

# Archimedean Adventures

Floor van Lamoen

**Abstract.** We explore the arbelos to find more Archimedean circles and several infinite families of Archimedean circles. To define these families we study another shape introduced by Archimedes, the *salinon*.

## 1. Preliminaries

We consider an arbelos, consisting of three semicircles  $(O_1)$ ,  $(O_2)$  and  $(O)$ , having radii  $r_1$ ,  $r_2$  and  $r = r_1 + r_2$ , mutually tangent in the points  $A$ ,  $B$  and  $C$  as shown in Figure 1 below.

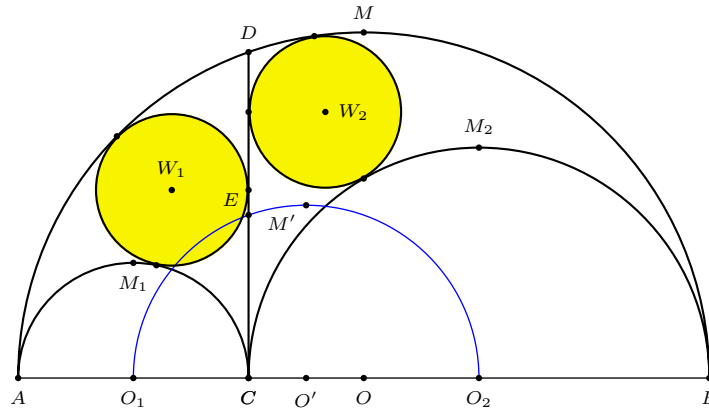


Figure 1. The arbelos with its Archimedean circles

It is well known that Archimedes has shown that the circles tangent to  $(O)$ ,  $(O_1)$  and  $CD$  and to  $(O)$ ,  $(O_2)$  and  $CD$  respectively are congruent. Their radius is  $r_A = \frac{r_1 r_2}{r}$ . See Figure 1. Thanks to [1, 2, 3] we know that these Archimedean twins are not only twins, but that there are many more Archimedean circles to be found in surprisingly beautiful ways. In their overview [4] Dodge et al have expanded the collection of Archimedean circles to huge proportions, 29 individual circles and an infinite family of Woo circles with centers on the Schoch line. Okumura and Watanabe [6] have added a family to the collection, which all pass through  $O$ , and have given new characterizations of the circles of Schoch and Woo.<sup>1</sup> Recently Power has added four Archimedean circles in a short note [7]. In this paper

Publication Date: March, 2006. Communicating Editor: Paul Yiu.

<sup>1</sup>The circles of Schoch and Woo often make use of tangent lines. The use of tangent lines in the arbelos, being a curvilinear triangle, may seem surprising, its relevance is immediately apparent when we realize that the common tangent of  $(O_1)$ ,  $(O_2)$ ,  $(O')$  and  $C(2r_A)$  ( $\mathcal{E}$  in [6]) and the common tangent of  $(O)$ ,  $(W_{21})$  and the incircle  $(O_3)$  of the arbelos meet on  $AB$ .

we introduce some new Archimedean circles and some in infinite families. The Archimedean circles ( $W_n$ ) are those appearing in [4]. New ones are labeled ( $K_n$ ).

We adopt the following notations.

$P(r)$	circle with center $P$ and radius $r$
$P(Q)$	circle with center $P$ and passing through $Q$
$(P)$	circle with center $P$ and radius clear from context
$(PQ)$	circle with diameter $PQ$
$(PQR)$	circle through $P, Q, R$

In the context of arbeloi, these are often interpreted as semicircles. Thus, an arbelos consists of three semicircles ( $O_1 = (AC)$ ), ( $O_2 = (CB)$ ), and ( $O = (AB)$ ) on the same side of the line  $AB$ , of radii  $r_1, r_2$ , and  $r = r_1 + r_2$ . The common tangent at  $C$  to ( $O_1$ ) and ( $O_2$ ) meets ( $O$ ) in  $D$ . We shall call the semicircle ( $O' = (O_1O_2)$ ) (on the same side of  $AB$ ) the *midway semicircle* of the arbelos.

It is convenient to introduce a cartesian coordinate system, with  $O$  as origin. Here are the coordinates of some basic points associated with the arbelos.

$A$	$(-(r_1 + r_2), 0)$	$B$	$(r_1 + r_2, 0)$	$C$	$(r_1 - r_2, 0)$
$O_1$	$(-r_2, 0)$	$O_2$	$(r_1, 0)$		
$M_1$	$(-r_2, r_1)$	$M_2$	$(r_1, r_2)$	$M$	$(0, r_1 + r_2)$
$O'$	$(\frac{r_1 - r_2}{2}, 0)$	$M'$	$(\frac{r_1 - r_2}{2}, \frac{r_1 + r_2}{2})$		
$D$	$(r_1 - r_2, 2\sqrt{r_1 r_2})$	$E$	$(r_1 - r_2, \sqrt{r_1 r_2})$		

## 2. New Archimedean circles

2.1. ( $K_1$ ) and ( $K_2$ ). The Archimedean circle ( $W_8$ ) has  $C$  as its center and is tangent to the tangents  $O_1K_2$  and  $O_2K_1$  to ( $O_1$ ) and ( $O_2$ ) respectively. By symmetry it is easy to find from these the Archimedean circles ( $K_1$ ) and ( $K_2$ ) tangent to  $CD$ , see Figure 2. A second characterization of the points  $K_1$  and  $K_2$ , clearly equivalent, is that these are the points of intersection of ( $O_1$ ) and ( $O_2$ ) with ( $O'$ ). We note that the tangent to  $A(C)$  at the point of intersection of  $A(C)$ , ( $W_{13}$ ) and ( $O$ ) passes through  $B$ , and is also tangent to ( $K_1$ ). A similar statement is true for the tangent to  $B(C)$  at its intersection with ( $W_{14}$ ) and ( $O$ ).

To prove the correctness, let  $T$  be the perpendicular foot of  $K_1$  on  $AB$ . Then, making use of the right triangle  $O_1O_2K_1$ , we find  $O_1T = \frac{r_1^2}{r_1 + r_2}$  and thus the radius of ( $K_1$ ) is equal to  $O_1C - O_1T = r_A$ . By symmetry the radius of ( $K_2$ ) equals  $r_A$  as well.

The circles ( $K_1$ ) and ( $K_2$ ) are closely related to ( $W_{13}$ ) and ( $W_{14}$ ), which are found by intersecting  $A(C)$  and  $B(C)$  with ( $O$ ) and then taking the smallest circles through the respective points of intersection tangent to  $CD$ . The relation is clear when we realize that  $A(C)$ ,  $B(C)$  and ( $O$ ) can be found by a homothety with center  $C$ , factor 2 applied to ( $O_1$ ), ( $O_2$ ) and ( $O'$ ).

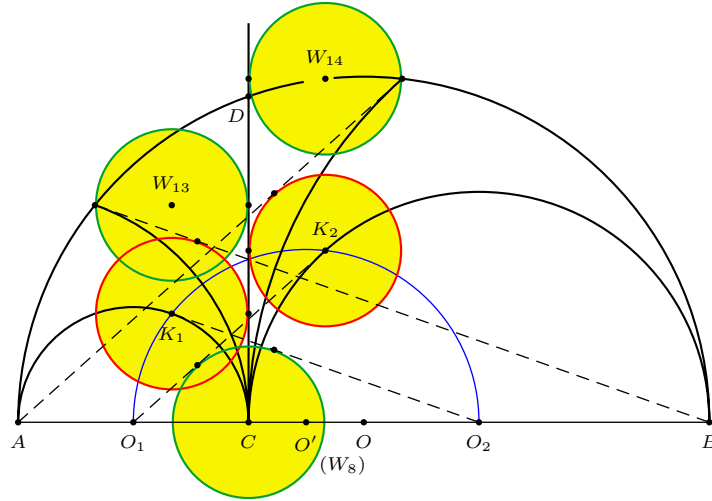


Figure 2. The Archimedean circles  $(K_1)$  and  $(K_2)$

2.2.  $(K_3)$ . In private correspondence [10] about  $(K_1)$  and  $(K_2)$ , Paul Yiu has noted that the circle with center on  $CD$  tangent to  $(O)$  and  $(O')$  is Archimedean. Indeed, if  $x$  is the radius of this circle, then we have

$$(r - x)^2 - (r_1 - r_2)^2 = \left(\frac{r}{2} + x\right)^2 - \left(\frac{r_1 - r_2}{2}\right)^2$$

and that yields  $x = r_A$ .

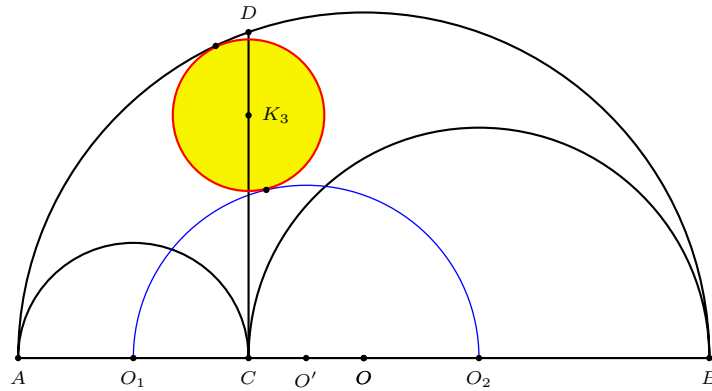


Figure 3. The Archimedean circle  $(K_3)$

2.3.  $(K_4)$  and  $(K_5)$ . There is an interesting similarity between  $(K_3)$  and Bankoff's triplet circle  $(W_3)$ . Recall that this is the circle that passes through  $C$  and the points

of tangency of  $(O_1)$  and  $(O_2)$  with the incircle  $(O_3)$  of the arbelos.<sup>2</sup> Just like  $(K_3)$ ,  $(W_3)$  has its center on  $CD$  and is tangent to  $(O')$ , a fact that seems to have been unnoted so far. The tangency can be shown by using Pythagoras' Theorem in triangle  $CO'W_3$ , yielding that  $O'W_3 = \frac{r_1^2 + r_2^2}{2r}$ . This leaves for the length of the radius of  $(O')$  beyond  $W_3$ :

$$\frac{r}{2} - \frac{r_1^2 + r_2^2}{2r} = \frac{r_1 r_2}{r}.$$

This also gives us in a simple way the new Archimedean circle  $(K_4)$ : the circle tangent to  $AB$  at  $O$  and to  $(O')$  is Archimedean by reflection of  $(W_3)$  through the perpendicular to  $AB$  in  $O'$ .

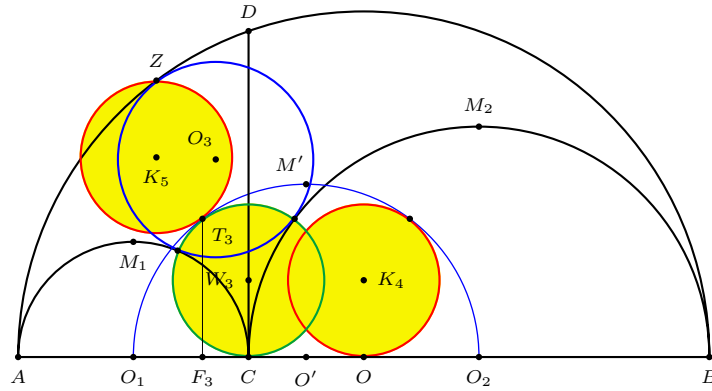


Figure 4. The Archimedean circles  $(K_4)$  and  $(K_5)$

Let  $T_3$  be the point of tangency of  $(O')$  and  $(W_3)$ , and  $F_3$  be the perpendicular foot of  $T_3$  on  $AB$ . Then

$$O'F_3 = \frac{\frac{r}{2}}{\frac{r}{2} - r_A} \cdot O'C = \frac{(r_2 - r_1)r^2}{2(r_1^2 + r_2^2)},$$

and from this we see that  $O_1F_3 : F_3O_2 = r_1^2 : r_2^2$ , so that  $O_1T_3 : T_3O_2 = r_1 : r_2$ , and hence the angle bisector of  $\angle O_1T_3O_2$  passes through  $C$ . If  $Z$  is the point of tangency of  $(O_3)$  and  $(O)$  then [9, Corollary 3] shows the same for the angle bisector of  $\angle AZB$ . But this means that the points  $C$ ,  $Z$  and  $T_3$  are collinear. This also gives us the circle  $(K_5)$  tangent to  $(O')$  and tangent to the parallel to  $AB$  through  $Z$  is Archimedean by reflection of  $(W_3)$  through  $T_3$ . See Figure 4.

There are many ways to find the interesting point  $T_3$ . Let me give two.

(1) Let  $M$ ,  $M'$ ,  $M_1$  and  $M_2$  be the midpoints of the semicircular arcs of  $(O)$ ,  $(O')$ ,  $(O_1)$  and  $(O_2)$  respectively.  $T_3$  is the second intersection of  $(O')$  with line  $M_1M'M_2$ , apart from  $M'$ . This line  $M_1M'M_2$  is an angle bisector of the angle formed by lines  $AB$  and the common tangent of  $(O)$ ,  $(O_3)$  and  $(W_{21})$ .

(2) The circle  $(AM_1O_2)$  intersects the semicircle  $(O_2)$  at  $Y$ , and the circle  $(BM_2O_1)$  intersects the semicircle  $(O_1)$  at  $X$ .  $X$  and  $Y$  are the points of tangency

<sup>2</sup>For simple constructions of  $(O_3)$ , see [9, 11].

of  $(O_3)$  with  $(O_1)$  and  $(O_2)$  respectively. Now, each of these circles intersects the midway semicircle  $(O')$  at  $T_3$ . See Figure 5.

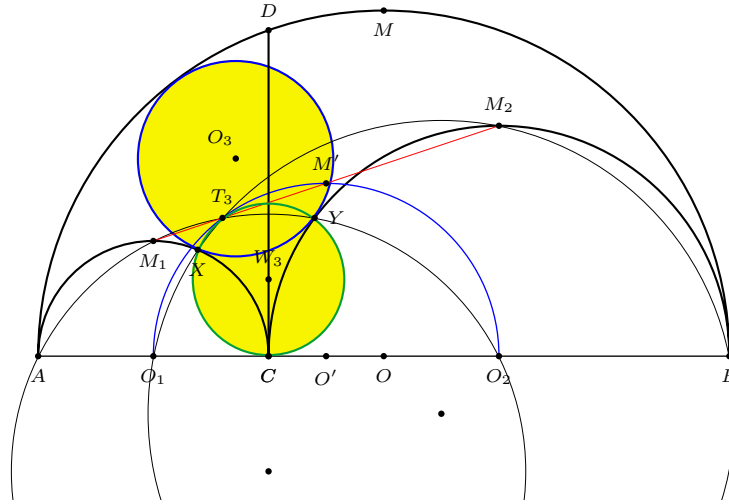


Figure 5. Construction of  $T_3$

2.4.  $(K_6)$  and  $(K_7)$ . Consider again the circles  $A(C)$  and  $B(C)$ . The circle with center  $K_6$  on the radical axis of  $A(C)$  and  $(O)$  and tangent to  $A(C)$  as well as  $(O')$  is the Archimedean circle  $(K_6)$ . This circle is easily constructed by noticing that the common tangent of  $A(C)$  and  $(K_6)$  passes through  $O_2$ . Let  $T_6$  be the point of tangency, then  $K_6$  is the intersection of the mentioned radical axis and  $AT_6$ . Similarly one finds  $(K_7)$ . See Figure 6.

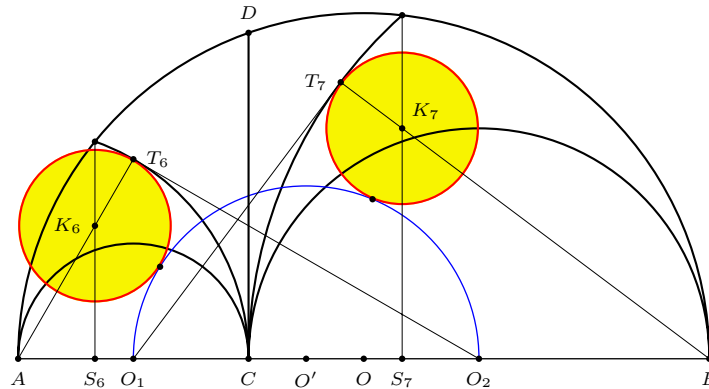


Figure 6. The Archimedean circles  $(K_6)$  and  $(K_7)$

To prove that  $(K_6)$  is indeed Archimedean, let  $S_6$  be the intersection of  $AB$  and the radical axis of  $A(C)$  and  $(O)$ , then  $O'S_6 = 2r_A + \frac{r_2 - r_1}{2}$  and  $AS_6 = 2(r_1 - r_A)$ .

If  $x$  is the radius of  $(K_6)$ , then

$$\left(\frac{r_1 + r_2}{2} + x\right)^2 - \left(2r_A + \frac{r_2 - r_1}{2}\right)^2 = (2r_1 - x)^2 - 4(r_1 - r_A)^2$$

which yields  $x = r_A$ .

For justification of the simple construction note that

$$\cos \angle S_6 K_6 A = \frac{AS_6}{AK_6} = \frac{2r_1 - 2r_A}{2r_1 - r_A} = \frac{2r_1}{2r_1 + r_2} = \frac{AT_6}{AO_2} = \cos \angle AO_2 T_6.$$

2.5.  $(K_8)$  and  $(K_9)$ . Let  $A'$  and  $B'$  be the reflections of  $C$  through  $A$  and  $B$  respectively. The circle with center on  $AB$ , tangent to the tangent from  $A'$  to  $(O_2)$  and to the radical axis of  $A(C)$  and  $(O)$  is the circle  $(K_8)$ . Similarly we find  $(K_9)$  from  $B'$  and  $(O_1)$  respectively.

Let  $x$  be the radius of  $(K_8)$ . We have

$$\frac{4r_1 + r_2}{r_2} = \frac{4r_1 - 2r_A - x}{x}$$

implying indeed that  $x = r_A$ . And  $(K_8)$  is Archimedean, while this follows for  $(K_9)$  as well by symmetry. See Figure 7.

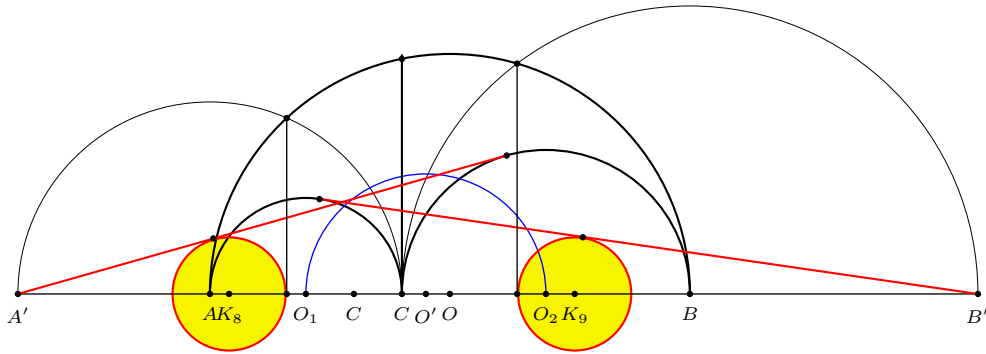


Figure 7. The Archimedean circles  $(K_8)$  and  $(K_9)$

2.6. *Four more Archimedean circles from the midway semicircle.* The points  $M$ ,  $M_1$ ,  $M_2$ ,  $O$ ,  $C$  and the point of tangency of  $(O)$  and the incircle  $(O_3)$  are concyclic, the center of their circle, the *mid-arc circle* is  $M'$ . This circle  $(M')$  meets  $(O')$  in two points  $K_{10}$  and  $K_{11}$ . The circles with  $K_{10}$  and  $K_{11}$  as centers and tangent to  $AB$  are Archimedean. To see this note that the radius of  $(MM_1M_2)$  equals  $\sqrt{\frac{r_1^2 + r_2^2}{2}}$ . Now we know the sides of triangle  $O'M'K_{10}$ . The altitude from  $K_{10}$  has length  $\frac{\sqrt{(r_1^2 + 4r_1r_2 + r_2^2)(r_1^2 + r_2^2)}}{2r}$  and divides  $O'M'$  indeed in segments of  $\frac{r_1^2 + r_2^2}{2r}$  and  $r_A$ . Of course by similar reasoning this holds for  $(K_{11})$  as well.

Now the circle  $M(C)$  meets  $(O)$  in two points. The smallest circles  $(K_{12})$  and  $(K_{13})$  through these points tangent to  $AB$  are Archimedean as well. This can be seen by applying homothety with center  $C$  with factor 2 to the points  $K_{10}$  and  $K_{11}$ .

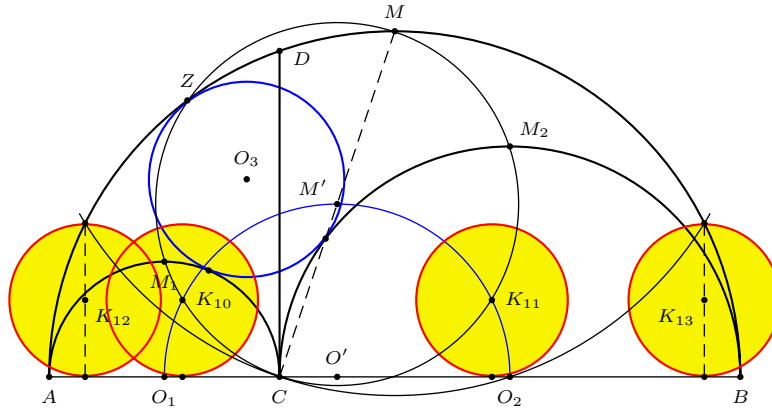


Figure 8. The Archimedean circles  $(K_{10})$ ,  $(K_{11})$ ,  $(K_{12})$  and  $(K_{13})$

The image points are the points where  $M(C)$  meets  $(O)$  and are at distance  $2r_A$  from  $AB$ . See Figure 8.

2.7.  $(K_{14})$  and  $(K_{15})$ . It is easy to see that the semicircles  $A(C)$ ,  $B(C)$  and  $(O)$  are images of  $(O_1)$ ,  $(O_2)$  and  $(O')$  after homothety through  $C$  with factor 2. This shows that  $A(C)$ ,  $B(C)$  and  $(O)$  have a common tangent parallel to the common tangent  $d$  of  $(O_1)$ ,  $(O_2)$  and  $(O')$ . As a result, the circles  $(K_{14})$  and  $(K_{15})$  tangent internally to  $A(C)$  and  $B(C)$  respectively and both tangent to  $d$  at the opposite of  $(O_1)$ ,  $(O_2)$  and  $(O')$ , are Archimedean circles, just as is Bankoff's quadruplet circle  $(W_4)$ .

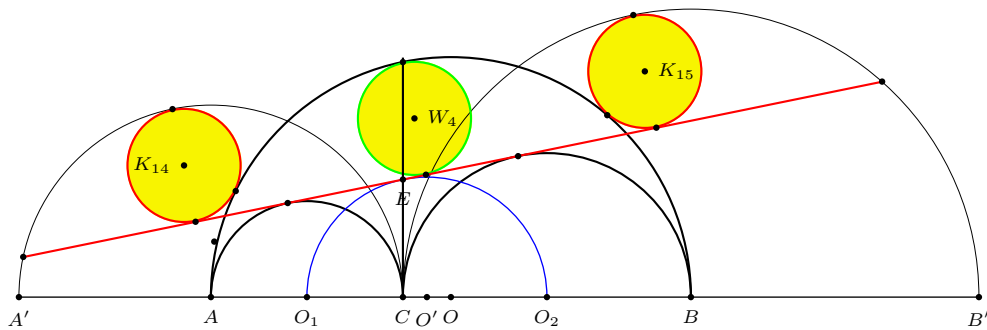


Figure 9. The Archimedean circles  $(K_{14})$  and  $(K_{15})$

An additional property of  $(K_{14})$  and  $(K_{15})$  is that these are tangent to  $(O)$  externally. To see this note that, using linearity, the distance from  $A$  to  $d$  equals  $\frac{2r_1^2}{r}$ , so  $AK_{14} = \frac{r_1(2r_1+r_2)}{r}$ . Let  $F_{14}$  be the perpendicular foot of  $K_{14}$  on  $AB$ . In triangle  $COD$  we see that  $CD = 2\sqrt{r_1r_2}$  and thus by similarity of  $COD$  and  $F_{14}AK_{14}$

we have

$$K_{14}F_{14} = \frac{2\sqrt{r_1r_2}}{r}AK_{14} = \frac{2r_1\sqrt{r_1r_2}(2r_1 + r_2)}{r^2},$$

$$OF_{14} = r + \frac{r_2 - r_1}{r}AK_{14} = \frac{r_2^3 + 4r_1r_2^2 + 4r_1^2r_2 - r_1^3}{r^2},$$

and now we see that  $K_{14}F_{14}^2 + OF_{14}^2 = (r + r_A)^2$ . In the same way it is shown  $(K_{15})$  is tangent to  $(O)$ . See Figure 9. Note that  $O_1K_{15}$  passes through the point of tangency of  $d$  and  $(O_2)$ , which also lies on  $O_1(D)$ . We leave details to the reader.

2.8.  $(K_{16})$  and  $(K_{17})$ . Apply the homothety  $h(A, \lambda)$  to  $(O)$  and  $(O_1)$  to get the circles  $(\Omega)$  and  $(\Omega_1)$ . Let  $U(\rho)$  be the circle tangent to these two circles and to  $CD$ , and  $U'$  the perpendicular foot of  $U$  on  $AB$ . Then  $|U'\Omega| = \lambda r - 2r_1 + \rho$  and  $|U'\Omega_1| = (\lambda - 2)r_1 + \rho$ . Using the Pythagorean theorem in triangle  $UU'\Omega$  and  $UU'\Omega_1$  we find

$$(\lambda r_1 + \rho)^2 - ((\lambda - 2)r_1 + \rho)^2 = (\lambda r - \rho)^2 - (\lambda r - 2r_1 + \rho)^2.$$

This yields  $\rho = r_A$ .

By symmetry, this shows that the twin circles of Archimedes are members of a family of Archimedean twin circles tangent to  $CD$ . In particular,  $(W_6)$  and  $(W_7)$  of [4] are limiting members of this family. As special members of this family we add  $(K_{16})$  as the circle tangent to  $C(A)$ ,  $B(A)$ , and  $CD$ , and  $(K_{17})$  tangent to  $C(B)$ ,  $A(B)$ , and  $CD$ . See Figure 10.

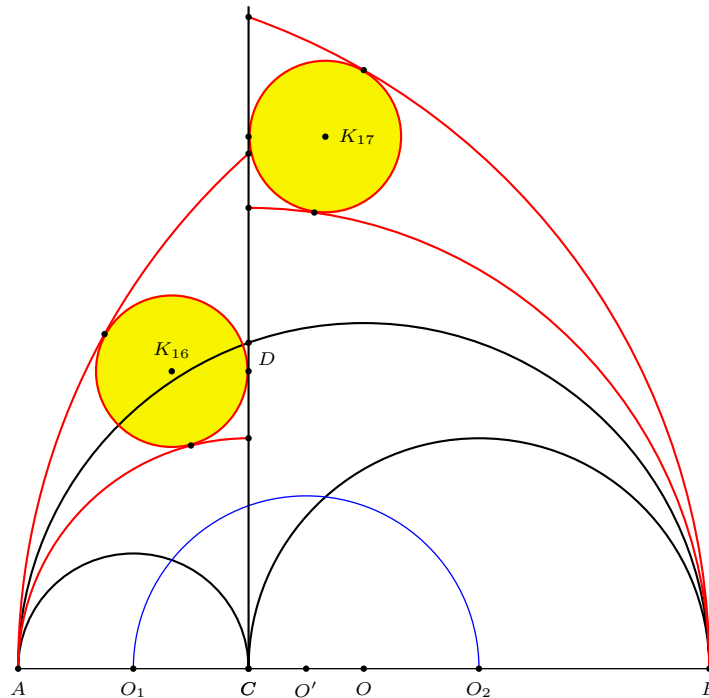


Figure 10. The Archimedean circles  $(K_{16})$  and  $(K_{17})$

### 3. Extending circles to families

There is a very simple way to turn each Archimedean circle into a member of an infinite Archimedean family, that is by attaching to  $(O)$  a semicircle  $(O')$  to be the two inner semicircles of a new arbelos. When  $(O')$  is chosen smartly, this new arbelos gives Archimedean circles exactly of the same radii as Archimedean circles of the original arbelos. By repetition this yields infinite families of Archimedean circles. If  $r''$  is the radius of  $(O')$ , then we must have

$$\frac{r_1 r_2}{r} = \frac{r r''}{r + r''}$$

which yields

$$r'' = \frac{r r_1 r_2}{r^2 - r_1 r_2},$$

surprisingly equal to  $r_3$ , the radius of the incircle  $(O_3)$  of the original arbelos, as derived in [4] or in generalized form in [6, Theorem 1]. See Figure 11.

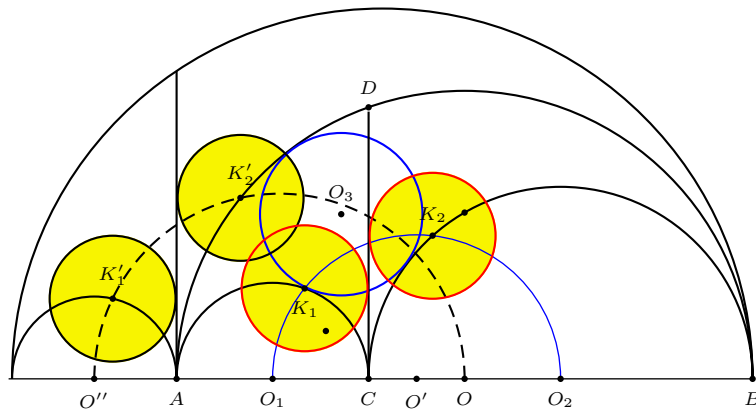


Figure 11

Now let  $(O_1(\lambda))$  and  $(O_2(\lambda))$  be semicircles with center on  $AB$ , passing through  $C$  and with radii  $\lambda r_1$  and  $\lambda r_2$  respectively. From the reasoning of §2.7 it is clear that the common tangent of  $(O_1(\lambda))$  and  $(O_2(\lambda))$  and the semicircles  $(O_1(\lambda + 1))$ ,  $(O_2(\lambda + 1))$  and  $(O_1(\lambda + 1)O_2(\lambda + 1))$  enclose Archimedean circles  $(K_{14}(\lambda))$ ,  $(K_{15}(\lambda))$  and  $(W_4(\lambda))$ . The result is that we have three families. By homothety the point of tangency of  $(K_{14}(\lambda))$  and  $(O_1(\lambda + 1))$  runs through a line through  $C$ , so that the centers  $K_{14}(\lambda)$  run through a line as well. This line and a similar line containing the centers of  $K_{15}(\lambda)$  are perpendicular. This is seen best by the well known observation that the two points of tangency of  $(O_1)$  and  $(O_2)$  with their common tangent together with  $C$  and  $D$  are the vertices of a rectangle, one of Bankoff's surprises [2]. Of course the centers  $W_4(\lambda)$  lie on a line perpendicular to  $AB$ . See Figure 12.

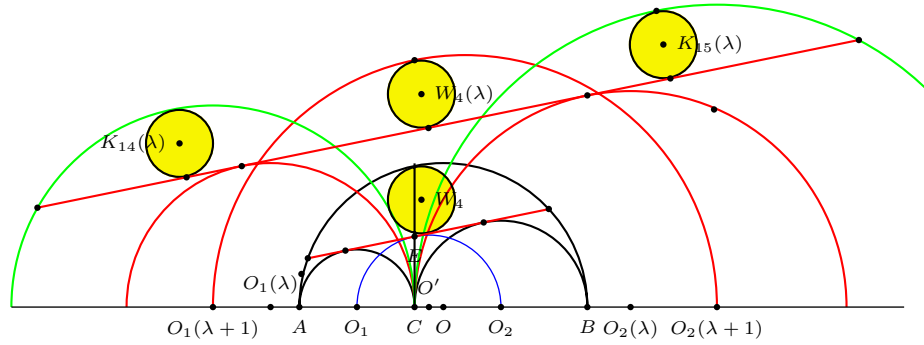


Figure 12

#### 4. A family of salina

We consider another way to generalize the Archimedean circles in infinite families. Our method of generalization is to translate the two basic semicircles ( $O_1$ ) and ( $O_2$ ) and build upon them a (skew) *salinon*. We do this in such a way that to each arbelos there is a family of salina that accompanies it. The family of salina and the arbelos are to have common tangents.

This we do by starting with a point  $O'_t$  that divides  $O_1O_2$  in the ratio  $O_1O'_t : O'_tO_2 = t : 1 - t$ . We create a semicircle ( $O'_t$ ) = ( $O_{t,1}O_{t,2}$ ) with radius  $r'_t = (1 - t)r_1 + tr_2$ , so that it is tangent to  $d$ . This tangent passes through  $E$ , see [6, Theorem 8], and meets  $AB$  in  $N$ , the external center of similitude of ( $O_1$ ) and ( $O_2$ ). Then we create semicircles ( $O_{t,1}$ ) and ( $O_{t,2}$ ) with radii  $r_1$  and  $r_2$  respectively. These two semicircles have a semicircular hull ( $O_t$ ) = ( $A_tB_t$ ) and meet  $AB$  as second points in  $C_{t,1}$  and  $C_{t,2}$  respectively. Through these we draw a semicircle ( $O_{t,4}$ ) = ( $C_{t,2}C_{t,1}$ ) opposite to the other semicircles with respect to  $AB$ . Assume  $r_1 < r_2$ .  $C_{t,1}$  is on the left of  $C_{t,2}$  if and only if  $t \geq \frac{1}{2}$ .<sup>3</sup> In this case we call the region bounded by the 4 semicircles the  $t$ -salinon of the arbelos. See Figure 13.

Here are the coordinates of the various points.

$$\begin{array}{ll}
 O'_t & (tr_1 + (t-1)r_2, 0) \\
 O_{t,1} & ((2t-1)r_1 - r_2, 0) \\
 A_t & ((2t-2)r_1 - r_2, 0) \\
 C_{t,1} & (2tr_1 - r_2, 0) \\
 O_t & \left( \left( t - \frac{1}{2} \right) r, 0 \right) \\
 C_t & \left( \frac{r_1^2 - r_2^2 + (2t-1)r_1r_2}{r}, 0 \right) \\
 O_{t,2} & (r_1 + (2t-1)r_2, 0) \\
 B_t & (r_1 + 2tr_2, 0) \\
 C_{t,2} & (r_1 + (2t-2)r_2, 0) \\
 O_{t,4} & \left( \left( t + \frac{1}{2} \right) r_1 + \left( t - 1\frac{1}{2} \right) r_2, 0 \right)
 \end{array}$$

The radical axis of the circles

$$(O'_t) : (x - tr_1 - (t-1)r_2)^2 + y^2 = ((1-t)r_1 + tr_2)^2$$

<sup>3</sup>If  $t < \frac{1}{2}$  we can still find valid results by drawing ( $O_{t,4}$ ) on the same side of  $AB$  as the other semicircles. In this paper we will not refer to these results, as the resulting figure is not really like Archimedes' salinon.

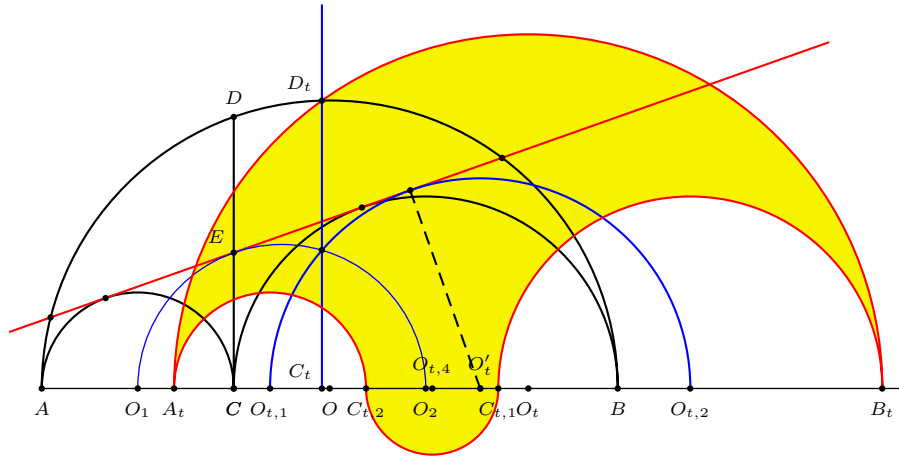


Figure 13. A  $t$ -salinon

and

$$(O') : \quad \left(x - \frac{r_1 - r_2}{2}\right)^2 + y^2 = \frac{r^2}{4}$$

is the line

$$l_t : \quad x = \frac{(2t - 1)r_1r_2 + r_1^2 - r_2^2}{r}.$$

So is the radical axis of

$$(O) : \quad x^2 + y^2 = r^2$$

and

$$(O_t) : \quad \left(x - \left(t - \frac{1}{2}\right)r\right)^2 + y^2 = \left(\left(\frac{1}{2} - t\right)r_1 + \left(t + \frac{1}{2}\right)r_2\right)^2.$$

On this common radical axis  $l_t$ , we define points the points  $C_t$  on  $AB$  and  $D_t$  on  $(O)$ . See Figure 13.

### 5. Archimedean circles in the $t$ -salinon

5.1. *The twin circles of Archimedes.* We can generalize the well known Adam and Eve of the Archimedean circles to adjoint salina in the following way: The circles  $W_{t,1}$  and  $W_{t,2}$  tangent to both  $(O_t)$  and  $l_t$  and to  $(O_1)$  and  $(O_2)$  respectively are Archimedean.

This can be proven with the above coordinates: The semicircles  $O_t((1\frac{1}{2} - t)r_1 + (t + \frac{1}{2})r_2 - r_A)$  and  $O_1(r_1 + r_A)$  intersect in the point

$$W_{t,1} \left( \frac{(2t - 2)r_1r_2 + r_1^2 - r_2^2}{r}, r_A \sqrt{(3 - 2t)(2t + 1 + \frac{2r_1}{r_2})} \right),$$

which lies indeed  $r_A$  left of  $\ell_t$ . Similarly we find for the intersection of  $O_t((1\frac{1}{2} - t)r_1 + (t + \frac{1}{2})r_2 - r_A)$  and  $O_2(r_2 + r_A)$

$$W_{t,2} \left( \frac{r_1^2 + 2r_1r_2t - r_2^2}{r}, r_A \sqrt{(2t+1)(3-2t + \frac{2r_2}{r_1})} \right),$$

which lies  $r_A$  right of  $\ell_t$ . See Figure 14.

These circles are real if and only if  $\frac{1}{2} \leq t \leq \frac{3}{2}$ .

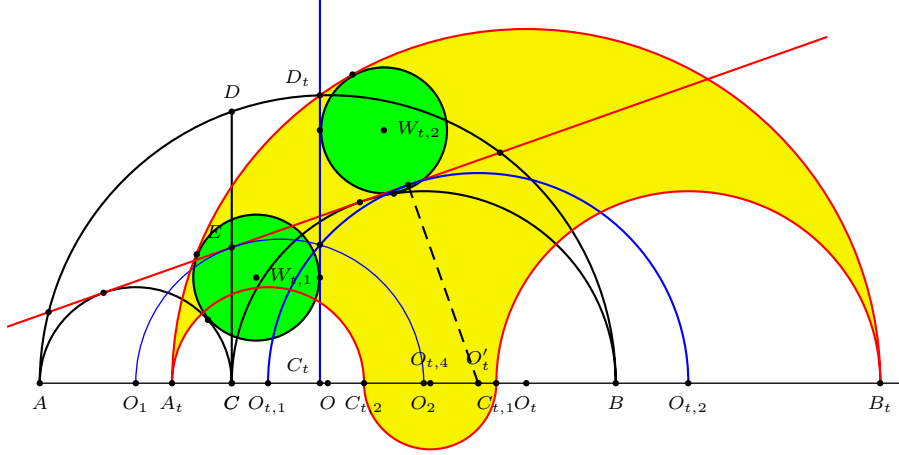


Figure 14. The Archimedean circles  $(W_{t,1})$  and  $(W_{t,2})$

Two properties of the twin circles of Archimedes can be generalized as well. Dodge et al [4] state that the circle  $A(D)$  passes through the point of tangency of  $(O_2)$  and  $(W_2)$ , while Wendijk [8] and d'Ignazio and Suppa in [5, p. 236] ask in a problem to show that the point  $N_2$  where  $O_2W_2$  meets  $CD$  lies on  $O_2(A)$  (reworded). We can generalize these to

- the circle  $A(D_t)$  passes through the point of tangency of  $(W_{t,2})$  and  $(O_2)$ ;
- the point  $N_{t,2}$  where  $O_2W_{t,2}$  meets the perpendicular to  $AB$  through  $C_{t,1}$  lies on  $O_2(A)$ .

Of course similar properties are found for  $W_{t,1}$ .

To verify this note that  $D_t$  has coordinates

$$D_t \left( \frac{r_1^2 - r_2^2 + (2t-1)r_1r_2}{r}, \frac{\sqrt{r_1r_2((3-2t)r_1 + 2r_2)(2r_1 + (2t+1)r_2)}}{r} \right),$$

while the point of contact  $R_2$  of  $(W_{t,2})$  and  $(O_2)$  is

$$\begin{aligned} & O_2 + \frac{r_2}{r_2 + r_A}(W_{t,2} - O_2) \\ &= \left( \frac{2r_1^2 - r_2^2 + 2tr_1r_2}{2r_1 + r_2}, \frac{\sqrt{(1+2t)r_1r_2^2((3-2t)r_1 + 2r_2)}}{2r_1 + r_2} \right). \end{aligned}$$

Straightforward verification now shows that

$$d(A, D_t)^2 = d(A, R_2)^2 = 2r_1(2r_1 + (1 + 2t)r_2).$$

Furthermore, we see that the point  $N_{t,2} = O_2 + \frac{2r_1+r_2}{r}(R_2 - O_2)$  has coordinates

$$N_{t,2} = \left( 2tr_1 - r_2, \sqrt{(1 + 2t)r_1((3 - 2t)r_1 + 2r_2)} \right)$$

so that this point lies indeed on  $O_2W_{t,2}$ , on  $O_2(A)$  and on the perpendicular to  $AB$  through  $C_{t,1}$ .

5.2. ( $W_{t,3}$ ). Consider the circle through  $C$  tangent to  $(O'_t)$  and with center on  $C_tD_t$ . When  $u$  is the radius of this circle we have

$$\begin{aligned} (r't - u)^2 - O'tCt^2 &= u^2 - CCt^2 \\ ((1 - t)r_1 + tr_2 - u)^2 - \left(\frac{(t - 1)r_1^2 + tr_2^2}{r}\right)^2 &= u^2 - ((2t - 1)r_A)^2 \end{aligned}$$

which yields  $u = r_A$ . This shows that this circle ( $W_{t,3}$ ) is an Archimedean circle and generalizes the Bankoff triplet circle ( $W_3$ ), using the tangency of ( $W_3$ ) to  $(O')$  shown above. See Figure 15. These circles are real if and only if  $\frac{1}{2} \leq t \leq 1$ .

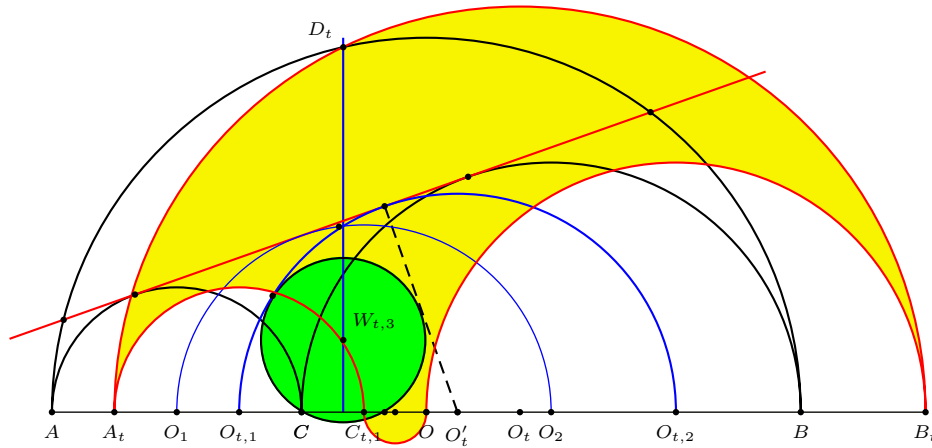


Figure 15. The Archimedean circle ( $W_{t,3}$ )

5.3. ( $W_{t,4}$ ). To generalize Bankoff's quadruplet circle ( $W_4$ ) we start with a lemma.

**Lemma 1.** *Let  $(K)$  be a circle with center  $K$  and  $\ell_1$  and  $\ell_2$  be two tangents to  $(K)$  meeting in a point  $P$ . Let  $L$  be a point travelling through the line  $PK$ . When  $L$  travels linearly, then the radical axis of  $(K)$  and the circle  $(L)$  tangent to  $\ell_1$  and  $\ell_2$  moves linearly as well. The speed relative to the speed of  $L$  depends only on the angle of  $\ell_1$  and  $\ell_2$ .*

*Proof.* Let  $P$  be the origin for Cartesian coordinates. Without loss of generality let  $\ell_1$  and  $\ell_2$  be lines making angles  $\pm\phi$  with the  $x$ -axis and let  $K(x_1, 0), L(x_2, 0)$ . With  $v = \sin \phi$  the circles  $(K)$  and  $(L)$  are given by

$$\begin{aligned} (x - x_1)^2 + y^2 &= v^2 x_1^2, \\ (x - x_2)^2 + y^2 &= v^2 x_2^2. \end{aligned}$$

The radical axis of these is  $x = \frac{1-v^2}{2}(x_1 + x_2) = \frac{1}{2} \cos^2 \phi(x_1 + x_2)$ . □

It is easy to check that the slope of the line through the midpoints of the semi-circular arcs  $(O)$  and  $(O_t)$  is equal to  $\frac{r_2-r_1}{r}$  and thus equal to the slope of the line through  $M_1$  and  $M_2$ . This shows that the common tangent of  $(O)$  and  $(O_t)$  is parallel to the common tangent of  $(O_1)$  and  $(O_2)$ , i.e.  $d$ , also the common tangent of  $(O')$  and  $(O'_t)$ . As a result of Lemma 1, it is now clear why the radical axes of  $(O)$  and  $(O_t)$  and of  $(O')$  and  $(O'_t)$  coincide for all  $t$ . Another consequence is that the greatest circle tangent to  $d$  at the opposite side of  $(O_1)$  and  $(O_2)$  and to  $(O_t)$  internally is, just as the famous example of Bankoff's quadruplet circle  $(W_4)$ , an Archimedean circle  $(W_{t,4})$ . See Figure 16. Of course this means that the circles  $(K_{12})$  and  $(K_{13})$  are found as members of the family  $(W_{t,4})$ .

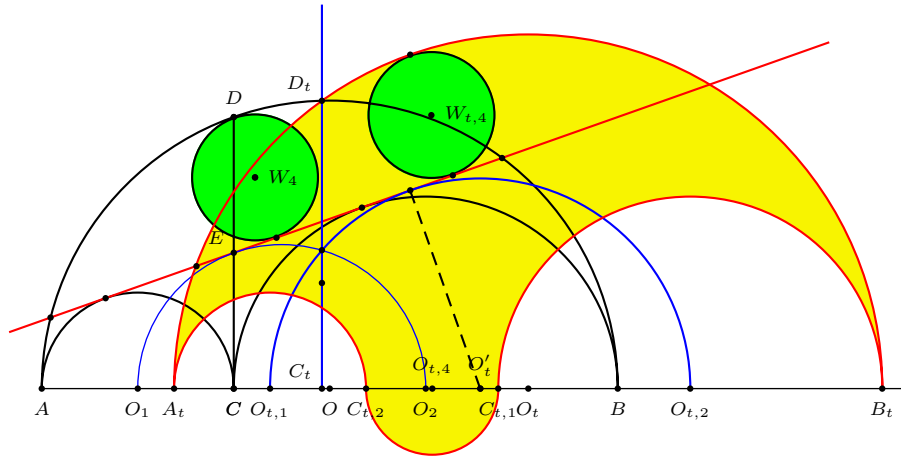


Figure 16. The Archimedean circle  $(W_{t,4})$

We note that the Archimedean circles  $(W_{t,4})$  can be constructed easily in any (skew) salinon without seeing the salinon as adjoint to an arbelos and without reconstructing (parts of) this arbelos. To see this we note that  $N$  is external center of similitude of  $(O_{t,1})$  and  $(O_{t,2})$ , and  $d$  is the tangent from  $N$  to  $(O'_t)$ .

5.4.  $(W_{t,6}), (W_{t,7}), (W_{t,13})$  and  $(W_{t,14})$ . Whereas  $(W_{13}), (W_{14}), (K_1)$  and  $(K_2)$  are defined in terms of intersections of (semi-)circles or radical axes, with Lemma 1 their generalizations are obvious:  $CD, (O)$  and  $(O')$  can be replaced by  $C_tD_t, (O_t)$  and  $(O'_t)$ . The generalization of  $(W_6)$  and  $(W_7)$  as Archimedean circles with

center on  $AB$  and tangent to  $CtDt$  keep having some interest as well. The tangents from  $A$  to  $(O_{t,2})$  and from  $B$  to  $(O_{t,1})$  are tangent to  $(W_{t,6})$  and  $(W_{t,7})$  respectively. This is seen by straightforward calculation, which is left to the reader. As a result we have two stacks of three families. See Figure 17.

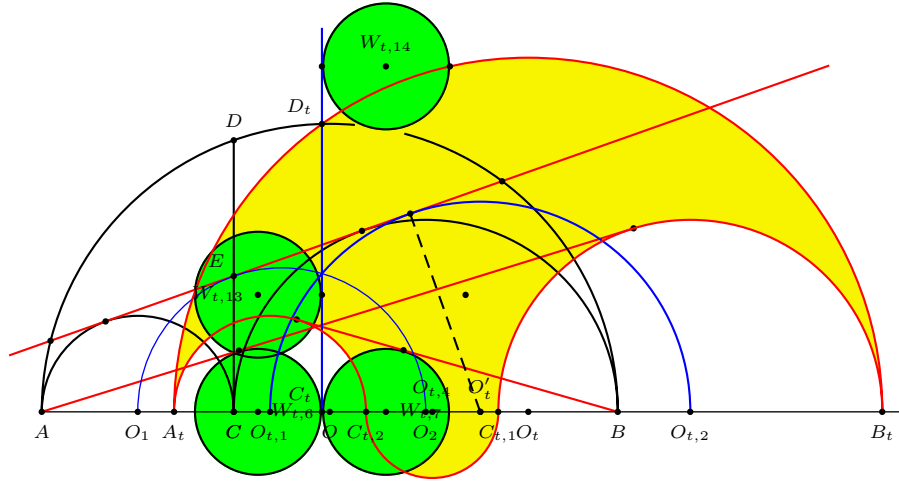


Figure 17. The Archimedean circles  $(W_{t,6})$ ,  $(W_{t,7})$ ,  $(W_{t,13})$  and  $(W_{t,14})$

We zoom in on the families  $(W_{t,13})$  and  $(W_{t,14})$ . Let  $T_{t,13}$  be the point where  $(W_{t,13})$ ,  $A(C)$  and  $(O_t)$  meet and similarly define  $T_{t,14}$ . We find that

$$T_{t,13} = \left( \frac{r_1^2 + (2t-3)r_1r_2 - r_2^2}{r}, \frac{r_1 \sqrt{-r_2((8t-12)r_1 + (4t^2 - 4t - 3)r_2)}}{r} \right),$$

$$T_{t,14} = \left( \frac{r_1^2 + (2t+1)r_1r_2 - r_2^2}{r}, \frac{r_2 \sqrt{-r_1((4t^2 - 4t - 3)r_1 - (8t+4)r_2)}}{r} \right).$$

The slope of the tangent to  $A(C)$  in  $T_{t,13}$  with respect to the  $x$ -axis is equal to

$$s_A = -\frac{x_A - x_{T_{t,13}}}{0 - y_{T_{t,13}}}$$

and the  $x$ -coordinate of the point  $R_t$  where this tangent meets  $AB$  is equal to

$$x_{R_t} = x_{T_{t,13}} - \frac{y_{T_{t,13}}}{s_A} = \frac{r(2r_1 + (1-2t)r_2)}{2r_1 + (2t-1)r_2}.$$

Similarly we find for the point  $L_t$  where the tangent to  $B(C)$  in  $T_{t,14}$  meets  $AB$

$$x_{L_t} = \frac{r((2t-1)r_1 + 2r_2)}{(2t-1)r_1 - 2r_2}.$$

The coordinates of  $Z$  are given by

$$\left( \frac{r^2(r_1 - r_2)}{r_1^2 + r_2^2}, \frac{2rr_1r_2}{r_1^2 + r_2^2} \right).$$

By straightforward calculation we can now verify that  $ZL_t^2 + ZR_t^2 = L_tR_t^2$  and we have a new characterization of the families  $(W_{t,13})$  and  $(W_{t,14})$ .

**Theorem 2.** *Let  $\mathcal{K}$  be a circle through  $Z$  with center on  $AB$ . This circle meets  $AB$  in two points  $L$  on the left hand side and  $R$  on the right hand side. Let the tangent to  $A(C)$  through  $R$  meet  $A(C)$  in  $P_1$  and similarly find  $P_2$  on  $B(C)$ . Let  $k$  be the line through the midpoint of  $P_1P_2$  perpendicular to  $AB$ . Then the smallest circles through  $P_1$  and  $P_2$  tangent to  $k$  are Archimedean.*

*Remark.* Similar characterizations can be found for  $(K_{t,1})$  and  $(K_{t,2})$ .

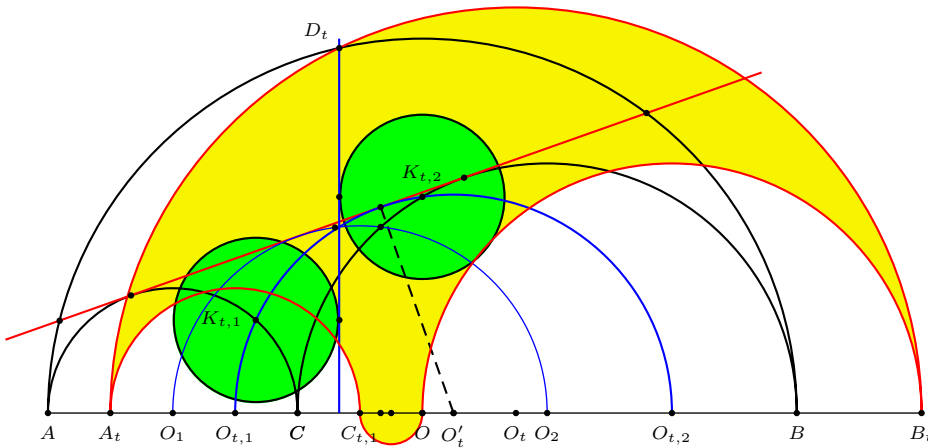


Figure 18. The Archimedean circles  $(K_{t,1})$  and  $(K_{t,2})$

5.5. *More corollaries.* We go back to Lemma 1. Since the distance between  $d$  and the common tangent of  $(O)$ ,  $(O_t)$ ,  $A(C)$  and  $B(C)$  is equal to  $2r_A$ , we notice that for instance  $A(2r_1 - r_A)$  and  $O'_t(r'_t + r_A)$  have a common tangent parallel to  $d$  as well. But that implies that we can use Lemma 1 on their intersection (and radical axis). This leads for instance to easy generalizations of  $(K_3)$  and  $(K_7)$ . See Figure 19. The lemma can also help to generalize for instance  $(K_8)$  and  $(K_9)$ , but then some more work has to be done. We leave this to the reader.

5.6. *Archimedean circles from the mid-arc circle of the salinon.* We end the adventures with a sole salinon. We note that the midpoints  $M_t$ ,  $M_{t,1}$ ,  $M_{t,2}$  and  $M_{t,4}$  of the semicircular arcs  $(O_t)$ ,  $(O_{t,1})$ ,  $(O_{t,2})$  and  $(O_{t,4})$  are concyclic. More precisely they are vertices of a rectangle. Their circle, the mid-arc circle  $(O_5)$  of the salinon and the circle  $(O_{t,1}O_{t,2})$  meet in two points, that are centers of Archimedean circles  $(K_{t,10})$  and  $(K_{t,11})$ . (Of course the parameter  $t$  does not really play a role here, but for reasons of uniformity we still use it in the naming).

To verify this, denote by  $u$  the distance between  $O_{t,1}$  and  $O_{t,2}$ . Then the distance between  $M_{t,1}$  and  $M_{t,2}$  equals  $\sqrt{u^2 + (r_2 - r_1)^2}$  and the distance from  $O'$  to  $AB$

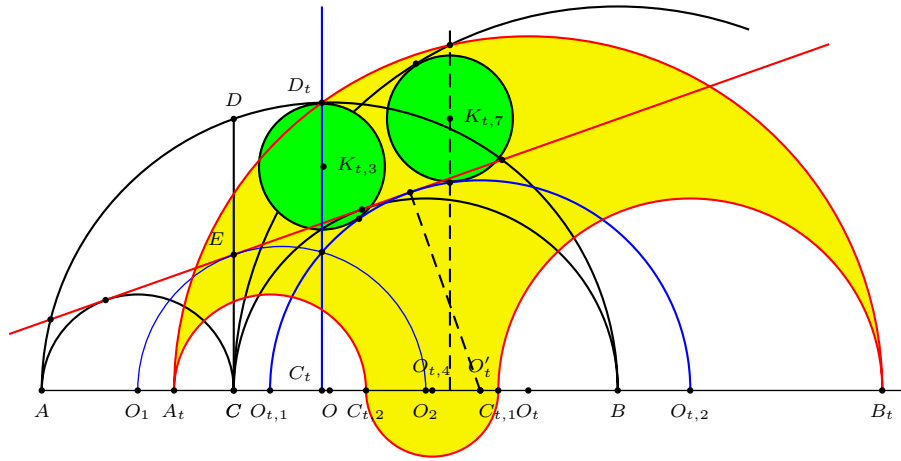


Figure 19. The Archimedean circles  $(K_{t,3})$  and  $(K_{t,7})$

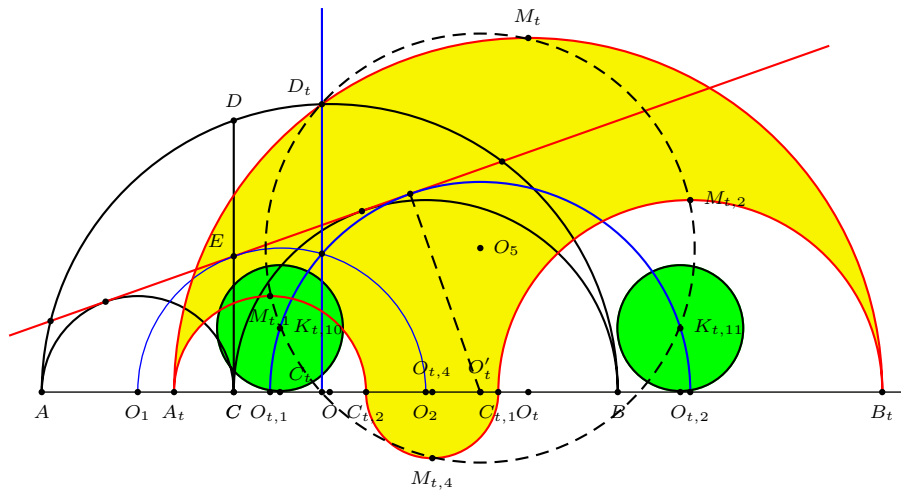


Figure 20. The Archimedean circles  $(K_{t,10})$  and  $(K_{t,11})$

equals  $\frac{r_1+r_2}{2}$ . If  $x$  is the distance from the intersections of  $(O_{t,1}O_{t,2})$  and  $(O_5)$  to  $AB$ , then

$$x^2 - \frac{u^2}{4} = \left( \frac{r_1 + r_2}{2} - x \right)^2 - \frac{u^2 + (r_2 - r_1)^2}{4}$$

which leads to  $x = \frac{r_1 r_2}{r_1 + r_2} = r_A$ . We can generalize  $(K_{12})$  and  $(K_{13})$  in a similar way. To see this we note that the circle with center on  $O_t M_t$  in the pencil generated by  $(M_{t,4})$  and  $(O_5)$  intersects  $(O_t)$  in two points at a distance of  $2r_A$  from  $AB$ . See Figure 21.

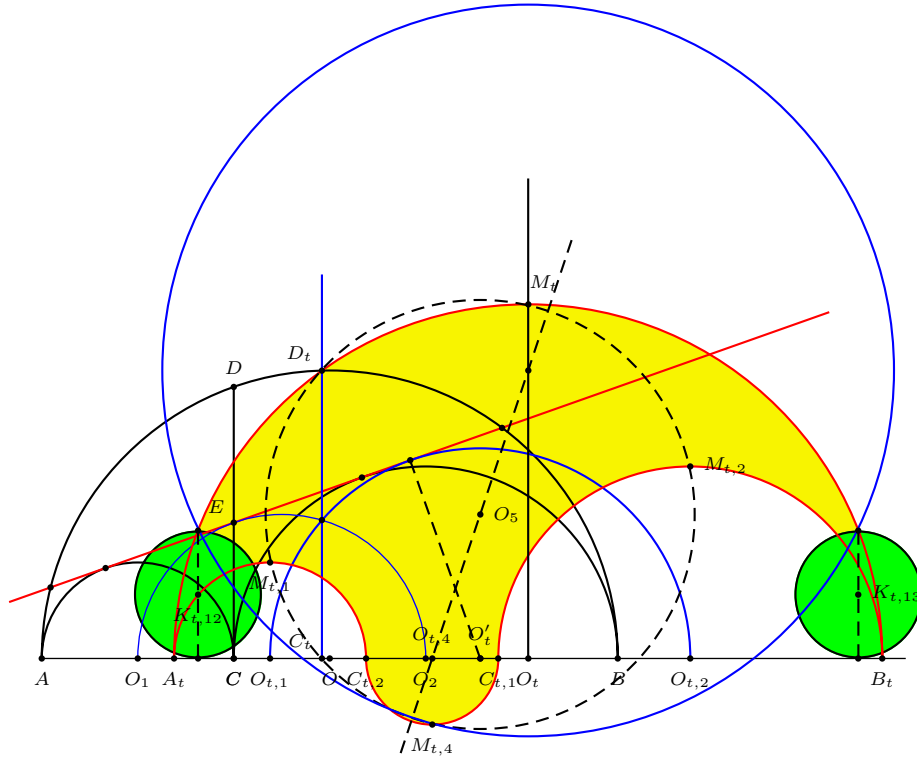


Figure 21. The Archimedean circles  $(K_{t,12})$  and  $(K_{t,13})$

## References

- [1] L. Bankoff, Are the twin circles of Archimedes really twins?, *Math. Mag.*, 47 (1974) 214–218.
- [2] L. Bankoff, A mere coincidence, *Mathematics Newsletter*, Los Angeles City College, Nov. 1954; reprinted in *College Math. Journal*, 23 (1992) 106.
- [3] L. Bankoff, The marvelous arbelos, in *The lighter Side of Mathematics*, 247–253, ed. R.K. Guy and R.E. Woodrow, Mathematical Association of America, 1994.
- [4] C.W. Dodge, T. Schoch, P.Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
- [5] I. d’Ignazio and E. Suppa, *Il Problema Geometrico, dal Compasso al Cabri*, Interlinea Editrice, Téramo, 2001.
- [6] H. Okumura and M. Watanabe, The Archimedean circles of Schoch and Woo, *Forum Geom.*, 4 (2004) 27–34.
- [7] F. Power, Some more Archimedean circles in the arbelos, *Forum Geom.*, 5 (2005) 133–134.
- [8] A. Wendijk and T. Hermann, Problem 10895, *Amer. Math. Monthly*, 108 (2001) 770; solution, 110 (2003) 63–64.
- [9] P.Y. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
- [10] P. Yiu, Private correspondence, 1999.
- [11] P. Yiu, Elegant geometric constructions, *Forum Geom.*, 5 (2005) 75–96.

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands  
 E-mail address: fvanlamoen@planet.nl