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A Characterization of the Centroid Using June Lester's Shape Function

Mowaffaq Hajja and Margarita Spirova

Abstract. The notion of triangle shape is used to give another proof of the fact that if P is a point inside triangle ABC and if the cevian triangle of P is similar to ABC in the natural order, then P is the centroid.

Identifying the Euclidean plane with the plane of complex numbers, we define a (non-degenerate) triangle to be any ordered triple (A, B, C) of distinct complex numbers, and we write it as ABC if no ambiguity may arise. According to this definition, there are in general six different triangles having the same set of vertices. We say that triangles ABC and $A'B'C'$ are *similar* if

$$\|A - B\| : \|A' - B'\| = \|B - C\| : \|B' - C'\| = \|C - A\| : \|C' - A'\|.$$

By the SAS similarity theorem and by the geometric interpretation of the quotient of two complex numbers, this is equivalent to the requirement that

$$\frac{A - B}{A - C} = \frac{A' - B'}{A' - C'}.$$

June A. Lester called the quantity $\dfrac{A - B}{A - C}$ the *shape* of triangle ABC and she studied properties and applications of this shape function in great detail in [4], [5], and [6].

In this note, we use this shape function to prove that if P is a point inside triangle ABC, and if AA', BB', and CC' are the cevians through P, then triangles ABC and $A'B'C'$ are similar if and only if P is the centroid of ABC. This has already appeared as Theorem 7 in [1], where three different proofs are given, and as a problem in the Problem Section of the *Mathematics Magazine* [2]. A generalization to d-simplices for all d is being considered in [3].

Our proof is an easy consequence of two lemmas that may prove useful in other contexts.

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Lemma 1. *Let ABC be a non-degenerate triangle, and let x, y, and z be real numbers such that*

$$x(A-B)^2 + y(B-C)^2 + z(C-A)^2 = 0. \tag{1}$$

Then either $x = y = z = 0$, or $xy + yz + zx > 0$.

Proof. Let $S = \frac{A-B}{A-C}$ be the shape of ABC. Dividing (1) by $(A-C)^2$, we obtain

$$(x+y)S^2 - 2yS + (y+z) = 0. \tag{2}$$

Since ABC is non-degenerate, S is not real. Thus if $x + y = 0$, then $y = 0$ and hence $x = y = z = 0$. Otherwise, $x + y \neq 0$ and the discriminant

$$4y^2 - 4(x+y)(y+z) = -(xy + yz + zx)$$

of (2) is negative, *i.e.*, $xy + yz + zx > 0$, as desired. □

Lemma 2. *Suppose that the cevians through an interior point P of a triangle divide the sides in the ratios $u : 1-u$, $v : 1-v$, and $w : 1-w$. Then*
(i) $uvw \leq \frac{1}{8}$*, with equality if and only if $u = v = w = \frac{1}{2}$, i.e., if and only if P is the centroid.*
(ii) $\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) \leq 0$*, with equality if and only if $u = v = w = \frac{1}{2}$,* i.e., *if and only if P is the centroid.*

Proof. Let $uvw = p$. Then using the cevian condition $uvw = (1-u)(1-v)(1-w)$, we see that

$$\begin{aligned} p &= \sqrt{u(1-u)}\sqrt{v(1-v)}\sqrt{w(1-w)} \\ &\leq \frac{u+(1-u)}{2}\,\frac{v+(1-v)}{2}\,\frac{w+(1-w)}{2}, \text{ by the AM-GM inequality} \\ &= \frac{1}{8}, \end{aligned}$$

with equality if and only if $u = \frac{1}{2}$, $v = \frac{1}{2}$, and $w = \frac{1}{2}$. This proves (i).

To prove (ii), note that

$$\begin{aligned} &\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) \\ &= (uv + vw + wu) - (u + v + w) + \frac{3}{4} \\ &= 2uvw - \frac{1}{4}, \end{aligned}$$

because $uvw = (1-u)(1-v)(1-w)$. Now use (i). □

We now use Lemmas 1 and 2 and the shape function to prove the main result.

Theorem 3. *Let AA', BB', and CC' be the cevians through an interior point P of triangle ABC. Then triangles ABC and $A'B'C'$ are similar if and only if P is the centroid of ABC.*

Proof. One direction being trivial, we assume that $A'B'C'$ and ABC are similar, and we prove that P is the centroid.

Suppose that the cevians AA', BB', and CC' through P divide the sides BC, CA, and AB in the ratios $u : 1-u$, $v : 1-v$, and $w : 1-w$, respectively. Since ABC and $A'B'C'$ are similar, it follows that they have equal shapes, *i.e.*,

$$\frac{A-B}{A-C} = \frac{A'-B'}{A'-C'}. \tag{3}$$

Substituting the values

$$A' = (1-u)B + uC, \quad B' = (1-v)C + vA, \quad C' = (1-w)A + wB$$

in (3) and simplifying, we obtain

$$\left(u-\frac{1}{2}\right)(A-B)^2 + \left(v-\frac{1}{2}\right)(B-C)^2 + \left(w-\frac{1}{2}\right)(C-A)^2 = 0.$$

By Lemma 1, either $u = v = w = \frac{1}{2}$, in which case P is the centroid, or

$$\left(u-\frac{1}{2}\right)\left(v-\frac{1}{2}\right) + \left(v-\frac{1}{2}\right)\left(w-\frac{1}{2}\right) + \left(w-\frac{1}{2}\right)\left(u-\frac{1}{2}\right) > 0,$$

in which case Lemma 2(ii) is contradicted. This completes the proof. □

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Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan.
E-mail address: mhajja@yu.edu.jo

Margarita Spirova: Faculty of Mathematics and Informatics, University of Sofia, 5 Yames Bourchier, 164 Sofia, Bulgaria.
E-mail address: spirova@fmi.uni-sofia.bg