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Archimedean Adventures

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Abstract. We explore the arbelos to find more Archimedean circles and several infinite families of Archimedean circles. To define these families we study another shape introduced by Archimedes, the *salinon*.

1. Preliminaries

We consider an arbelos, consisting of three semicircles (O_1), (O_2) and (O), having radii r_1, r_2 and $r = r_1 + r_2$, mutually tangent in the points A, B and C as shown in Figure 1 below.

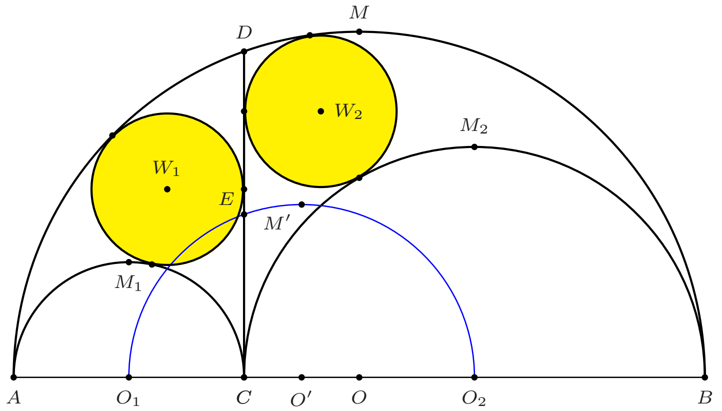

Figure 1. The arbelos with its Archimedean circles

It is well known that Archimedes has shown that the circles tangent to (O), (O_1) and CD and to (O), (O_2) and CD respectively are congruent. Their radius is $r_A = \frac{r_1 r_2}{r}$. See Figure 1. Thanks to [1, 2, 3] we know that these Archimedean twins are not only twins, but that there are many more Archimedean circles to be found in surprisingly beautiful ways. In their overview [4] Dodge et al have expanded the collection of Archimedean circles to huge proportions, 29 individual circles and an infinite family of Woo circles with centers on the Schoch line. Okumura and Watanabe [6] have added a family to the collection, which all pass through O, and have given new characterizations of the circles of Schoch and Woo. [1] Recently Power has added four Archimedean circles in a short note [7]. In this paper

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[1]The circles of Schoch and Woo often make use of tangent lines. The use of tangent lines in the arbelos, being a curvilinear triangle, may seem surprising, its relevance is immediately apparent when we realize that the common tangent of (O_1), (O_2), (O') and $C(2r_A)$ ($\mathcal{E}$ in [6]) and the common tangent of (O), (W_{21}) and the incircle (O_3) of the arbelos meet on AB.

we introduce some new Archimedean circles and some in infinite families. The Archimedean circles (W_n) are those appearing in [4]. New ones are labeled (K_n).

We adopt the following notations.

$P(r)$	circle with center P and radius r
$P(Q)$	circle with center P and passing through Q
(P)	circle with center P and radius clear from context
(PQ)	circle with diameter PQ
(PQR)	circle through P, Q, R

In the context of arbeloi, these are often interpreted as semicircles. Thus, an arbelos consists of three semicircles $(O_1) = (AC)$, $(O_2) = (CB)$, and $(O) = (AB)$ on the same side of the line AB, of radii r_1, r_2, and $r = r_1+r_2$. The common tangent at C to (O_1) and (O_2) meets (O) in D. We shall call the semicircle $(O') = (O_1O_2)$ (on the same side of AB) the *midway semicircle* of the arbelos.

It is convenient to introduce a cartesian coordinate system, with O as origin. Here are the coordinates of some basic points associated with the arbelos.

$A\ (-(r_1+r_2), 0)$	$B\ (r_1+r_2, 0)$	$C\ (r_1-r_2, 0)$
$O_1\ (-r_2, 0)$	$O_2\ (r_1, 0)$	
$M_1\ (-r_2,\ r_1)$	$M_2\ (r_1,\ r_2)$	$M\ (0, r_1+r_2)$
$O'\ \left(\frac{r_1-r_2}{2}, 0\right)$	$M'\ \left(\frac{r_1-r_2}{2},\ \frac{r_1+r_2}{2}\right)$	
$D\ (r_1-r_2,\ 2\sqrt{r_1r_2})$	$E\ (r_1-r_2,\ \sqrt{r_1r_2})$	

2. New Archimedean circles

2.1. (K_1) *and* (K_2). The Archimedean circle (W_8) has C as its center and is tangent to the tangents O_1K_2 and O_2K_1 to (O_1) and (O_2) respectively. By symmetry it is easy to find from these the Archimedean circles (K_1) and (K_2) tangent to CD, see Figure 2. A second characterization of the points K_1 and K_2, clearly equivalent, is that these are the points of intersection of (O_1) and (O_2) with (O'). We note that the tangent to $A(C)$ at the point of intersection of $A(C)$, (W_{13}) and (O) passes through B, and is also tangent to (K_1). A similar statement is true for the tangent to $B(C)$ at its intersection with (W_{14}) and (O).

To prove the correctness, let T be the perpendicular foot of K_1 on AB. Then, making use of the right triangle $O_1O_2K_1$, we find $O_1T = \frac{r_1^2}{r_1+r_2}$ and thus the radius of (K_1) is equal to $O_1C - O_1T = r_A$. By symmetry the radius of (K_2) equals r_A as well.

The circles (K_1) and (K_2) are closely related to (W_{13}) and (W_{14}), which are found by intersecting $A(C)$ and $B(C)$ with (O) and then taking the smallest circles through the respective points of intersection tangent to CD. The relation is clear when we realize that $A(C)$, $B(C)$ and (O) can be found by a homothety with center C, factor 2 applied to (O_1), (O_2) and (O').

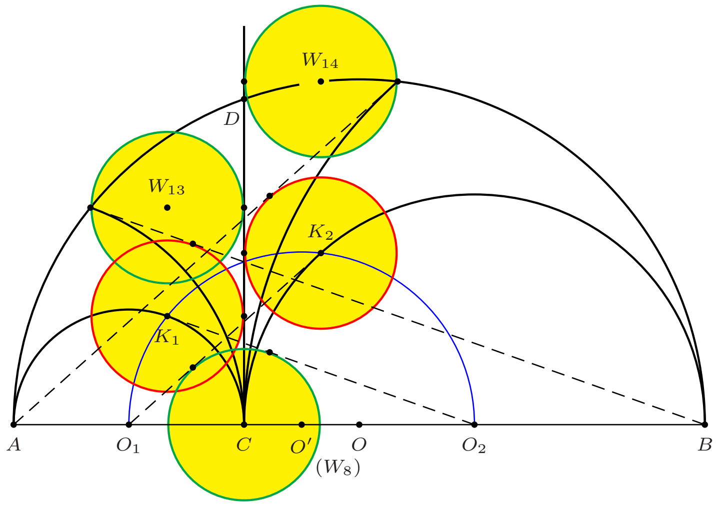

Figure 2. The Archimedean circles (K_1) and (K_2)

2.2. (K_3). In private correspondence [10] about (K_1) and (K_2), Paul Yiu has noted that the circle with center on CD tangent to (O) and (O') is Archimedean. Indeed, if x is the radius of this circle, then we have

$$(r-x)^2 - (r_1 - r_2)^2 = \left(\frac{r}{2} + x\right)^2 - \left(\frac{r_1 - r_2}{2}\right)^2$$

and that yields $x = r_A$.

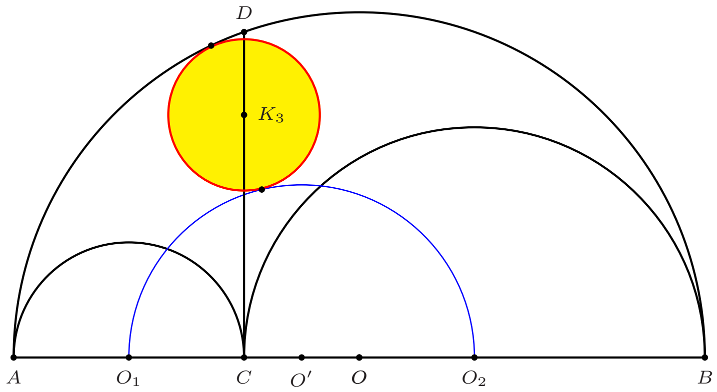

Figure 3. The Archimedean circle (K_3)

2.3. (K_4) *and* (K_5). There is an interesting similarity between (K_3) and Bankoff's triplet circle (W_3). Recall that this is the circle that passes through C and the points of tangency of (O_1) and (O_2) with the incircle (O_3) of the arbelos. [2] Just like (K_3), (W_3) has its center on CD and is tangent to (O'), a fact that seems to have been unnoted so far. The tangency can be shown by using Pythagoras' Theorem in triangle $CO'W_3$, yielding that $O'W_3 = \frac{r_1^2+r_2^2}{2r}$. This leaves for the length of the radius of (O') beyond W_3:

$$\frac{r}{2} - \frac{r_1^2 + r_2^2}{2r} = \frac{r_1 r_2}{r}.$$

This also gives us in a simple way the new Archimedean circle (K_4): the circle tangent to AB at O and to (O') is Archimedean by reflection of (W_3) through the perpendicular to AB in O'.

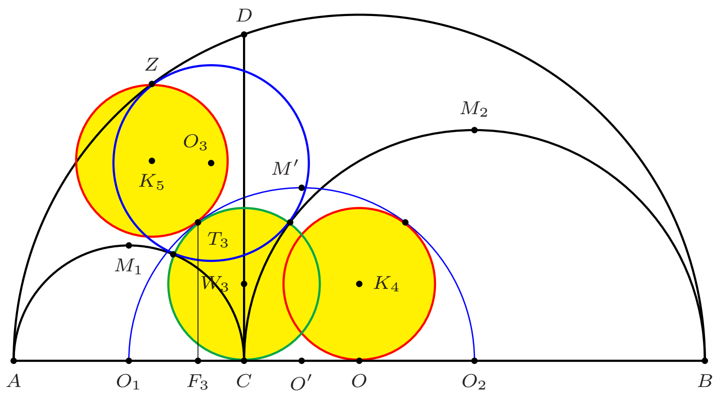

Figure 4. The Archimedean circles (K_4) and (K_5)

Let T_3 be the point of tangency of (O') and (W_3), and F_3 be the perpendicular foot of T_3 on AB. Then

$$O'F_3 = \frac{\frac{r}{2}}{\frac{r}{2} - r_A} \cdot O'C = \frac{(r_2 - r_1)r^2}{2(r_1^2 + r_2^2)},$$

and from this we see that $O_1F_3 : F_3O_2 = r_1^2 : r_2^2$, so that $O_1T_3 : T_3O_2 = r_1 : r_2$, and hence the angle bisector of $\angle O_1T_3O_2$ passes through C. If Z is the point of tangency of (O_3) and (O) then [9, Corollary 3] shows the same for the angle bisector of $\angle AZB$. But this means that the points C, Z and T_3 are collinear. This also gives us the circle (K_5) tangent to (O') and tangent to the parallel to AB through Z is Archimedean by reflection of (W_3) through T_3. See Figure 4.

There are many ways to find the interesting point T_3. Let me give two.

(1) Let M, M', M_1 and M_2 be the midpoints of the semicircular arcs of (O), (O'), (O_1) and (O_2) respectively. T_3 is the second intersection of (O') with line $M_1M'M_2$, apart from M'. This line $M_1M'M_2$ is an angle bisector of the angle formed by lines AB and the common tangent of (O), (O_3) and (W_{21}).

(2) The circle (AM_1O_2) intersects the semicircle (O_2) at Y, and the circle (BM_2O_1) intersects the semicircle (O_1) at X. X and Y are the points of tangency

[2] For simple constructions of (O_3), see [9, 11].

of (O_3) with (O_1) and (O_2) respectively. Now, each of these circles intersects the midway semicircle (O') at T_3. See Figure 5.

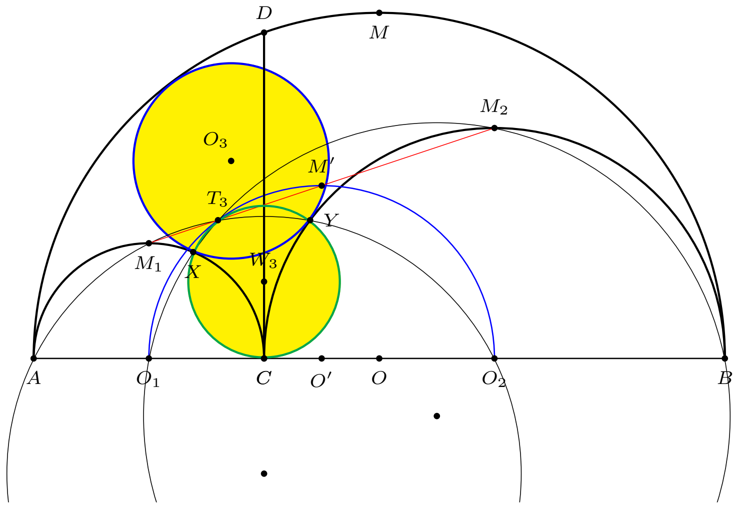

Figure 5. Construction of T_3

2.4. (K_6) *and* (K_7). Consider again the circles $A(C)$ and $B(C)$. The circle with center K_6 on the radical axis of $A(C)$ and (O) and tangent to $A(C)$ as well as (O') is the Archimedean circle (K_6). This circle is easily constructed by noticing that the common tangent of $A(C)$ and (K_6) passes through O_2. Let T_6 be the point of tangency, then K_6 is the intersection of the mentioned radical axis and AT_6. Similarly one finds (K_7). See Figure 6.

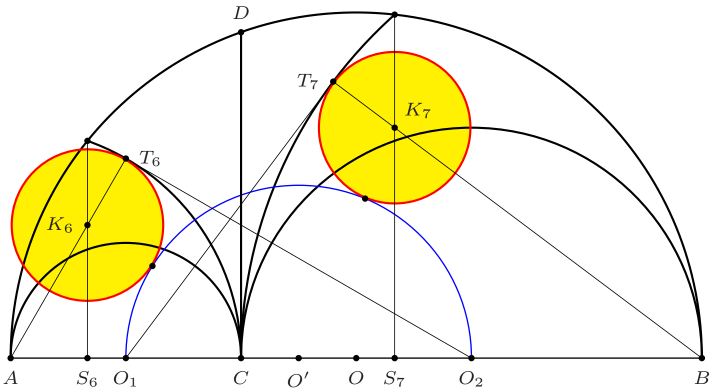

Figure 6. The Archimedean circles (K_6) and (K_7)

To prove that (K_6) is indeed Archimedean, let S_6 be the intersection of AB and the radical axis of $A(C)$ and (O), then $O'S_6 = 2r_A + \frac{r_2 - r_1}{2}$ and $AS_6 = 2(r_1 - r_A)$.

If x is the radius of (K_6), then

$$\left(\frac{r_1+r_2}{2}+x\right)^2-\left(2r_A+\frac{r_2-r_1}{2}\right)^2=(2r_1-x)^2-4(r_1-r_A)^2$$

which yields $x=r_A$.

For justification of the simple construction note that

$$\cos\angle S_6K_6A=\frac{AS_6}{AK_6}=\frac{2r_1-2r_A}{2r_1-r_A}=\frac{2r_1}{2r_1+r_2}=\frac{AT_6}{AO_2}=\cos\angle AO_2T_6.$$

2.5. (K_8) *and* (K_9). Let A' and B' be the reflections of C through A and B respectively. The circle with center on AB, tangent to the tangent from A' to (O_2) and to the radical axis of $A(C)$ and (O) is the circle (K_8). Similarly we find (K_9) from B' and (O_1) respectively.

Let x be the radius of (K_8). We have

$$\frac{4r_1+r_2}{r_2}=\frac{4r_1-2r_A-x}{x}$$

implying indeed that $x=r_A$. And (K_8) is Archimedean, while this follows for (K_9) as well by symmetry. See Figure 7.

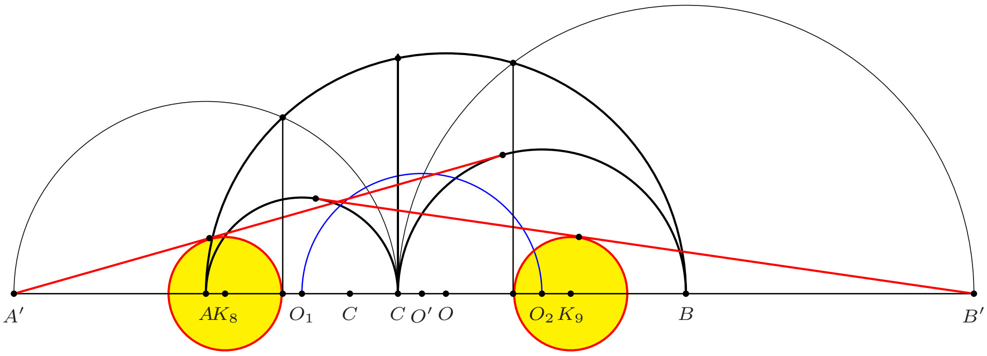

Figure 7. The Archimedean circles (K_8) and (K_9)

2.6. *Four more Archimedean circles from the midway semicircle.* The points M, M_1, M_2, O, C and the point of tangency of (O) and the incircle (O_3) are concyclic, the center of their circle, the *mid-arc circle* is M'. This circle (M') meets (O') in two points K_{10} and K_{11}. The circles with K_{10} and K_{11} as centers and tangent to AB are Archimedean. To see this note that the radius of (MM_1M_2) equals $\sqrt{\frac{r_1^2+r_2^2}{2}}$. Now we know the sides of triangle $O'M'K_{10}$. The altitude from K_{10} has length $\frac{\sqrt{(r_1^2+4r_1r_2+r_2^2)(r_1^2+r_2^2)}}{2r}$ and divides $O'M'$ indeed in segments of $\frac{r_1^2+r_2^2}{2r}$ and r_A. Of course by similar reasoning this holds for (K_{11}) as well.

Now the circle $M(C)$ meets (O) in two points. The smallest circles (K_{12}) and (K_{13}) through these points tangent to AB are Archimedean as well. This can be seen by applying homothety with center C with factor 2 to the points K_{10} and K_{11}.

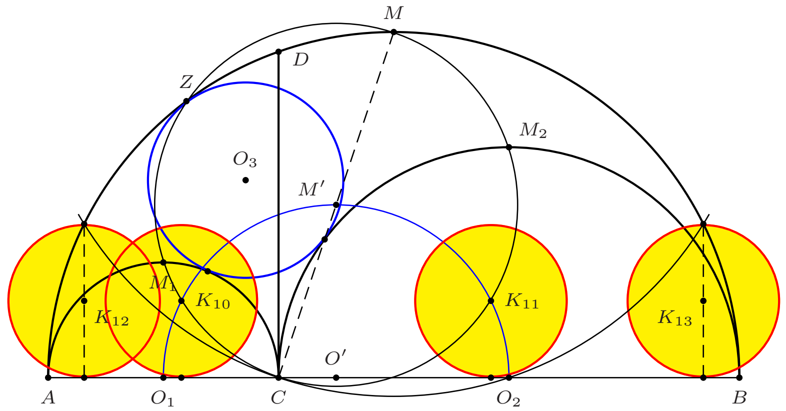

Figure 8. The Archimedean circles (K_{10}), (K_{11}), (K_{12}) and (K_{13})

The image points are the points where $M(C)$ meets (O) and are at distance $2r_A$ from AB. See Figure 8.

2.7. (K_{14}) *and* (K_{15}). It is easy to see that the semicircles $A(C)$, $B(C)$ and (O) are images of (O_1), (O_2) and (O') after homothety through C with factor 2. This shows that $A(C)$, $B(C)$ and (O) have a common tangent parallel to the common tangent d of (O_1), (O_2) and (O'). As a result, the circles (K_{14}) and (K_{15}) tangent internally to $A(C)$ and $B(C)$ respectively and both tangent to d at the opposite of (O_1), (O_2) and (O'), are Archimedean circles, just as is Bankoff's quadruplet circle (W_4).

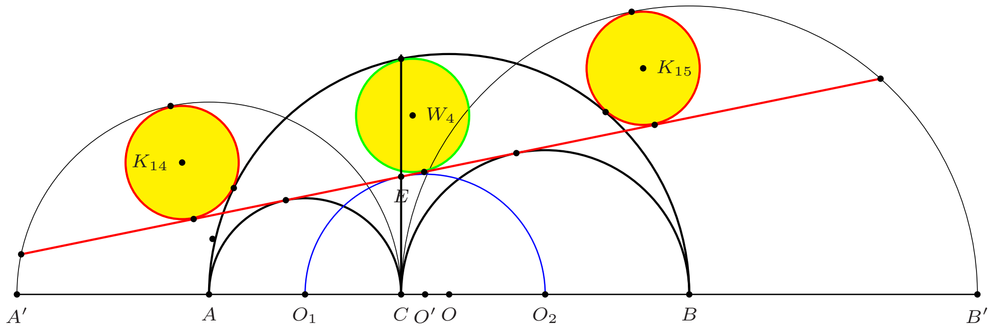

Figure 9. The Archimedean circles (K_{14}) and (K_{15})

An additional property of (K_{14}) and (K_{15}) is that these are tangent to (O) externally. To see this note that, using linearity, the distance from A to d equals $\frac{2r_1^2}{r}$, so $AK_{14} = \frac{r_1(2r_1+r_2)}{r}$. Let F_{14} be the perpendicular foot of K_{14} on AB. In triangle COD we see that $CD = 2\sqrt{r_1 r_2}$ and thus by similarity of COD and $F_{14}AK_{14}$

we have

$$K_{14}F_{14} = \frac{2\sqrt{r_1 r_2}}{r}AK_{14} = \frac{2r1\sqrt{r_1 r_2}(2r_1 + r_2)}{r^2},$$
$$OF_{14} = r + \frac{r_2 - r_1}{r}AK_{14} = \frac{r_2^3 + 4r_1 r_2^2 + 4r_1^2 r_2 - r_1^3}{r^2},$$

and now we see that $K_{14}F_{14}^2 + OF_{14}^2 = (r + r_A)^2$. In the same way it is shown (K_{15}) is tangent to (O). See Figure 9. Note that O_1K_{15} passes through the point of tangency of d and (O_2), which also lies on $O_1(D)$. We leave details to the reader.

2.8. (K_{16}) *and* (K_{17}). Apply the homothety $\mathsf{h}(A, \lambda)$ to (O) and (O_1) to get the circles (Ω) and (Ω_1). Let $U(\rho)$ be the circle tangent to these two circles and to CD, and U' the perpendicular foot of U on AB. Then $|U'\Omega| = \lambda r - 2r_1 + \rho$ and $|U'\Omega_1| = (\lambda - 2)r_1 + \rho$. Using the Pythagorean theorem in triangle $UU'\Omega$ and $UU'\Omega_1$ we find

$$(\lambda r_1 + \rho)^2 - ((\lambda - 2)r_1 + \rho)^2 = (\lambda r - \rho)^2 - (\lambda r - 2r_1 + \rho)^2.$$

This yields $\rho = r_A$.

By symmetry, this shows that the twin circles of Archimedes are members of a family of Archimedean twin circles tangent to CD. In particular, (W_6) and (W_7) of [4] are limiting members of this family. As special members of this family we add (K_{16}) as the circle tangent to $C(A)$, $B(A)$, and CD, and (K_{17}) tangent to $C(B)$, $A(B)$, and CD. See Figure 10.

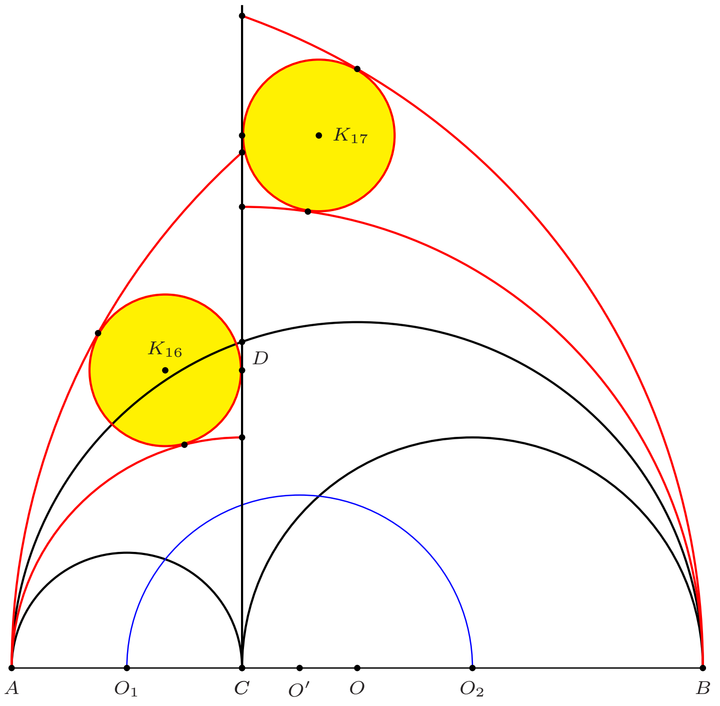

Figure 10. The Archimedean circles (K_{16}) and (K_{17})

3. Extending circles to families

There is a very simple way to turn each Archimedean circle into a member of an infinite Archimedean family, that is by attaching to (O) a semicircle (O'') to be the two inner semicircles of a new arbelos. When (O'') is chosen smartly, this new arbelos gives Archimedean circles exactly of the same radii as Archimedean circles of the original arbelos. By repetition this yields infinite families of Archimedean circles. If r'' is the radius of (O''), then we must have

$$\frac{r_1 r_2}{r} = \frac{r r''}{r + r''}$$

which yields

$$r'' = \frac{r r_1 r_2}{r^2 - r_1 r_2},$$

surprisingly equal to r_3, the radius of the incircle (O_3) of the original arbelos, as derived in [4] or in generalized form in [6, Theorem 1]. See Figure 11.

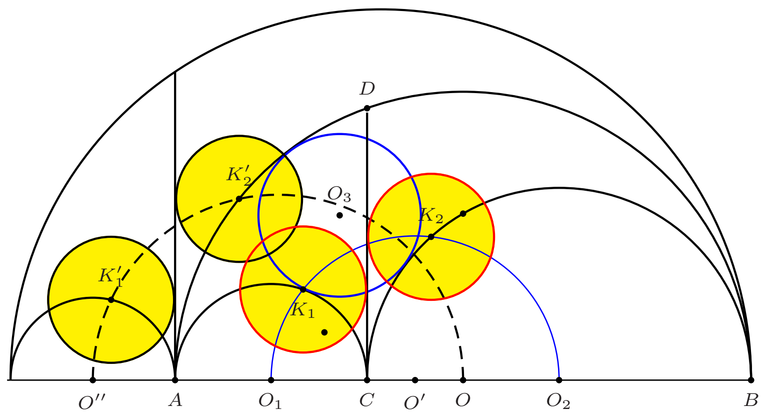

Figure 11

Now let $(O_1(\lambda))$ and $(O_2(\lambda))$ be semicircles with center on AB, passing through C and with radii λr_1 and λr_2 respectively. From the reasoning of §2.7 it is clear that the common tangent of $(O_1(\lambda))$ and $(O_2(\lambda))$ and the semicircles $(O_1(\lambda+1))$, $(O_2(\lambda+1))$ and $(O_1(\lambda+1)O_2(\lambda+1))$ enclose Archimedean circles $(K_{14}(\lambda))$, $(K_{15}(\lambda))$ and $(W_4(\lambda))$. The result is that we have three families. By homothety the point of tangency of $(K_{14}(\lambda))$ and $(O_1(\lambda+1))$ runs through a line through C, so that the centers $K_{14}(\lambda)$ run through a line as well. This line and a similar line containing the centers of $K_{15}(\lambda)$ are perpendicular. This is seen best by the well known observation that the two points of tangency of (O_1) and (O_2) with their common tangent together with C and D are the vertices of a rectangle, one of Bankoff's surprises [2]. Of course the centers $W_4(\lambda)$ lie on a line perpendicular to AB. See Figure 12.

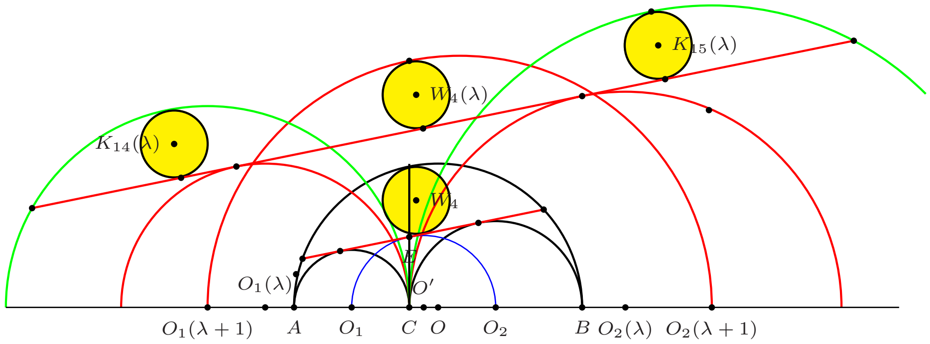

Figure 12

4. A family of salina

We consider another way to generalize the Archimedean circles in infinite families. Our method of generalization is to translate the two basic semicircles (O_1) and (O_2) and build upon them a (skew) *salinon*. We do this in such a way that to each arbelos there is a family of salina that accompanies it. The family of salina and the arbelos are to have common tangents.

This we do by starting with a point O'_t that divides O_1O_2 in the ratio $O_1O'_t : O'_tO_2 = t : 1-t$. We create a semicircle $(O'_t) = (O_{t,1}O_{t,2})$ with radius $r'_t = (1-t)r_1 + tr_2$, so that it is tangent to d. This tangent passes through E, see [6, Theorem 8], and meets AB in N, the external center of similitude of (O_1) and (O_2). Then we create semicircles $(O_{t,1})$ and $(O_{t,2})$ with radii r_1 and r_2 respectively. These two semicircles have a semicircular hull $(O_t) = (A_tB_t)$ and meet AB as second points in $C_{t,1}$ and $C_{t,2}$ respectively. Through these we draw a semicircle $(O_{t,4}) = (C_{t,2}C_{t,1})$ opposite to the other semicircles with respect to AB. Assume $r_1 < r_2$. $C_{t,1}$ is on the left of $C_{t,2}$ if and only if $t \geq \frac{1}{2}$.[3] In this case we call the region bounded by the 4 semicircles the t-salinon of the arbelos. See Figure 13.

Here are the coordinates of the various points.

O'_t	$(tr_1 + (t-1)r_2, 0)$		
$O_{t,1}$	$((2t-1)r_1 - r_2, 0)$	$O_{t,2}$	$(r_1 + (2t-1)r_2, 0)$
A_t	$((2t-2)r_1 - r_2, 0)$	B_t	$(r_1 + 2tr_2, 0)$
$C_{t,1}$	$(2tr_1 - r_2, 0)$	$C_{t,2}$	$(r_1 + (2t-2)r_2, 0)$
O_t	$\left(\left(t - \frac{1}{2}\right) r,\ 0\right)$	$O_{t,4}$	$\left((t+\frac{1}{2})r_1 + (t - 1\frac{1}{2})r_2, 0\right)$
C_t	$\left(\frac{r_1^2 - r_2^2 + (2t-1)r_1r_2}{r}, 0\right)$		

The radical axis of the circles

$$(O'_t): \qquad (x - tr_1 - (t-1)r_2)^2 + y^2 = ((1-t)r_1 + tr_2)^2$$

[3] If $t < \frac{1}{2}$ we can still find valid results by drawing $(O_{t,4})$ on the same side of AB as the other semicircles. In this paper we will not refer to these results, as the resulting figure is not really like Archimedes' salinon.

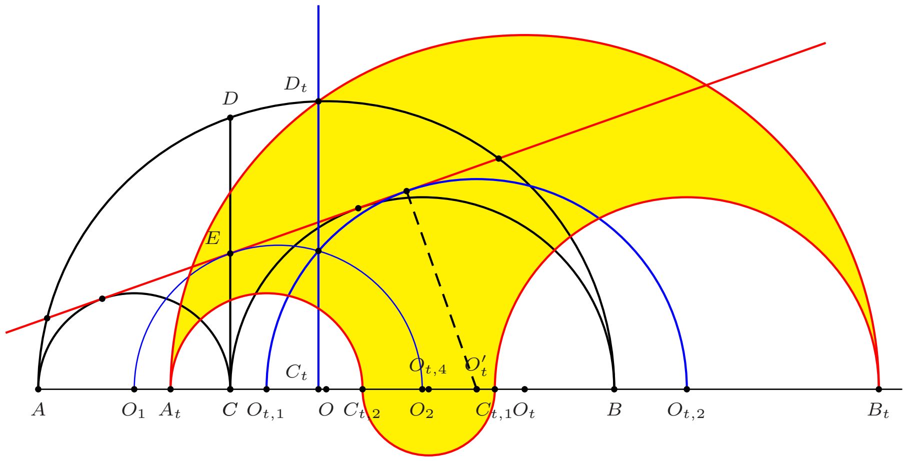

Figure 13. A t-salinon

and

$$(O'): \qquad \left(x - \frac{r_1 - r_2}{2}\right)^2 + y^2 = \frac{r^2}{4}$$

is the line

$$\ell_t: \qquad x = \frac{(2t-1)r_1r_2 + r_1^2 - r_2^2}{r}.$$

So is the radical axis of

$$(O): \qquad x^2 + y^2 = r^2$$

and

$$(O_t): \qquad \left(x - \left(t - \frac{1}{2}\right)r\right)^2 + y^2 = \left(\left(1\frac{1}{2} - t\right)r_1 + \left(t + \frac{1}{2}\right)r_2\right)^2.$$

On this common radical axis ℓ_t, we define points the points C_t on AB and D_t on (O). See Figure 13.

5. Archimedean circles in the t-salinon

5.1. *The twin circles of Archimedes.* We can generalize the well known Adam and Eve of the Archimedean circles to adjoint salina in the following way: The circles $W_{t,1}$ and $W_{t,2}$ tangent to both (O_t) and ℓ_t and to (O_1) and (O_2) respectively are Archimedean.

This can be proven with the above coordinates: The semicircles $O_t((1\frac{1}{2}-t)r_1 + (t+\frac{1}{2})r_2 - r_A))$ and $O_1(r_1 + r_A)$ intersect in the point

$$W_{t,1}\ \left(\frac{(2t-2)r_1r_2 + r_1^2 - r_2^2}{r},\ r_A\sqrt{(3-2t)(2t+1+\frac{2r_1}{r_2})}\right),$$

which lies indeed r_A left of ℓ_t. Similarly we find for the intersection of $O_t((1\frac{1}{2} - t)r_1 + (t + \frac{1}{2})r_2 - r_A))$ and $O_2(r_2 + r_A)$

$$W_{t,2}\left(\frac{r_1^2 + 2r_1r_2t - r_2^2}{r},\ r_A\sqrt{(2t+1)(3-2t+\frac{2r_2}{r_1})}\right),$$

which lies r_A right of ℓ_t. See Figure 14.

These circles are real if and only if $\frac{1}{2} \le t \le \frac{3}{2}$.

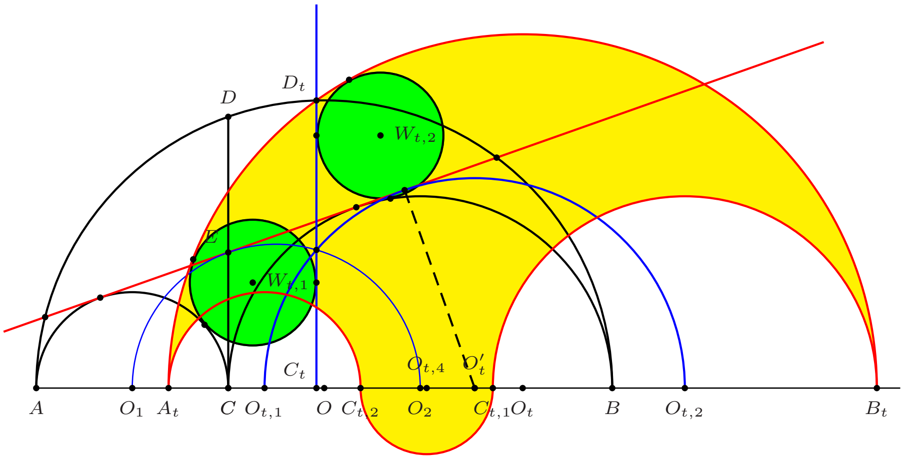

Figure 14. The Archimedean circles $(W_{t,1})$ and $(W_{t,2})$

Two properties of the twin circles of Archimedes can be generalized as well. Dodge et al [4] state that the circle $A(D)$ passes through the point of tangency of (O_2) and (W_2), while Wendijk [8] and d'Ignazio and Suppa in [5, p. 236] ask in a problem to show that the point N_2 where O_2W_2 meets CD lies on $O_2(A)$ (reworded). We can generalize these to

- the circle $A(D_t)$ passes through the point of tangency of $(W_{t,2})$ and (O_2);
- the point $N_{t,2}$ where $O_2W_{t,2}$ meets the perpendicular to AB through $C_{t,1}$ lies on $O_2(A)$.

Of course similar properties are found for $W_{t,1}$.

To verify this note that D_t has coordinates

$$D_t\left(\frac{r_1^2 - r_2^2 + (2t-1)r_1r_2}{r}, \frac{\sqrt{r_1r_2((3-2t)r_1 + 2r_2)(2r_1 + (2t+1)r_2)}}{r}\right),$$

while the point of contact R_2 of $(W_{t,2})$ and (O_2) is

$$\begin{aligned} & O_2 + \frac{r_2}{r_2 + r_A}(W_{t,2} - O_2) \\ = & \left(\frac{2r_1^2 - r_2^2 + 2tr_1r_2}{2r_1 + r_2}, \frac{\sqrt{(1+2t)r_1r_2^2((3-2t)r_1 + 2r_2)}}{2r_1 + r_2}\right). \end{aligned}$$

Straightforward verification now shows that

$$d(A, D_t)^2 = d(A, R_2)^2 = 2r_1(2r_1 + (1 + 2t)r_2).$$

Furthermore, we see that the point $N_{t,2} = O_2 + \frac{2r_1+r2}{r}(R_2 - O_2)$ has coordinates

$$N_{t,2} = \left(2tr_1 - r_2, \sqrt{(1 + 2t)r_1((3 - 2t)r_1 + 2r_2)}\right)$$

so that this point lies indeed on $O_2W_{t,2}$, on $O_2(A)$ and on the perpendicular to AB through $C_{t,1}$.

5.2. $(W_{t,3})$. Consider the circle through C tangent to (O'_t) and with center on C_tD_t. When u is the radius of this circle we have

$$\begin{aligned} (r't - u)^2 - O'tCt^2 &= u^2 - CCt^2 \\ ((1 - t)r_1 + tr_2 - u)^2 - (\frac{(t - 1)r_1^2 + tr_2^2}{r})^2 &= u^2 - ((2t - 1)r_A)^2 \end{aligned}$$

which yields $u = r_A$. This shows that this circle $(W_{t,3})$ is an Archimedean circle and generalizes the Bankoff triplet circle (W_3), using the tangency of (W_3) to (O') shown above. See Figure 15. These circles are real if and only if $\frac{1}{2} \leq t \leq 1$.

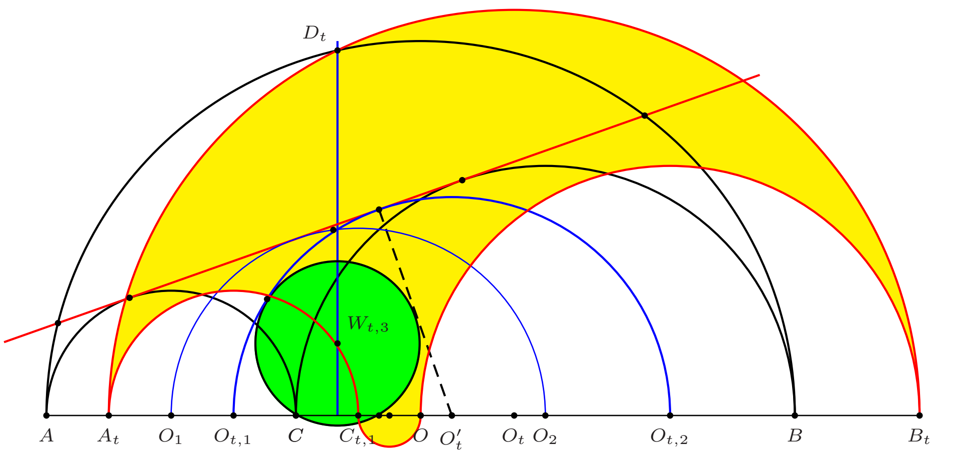

Figure 15. The Archimedean circle $(W_{t,3})$

5.3. $(W_{t,4})$. To generalize Bankoff's quadruplet circle (W_4) we start with a lemma.

Lemma 1. *Let (K) be a circle with center K and ℓ_1 and ℓ_2 be two tangents to (K) meeting in a point P. Let L be a point travelling through the line PK. When L travels linearly, then the radical axis of (K) and the circle (L) tangent to ℓ_1 and ℓ_2 moves linearly as well. The speed relative to the speed of L depends only on the angle of ℓ_1 and ℓ_2.*

Proof. Let P be the origin for Cartesian coordinates. Without loss of generality let ℓ_1 and ℓ_2 be lines making angles $\pm\phi$ with the $x-$axis and let $K(x_1, 0)$, $L(x_2, 0)$. With $v = \sin\phi$ the circles (K) and (L) are given by

$$\begin{aligned}(x - x_1)^2 + y^2 &= v^2x_1^2,\\ (x - x_2)^2 + y^2 &= v^2x_2^2.\end{aligned}$$

The radical axis of these is $x = \frac{1-v^2}{2}(x_1 + x_2) = \frac{1}{2}\cos^2\phi(x_1 + x_2)$. $\square$

It is easy to check that the slope of the line through the midpoints of the semicircular arcs (O) and (O_t) is equal to $\frac{r_2-r_1}{r}$ and thus equal to te slope of the line through M_1 and M_2. This shows that the common tangent of (O) and (O_t) is parallel to the common tangent of (O_1) and (O_2), i.e. d, also the common tangent of (O') and (O'_t). As a result of Lemma 1, it is now clear why the the radical axes of (O) and (O_t) and of (O') and (O'_t) coincide for all t. Another consequence is that the greatest circle tangent to d at the opposite side of (O_1) and (O_2) and to (O_t) internally is, just as the famous example of Bankoff's quadruplet circle (W_4), an Archimedean circle $(W_{t,4})$. See Figure 16. Of course this means that the circles (K_{12}) and (K_{13}) are found as members of the family $(W_{t,4})$.

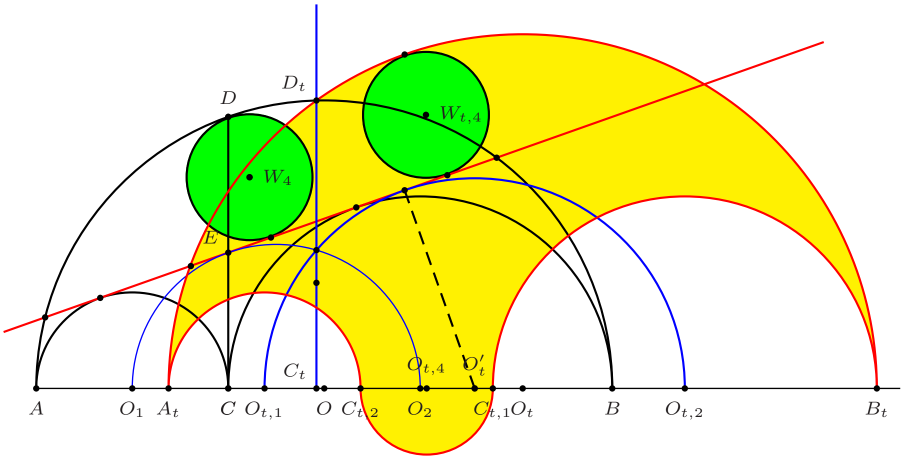

Figure 16. The Archimedean circle $(W_{t,4})$

We note that the the Archimedean circles $(W_{t,4})$ can be constructed easily in any (skew) salinon without seeing the salinon as adjoint to an arbelos and without reconstructing (parts of) this arbelos. To see this we note that N is external center of similitude of $(O_{t,1})$ and $(O_{t,2})$, and d is the tangent from N to (O'_t).

5.4. $(W_{t,6})$, $(W_{t,7})$, $(W_{t,13})$ *and* $(W_{t,14})$. Whereas (W_{13}), (W_{14}), (K_1) and (K_2) are defined in terms of intersections of (semi-)circles or radical axes, with Lemma 1 their generalizations are obvious: CD, (O) and (O') can be replaced by C_tD_t, (O_t) and (O'_t). The generalization of (W_6) and (W_7) as Archimedean circles with center on AB and tangent to $CtDt$ keep having some interest as well. The tangents from A to $(O_{t,2})$ and from B to $(O_{t,1})$ are tangent to $(W_{t,6})$ and $(W_{t,7})$ respectively. This is seen by straightforward calculation, which is left to the reader. As a result we have two stacks of three families. See Figure 17.

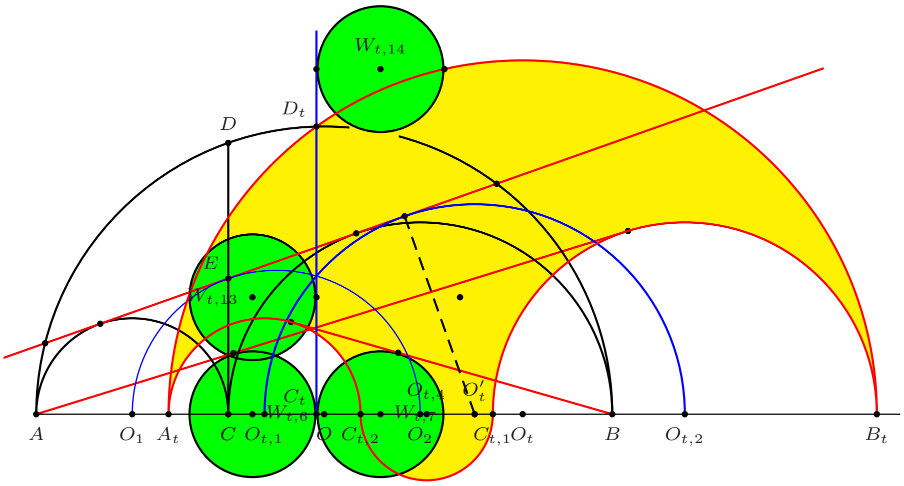

Figure 17. The Archimedean circles $(W_{t,6})$, $(W_{t,7})$, $(W_{t,13})$ and $(W_{t,14})$

We zoom in on the families $(W_{t,13})$ and $(W_{t,14})$. Let $T_{t,13}$ be the point where $(W_{t,13})$, $A(C)$ and (O_t) meet and similarly define $T_{t,14}$. We find that

$$\begin{aligned} T_{t,13} &\quad \left(\frac{r_1^2+(2t-3)r_1r_2-r_2^2}{r}, \frac{r_1\sqrt{-r_2((8t-12)r_1+(4t^2-4t-3)r_2)}}{r}\right), \\ T_{t,14} &\quad \left(\frac{r_1^2+(2t+1)r_1r_2-r_2^2}{r}, \frac{r_2\sqrt{-r_1((4t^2-4t-3)r_1-(8t+4)r_2)}}{r}\right). \end{aligned}$$

The slope of the tangent to $A(C)$ in $T_{t,13}$ with respect to the $x-$axis is equal to

$$s_A = -\frac{x_A - x_{T_{t,13}}}{0 - y_{T_{t,13}}}$$

and the $x-$coordinate of the point R_t where this tangent meets AB is equal to

$$x_{R_t} = x_{T_{t,13}} - \frac{y_{T_{t,13}}}{s_A} = \frac{r(2r_1 + (1-2t)r_2)}{2r_1 + (2t-1)r_2}.$$

Similarly we find for the point L_t where the tangent to $B(C)$ in $T_{t,14}$ meets AB

$$x_{L_t} = \frac{r((2t-1)r_1 + 2r_2)}{(2t-1)r_1 - 2r_2}.$$

The coordinates of Z are given by

$$\left(\frac{r^2(r_1 - r_2)}{r_1^2 + r_2^2}, \frac{2rr_1r_2}{r_1^2 + r_2^2}\right).$$

By straightforward calculation we can now verify that $ZL_t^2 + ZR_t^2 = L_tR_t^2$ and we have a new characterization of the families $(W_{t,13})$ and $(W_{t,14})$.

Theorem 2. *Let $\mathcal{K}$ be a circle through Z with center on AB. This circle meets AB in two points L on the left hand side and R on the right hand side. Let the tangent to $A(C)$ through R meet $A(C)$ in P_1 and similarly find P_2 on $B(C)$. Let k be the line through the midpoint of P_1P_2 perpendicular to AB. Then the smallest circles through P_1 and P_2 tangent to k are Archimedean.*

Remark. Similar characterizations can be found for $(K_{t,1})$ and $(K_{t,2})$.

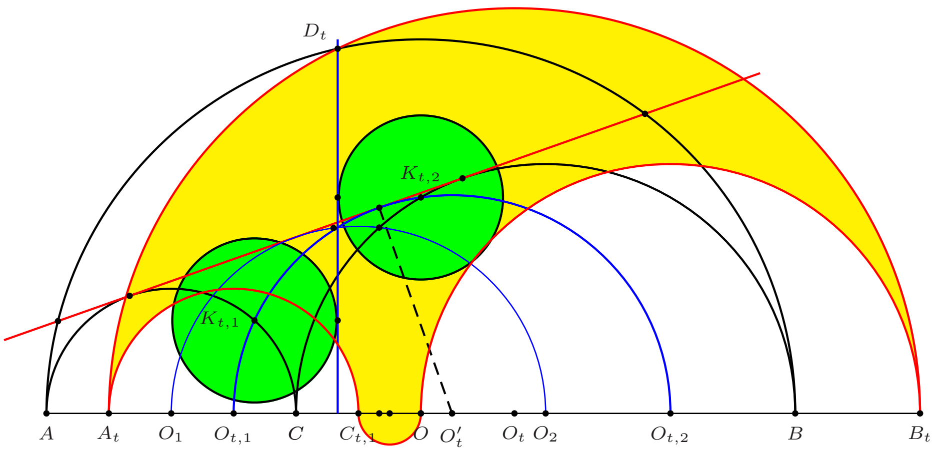

Figure 18. The Archimedean circles $(K_{t,1})$ and $(K_{t,2})$

5.5. *More corollaries.* We go back to Lemma 1. Since the distance between d and the common tangent of (O), (O_t), $A(C)$ and $B(C)$ is equal to $2r_A$, we notice that for instance $A(2r_1 - r_A)$ and $O'_t(r'_t + r_A)$ have a common tangent parallel to d as well. But that implies that we can use Lemma 1 on their intersection (and radical axis). This leads for instance to easy generalizations of (K_3) and (K_7). See Figure 19. The lemma can also help to generalize for instance (K_8) and (K_9), but then some more work has to be done. We leave this to the reader.

5.6. *Archimedean circles from the mid-arc circle of the salinon.* We end the adventures with a sole salinon. We note that the midpoints M_t, $M_{t,1}$, $M_{t,2}$ and $M_{t,4}$ of the semicircular arcs (O_t), $(O_{t,1})$, $(O_{t,2})$ and $(O_{t,4})$ are concyclic. More precisely they are vertices of a rectangle. Their circle, the mid-arc circle (O_5) of the salinon and the circle $(O_{t,1}O_{t,2})$ meet in two points, that are centers of Archimedean circles $(K_{t,10})$ and $(K_{t,11})$. (Of course the parameter t does not really play a role here, but for reasons of uniformity we still use it in the naming).

To verify this, denote by u the distance between $O_{t,1}$ and $O_{t,2}$. Then the distance between $M_{t,1}$ and $M_{t,2}$ equals $\sqrt{u^2 + (r_2 - r_1)^2}$ and the distance from O'' to AB

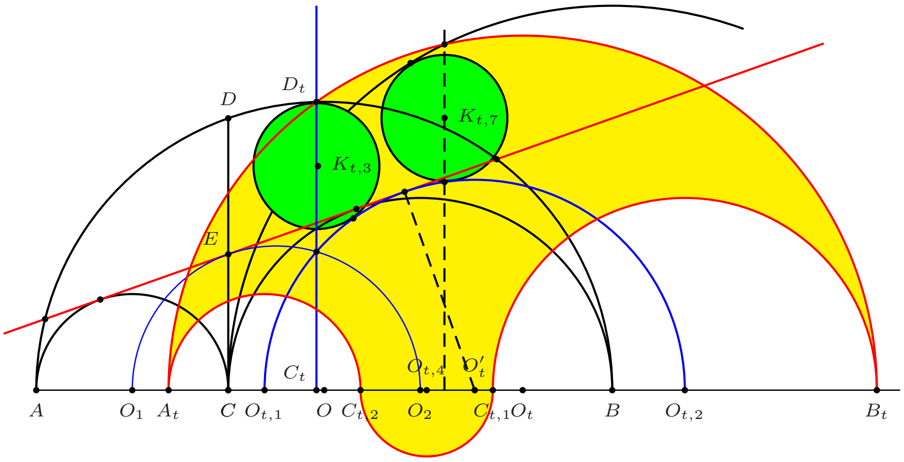

Figure 19. The Archimedean circles $(K_{t,3})$ and $(K_{t,7})$

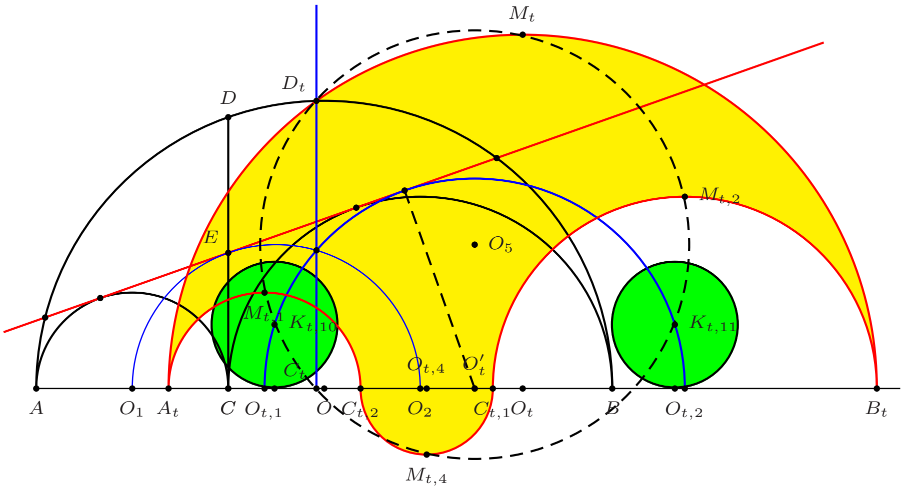

Figure 20. The Archimedean circles $(K_{t,10})$ and $(K_{t,11})$

equals $\frac{r_1+r_2}{2}$. If x is the distance from the intersections of $(O_{t,1}O_{t,2})$ and (O_5) to AB, then

$$x^2 - \frac{u^2}{4} = \left(\frac{r_1+r_2}{2} - x\right)^2 - \frac{u^2+(r_2-r_1)^2}{4}$$

which leads to $x = \frac{r_1 r_2}{r_1+r_2} = r_A$. We can generalize (K_{12}) and (K_{13}) in a similar way. To see this we note that the circle with center on O_tM_t in the pencil generated by $(M_{t,4})$ and (O_5) intersects (O_t) in two points at a distance of $2r_A$ from AB. See Figure 21.

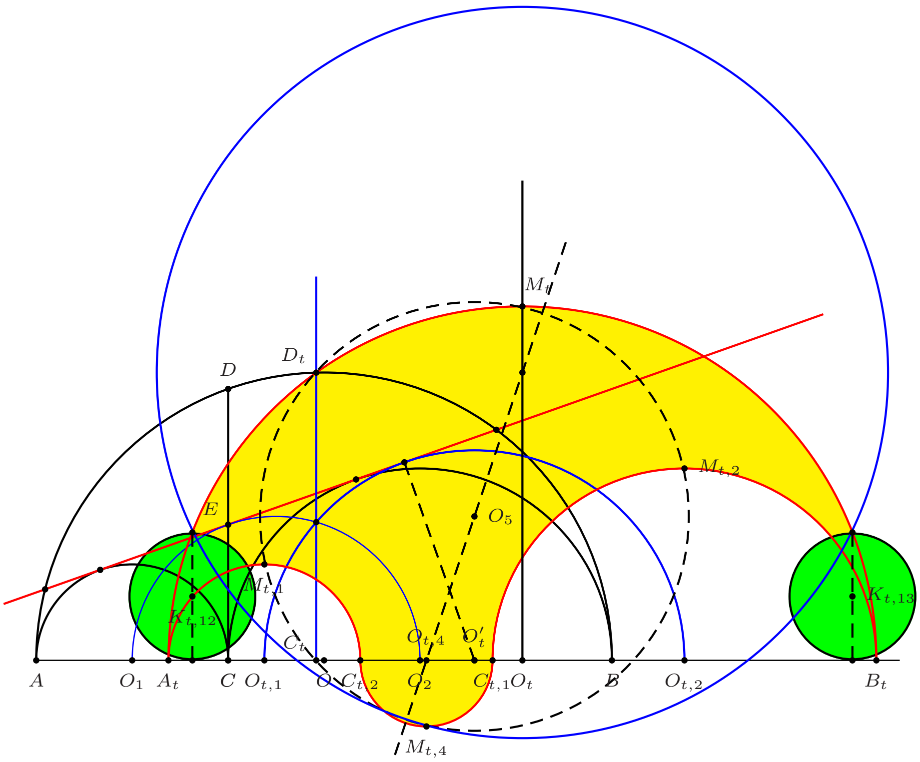

Figure 21. The Archimedean circles $(K_{t,12})$ and $(K_{t,13})$

References

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, *Math. Mag.*, 47 (1974) 214–218.
[2] L. Bankoff, A mere coincidence, Mathematics Newsletter, Los Angeles City College, Nov. 1954; reprinted in *College Math. Journal*, 23 (1992) 106.
[3] L. Bankoff, The marvelous arbelos, in *The lighter Side of Mathematics*, 247–253, ed. R.K. Guy and R.E. Woodrow, Mathematical Association of America, 1994.
[4] C.W. Dodge, T. Schoch, P.Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.
[5] I. d'Ignazio and E. Suppa, *Il Problema Geometrico, dal Compasso al Cabri*, Interlinea Editrice, Téramo, 2001.
[6] H. Okumura and M. Watanabe, The Archimedean circles of Schoch and Woo, *Forum Geom.*, 4 (2004) 27–34.
[7] F. Power, Some more Archimedean circles in the arbelos, *Forum Geom.*, 5 (2005) 133–134.
[8] A. Wendijk and T. Hermann, Problem 10895, *Amer. Math. Monthly*, 108 (2001) 770; solution, 110 (2003) 63–64.
[9] P.Y. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
[10] P. Yiu, Private correspondence, 1999.
[11] P. Yiu, Elegant geometric constructions, *Forum Geom.*, 5 (2005) 75–96.

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