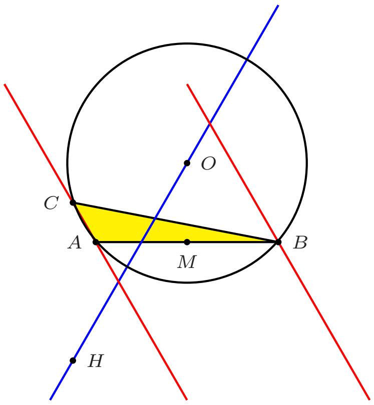

Figure 3B

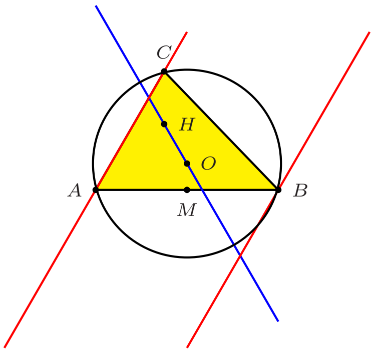

Figure 4A

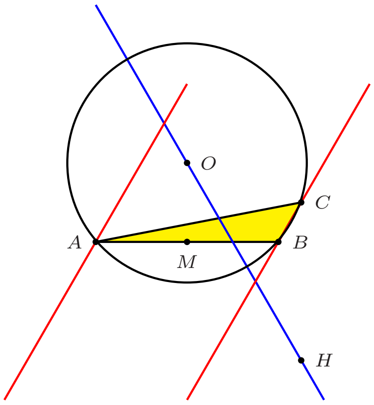

Figure 4B

3. Triangles with given Euler line

In this section we find the cartesian coordinates of the third vertex C in order that a given line be the Euler line of the triangle ABC with vertices $A = (-1, 0)$ and $B = (1, 0)$.

Lemma 4. *The Euler line of triangle ABC is perpendicular to AB if and only if $AB = AC$. In this case, the Euler line is the perpendicular bisector of AB.*

We shall henceforth assume that the Euler line is not perpendicular to AB. It therefore has an equation of the form

$$y = mx + k.$$

The circumcenter is the intersection of the Euler line with the line $x = 0$, the perpendicular bisector of AB. It is the point $O = (0, k)$. The circumcircle is

$$x^2 + (y - k)^2 = k^2 + 1$$

or

$$x^2 + y^2 - 2ky - 1 = 0. \tag{4}$$

Let M be the midpoint of AB; it is the origin of the Cartesian system. If G is the centroid, the vertex C is such that $MC : MG = 3 : 1$. Since G lies on the line $y = mx + k$, C lies on the line $y = mx + 3k$. It can therefore be constructed as the intersection of this line with the circle (4).

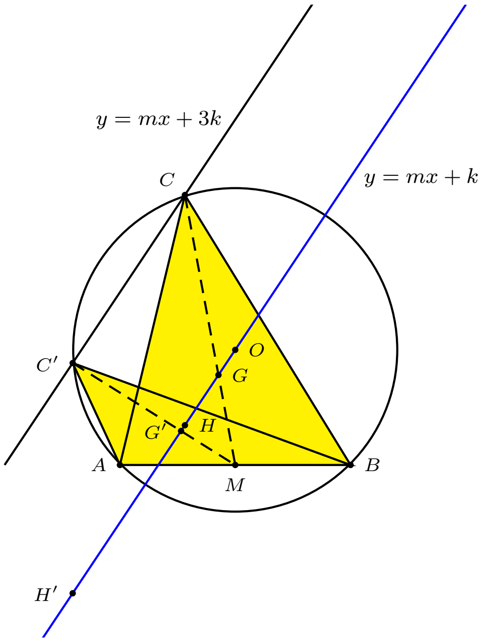

Figure 5

Proposition 5. *The number of points C for which triangle ABC has Euler line $y = mx + k$ is 0, 1, or 2 according as $(m^2 - 3)(k^2 + 1) <, =,$ or > -4.*

In the hyperbolic and degenerate cases $m^2 \geq 3$, there are always two such triangles. In the elliptic case, $m^2 < 3$. There are two such triangles if and only if $k^2 < \frac{m^2+1}{3-m^2}$.

Corollary 6. *For $m^2 < 3$ and $k = \pm\sqrt{\frac{m^2+1}{3-m^2}}$, there is a unique triangle ABC whose Euler line is the line $y = mx + k$. The lines $y = mx + 3k$ are tangent to the Euler ellipse* (3) *at the points*

$$\pm\left(\frac{-2m}{\sqrt{(m^2+1)(3-m^2)}}, \frac{3+m^2}{\sqrt{(m^2+1)(3-m^2)}}\right).$$

Figure 6 shows the configuration corresponding to $k = \sqrt{\frac{m^2+1}{3-m^2}}$. The other one can be obtained by reflection in M, the midpoint of AB.

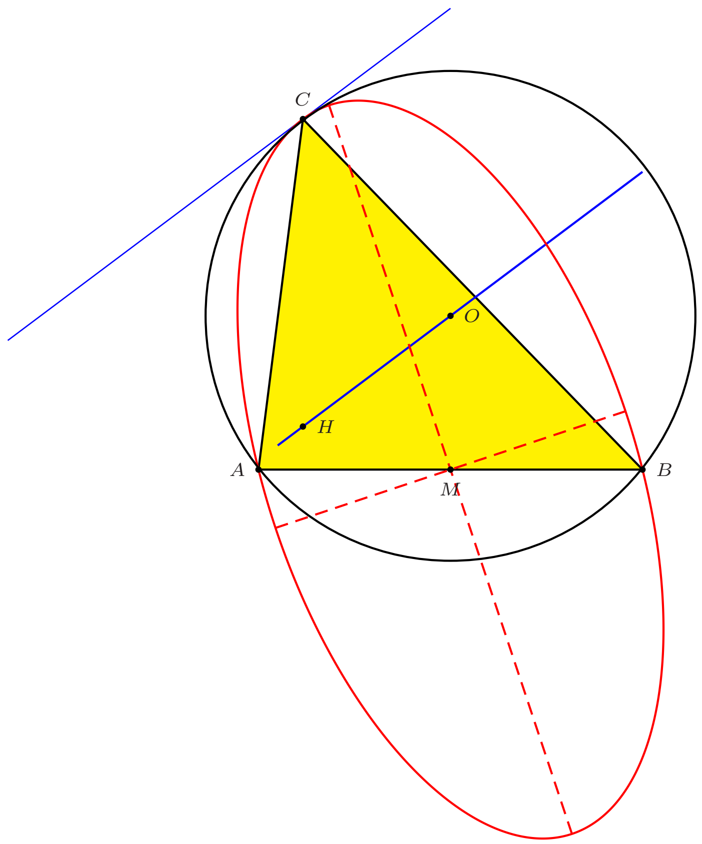

Figure 6

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