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# A Generalization of Power's Archimedean Circles

Hiroshi Okumura and Masayuki Watanabe

**Abstract**. We generalize the Archimedean circles in an arbelos given by Frank Power.

Let three semicircles $\alpha$, $\beta$ and $\gamma$ form an arbelos with inner semicircles $\alpha$ and $\beta$ with diameters $PA$ and $PB$ respectively. Let $a$ and $b$ be the radii of the circles $\alpha$ and $\beta$. Circles with radii $t = \dfrac{ab}{a+b}$ are called Archimedean circles. Frank Power [2] has shown that for "highest" points $Q_1$ and $Q_2$ of $\alpha$ and $\beta$ respectively, the circles touching $\gamma$ and the line $OQ_1$ (respectively $OQ_2$) at $Q_1$ (respectively $Q_2$) are Archimedean (see Figure 1). We generalize these Archimedean circles.

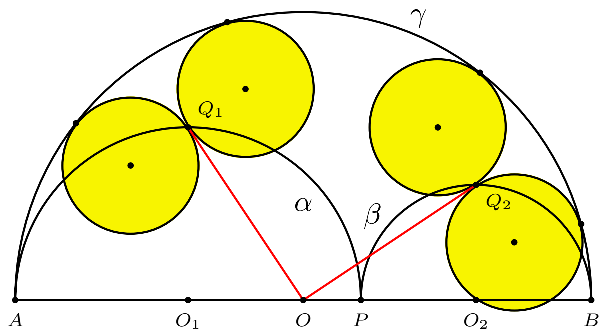


Figure 1

We denote the center of $\gamma$ by $O$. Let $Q$ be the intersection of the circle $\gamma$ and the perpendicular of $AB$ through $P$, and let $\delta$ be a circle touching $\gamma$ at the point $Q$ from the inside of $\gamma$. The radius of $\delta$ is expressed by $k(a+b)$ for a real number $k$ satisfying $0 \leq k < 1$. The tangents of $\delta$ perpendicular to $AB$ intersect $\alpha$ and $\beta$ at points $Q_1$ and $Q_2$ respectively, and intersect the line $AB$ at points $P_1$ and $P_2$ respectively (see Figures 2 and 3).

**Theorem.** (1) *The radii of the circles touching the circle $\gamma$ and the line $OQ_1$ (respectively $OQ_2$) at the point $Q_1$ (respectively $Q_2$) are $2(1-k)t$.*
(2) *The circle touching the circles $\gamma$ and $\alpha$ at points different from $A$ and the line $P_1Q_1$ from the opposite side of $B$ and the circle touching the circles $\gamma$ and $\beta$ at points different from $B$ and the line $P_2Q_2$ from the opposite side of $A$ are congruent with common radii $(1-k)t$.*

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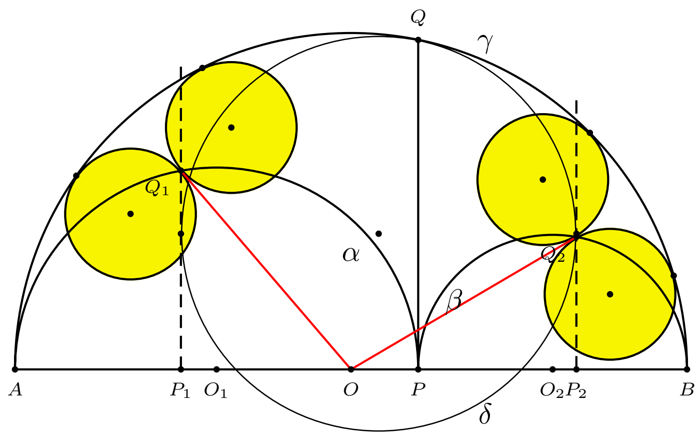


Figure 2

*Proof.* (1) Since $|PP_1| = 2ka$, $|OP_1| = (b-a) + 2ka$. While

$$|P_1Q_1|^2 = |PP_1||P_1A| = 2ka(2a-2ka) = 4k(1-k)a^2.$$

Hence $|OQ_1|^2 = ((b-a)+2ka)^2 + 4k(1-k)a^2 = (a-b)^2 + 4kab$. Let $x$ be the radius of one of the circles touching $\gamma$ and the line $OQ_1$ at $Q_1$. From the right triangle formed by $O$, $Q_1$ and the center of this circle, we get

$$(a+b-x)^2 = x^2 + (a-b)^2 + 4kab$$

Solving the equation for $x$, we get $x = \frac{2(1-k)ab}{a+b} = 2(1-k)t$. The other case can be proved similarly.

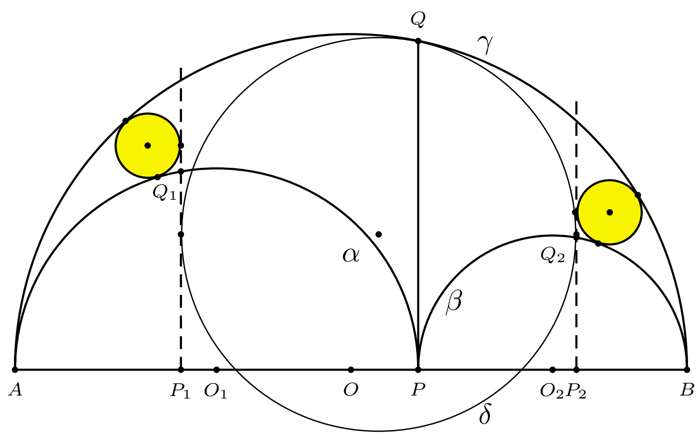


Figure 3

(2) The radius of the circle touching $\alpha$ externally and $\gamma$ internally is proportional to the distance between the center of this circle and the radical axis of $\alpha$ and $\gamma$ [1, p. 108]. Hence its radius is $(1-k)$ times of the radii of the twin circles of Archimedes. □

The Archimedean circles of Power are obtained when $\delta$ is the circle with a diameter $OQ$. The twin circles with the half the size of the Archimedean circles in [4] are also obtained in this case. The statement (2) is a generalization of the twin circles of Archimedes, which are obtained when $\delta$ is the point circle. In this case the points $Q_1$, $Q_2$ and $P$ coincide, and we get the circle with radius $2t$ touching the line $AB$ at $P$ and the circle $\gamma$ by (1) [3].

## References

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Hiroshi Okumura: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

*E-mail address*: `okumura@maebashi-it.ac.jp`

Masayuki Watanabe: Department of Information Engineering, Maebashi Institute of Technology, 460-1 Kamisadori Maebashi Gunma 371-0816, Japan

*E-mail address*: `watanabe@maebashi-it.ac.jp`