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Pseudo-Incircles

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Abstract. This paper generalizes properties of mixtilinear incircles. Let (S) be any circle in the plane of triangle ABC. Suppose there are circles (S_a), (S_b), and (S_c) each tangent internally to (S); and (S_a) is inscribed in angle BAC (similarly for (S_b) and (S_c)). Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Then it is shown that the lines AX, BY, and CZ meet in a point.

1. Introduction

A mixtilinear incircle of a triangle ABC is a circle tangent to two sides of the triangle and also internally tangent to the circumcircle of that triangle. In 1999, Paul Yiu discovered an interesting property of these mixtilinear incircles.

Proposition 1 (Yiu [8])**.** *If the points of contact of the mixtilinear incircles of $\triangle ABC$ with the circumcircle are X, Y, and Z, then the lines AX, BY, and CZ are concurrent* (Figure 1). *The point of concurrence is the external center of similitude of the incircle and the circumcircle.* [1]

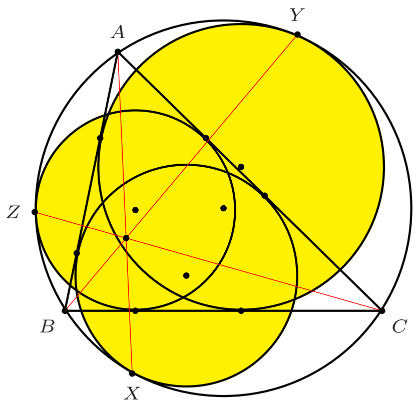

Figure 1

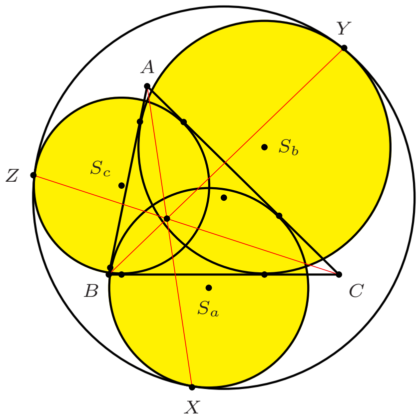

Figure 2

I wondered if there was anything special about the circumcircle in Proposition 1. After a little experimentation, I discovered that the result would remain true if the circumcircle was replaced with any circle in the plane of $\triangle ABC$.

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[1]This is the triangle center X_{56} in Kimberling's list [5].

Theorem 2. *Let (S) be any circle in the plane of $\triangle ABC$. Suppose that there are three circles, (S_a), (S_b), and (S_c), each internally (respectively externally) tangent to (S). Furthermore, suppose (S_a), (S_b), (S_c) are inscribed in $\angle BAC$, $\angle ABC$, $\angle ACB$ respectively* (Figure 2). *Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Then the lines AX, BY, and CZ are concurrent at a point P. The point P is the external (respectively internal) center of similitude of the incircle of $\triangle ABC$ and circle (S).*

When we say that a circle is *inscribed* in an angle ABC, we mean that the circle is tangent to the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Definitions. Given a triangle and a circle, a *pseudo-incircle* of the triangle is a circle that is tangent to two sides of the given triangle and internally tangent to the given circle. A *pseudo-excircle* of the triangle is a circle that is tangent to two sides of the given triangle and externally tangent to the given circle.

There are many configurations that meet the requirements of Theorem 2. Figure 2 shows an example where (S) surrounds the triangle and the three circles are all internally tangent to (S). Figure 3a shows an example of pseudo-excircles where (S) lies inside the triangle. Figure 3b shows an example where (S) intersects the triangle. Figure 3c shows an example where (S) surrounds the triangle and the three circles are all externally tangent to (S).

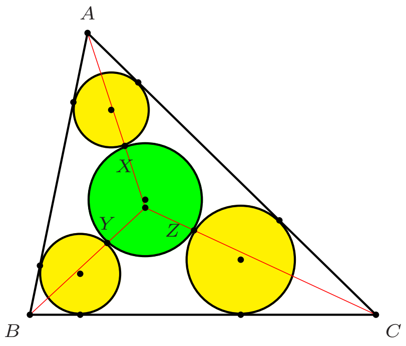

Figure 3a. Pseudo-excircles

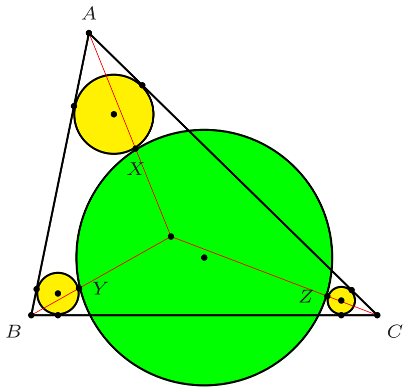

Figure 3b. Pseudo-excircles

There is another way of viewing Figure 3a. Instead of starting with the triangle and the circle (S), we can start with the other three circles. Then we get the following proposition.

Proposition 3. *Let there be given three circles in the plane, each external to the other two. Let triangle ABC be the triangle that circumscribes these three circles (that is, the three circles are inside the triangle and each side of the triangle is a common external tangent to two of the circles). Let (S) be the circle that is externally tangent to all three circles. Let the points of tangency of (S) with the three circles be X, Y, and Z* (Figure 3a). *Then lines AX, BY, and CZ are concurrent.*

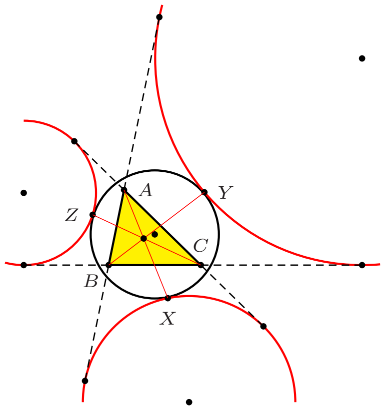

Figure 3c. Pseudo-excircles

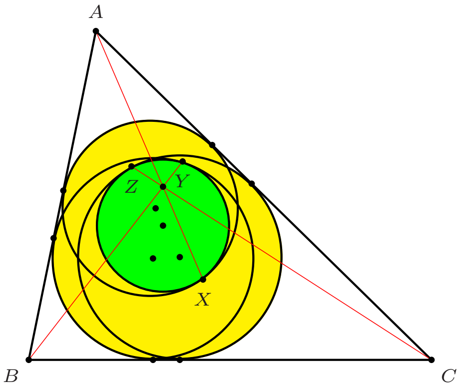

Figure 4. Pseudo-incircles

Figure 4 also shows pseudo-incircles (as in Figure 2), but in this case, the circle (S) lies inside the triangle. The other three circles are internally tangent to (S) and again, AX, BY, and CZ are concurrent. This too can be looked at from the point of view of the circles, giving the following proposition.

Proposition 4. *Let there be given three mutually intersecting circles in the plane. Let triangle ABC be the triangle that circumscribes these three circles* (Figure 4). *Let (S) be the circle that is internally tangent to all three circles. Let the points of tangency of (S) with the three circles be X, Y, and Z. Then lines AX, BY, and CZ are concurrent.*

We need the following result before proving Theorem 2. It is a generalization of Monge's three circle theorem ([7, p.1949]).

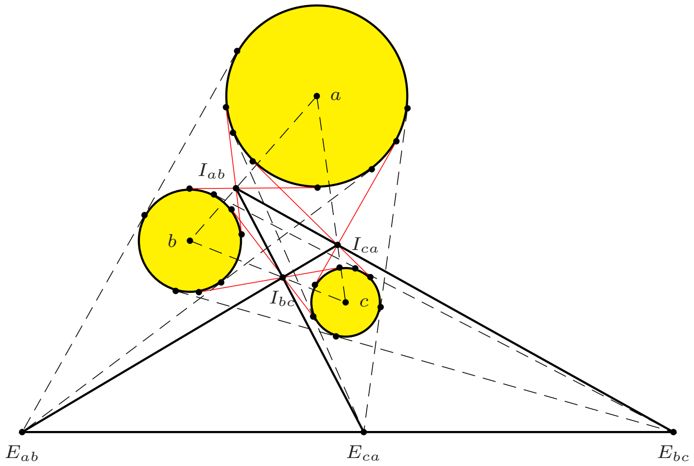

Figure 5. Six centers of similitude

Proposition 5 ([1, p.188], [2, p.151], [6])**.** *The six centers of similitude of three circles taken in pairs lie by threes on four straight lines* (Figure 5)*. In particular, the three external centers of similitude are collinear; and any two internal centers of similitude are collinear with the third external one.*

2. Proof of Theorem 2

Figure 6 shows an example where the three circles are all externally tangent to (S), but the proof holds for the internally tangent case as well. Let I be the center of the incircle.

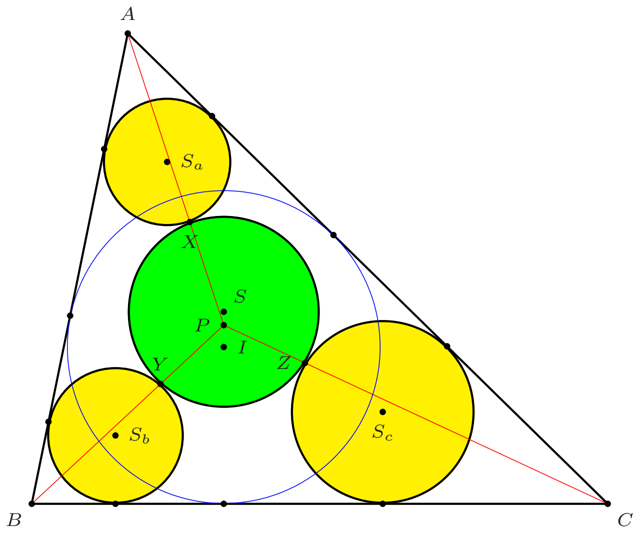

Figure 6

Consider the three circles (S), (I), and (S_c). The external common tangents of circles (J) and (S_c) are sides AC and BC, so C is their external center of similitude. Circles (S) and (S_c) are tangent externally (respectively internally), so their point of contact, Z, is their internal (respectively external) center of similitude. By Proposition 5, points C and Z are collinear with P, the internal (respectively external) center of similitude of circles (S) and (I). That is, line CZ passes through P. Similarly, lines AX and BY also pass through P.

Corollary 6. *In Theorem 2, the points S, P, and I are collinear* (Figure 6)*.*

Proof. Since P is a center of similitude of circles (I) and (S), P must be collinear with the centers of the two circles. □

3. Special Cases

Theorem 2 holds for any circle, (S), in the plane of the triangle. We can get interesting special cases for particular circles. We have already seen a special case in Proposition 1, where (S) is the circumcircle of $\triangle ABC$.

3.1. *Mixtilinear excircles.*

Corollary 7 (Yiu [9])**.** *The circle tangent to sides AB and AC of $\triangle ABC$ and also externally tangent to the circumcircle of $\triangle ABC$ touches the circumcircle at point X. In a similar fashion, points Y and Z are determined* (Figure 7)*. Then the lines AX, BY, and CZ are concurrent. The point of concurrence is the internal center of similitude of the incircle and circumcircle of $\triangle ABC$.* [2]

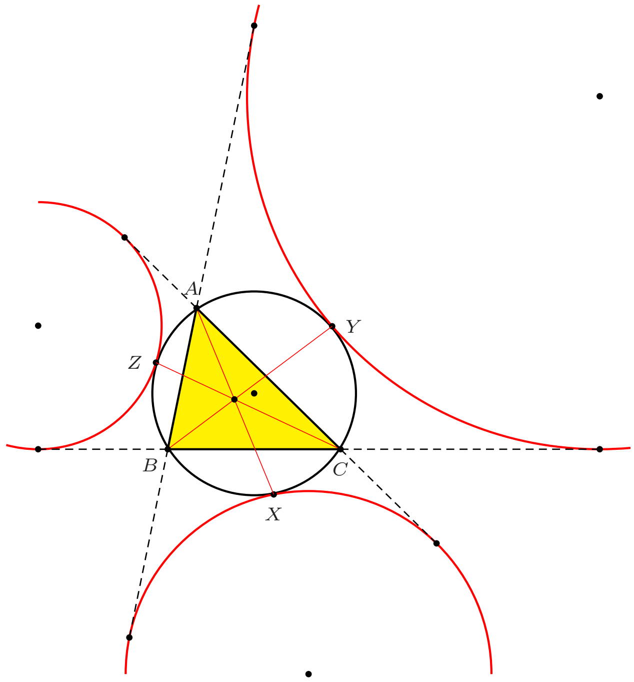

Figure 7. Mixtilinear excircles

3.2. *Malfatti circles.* Consider the Malfatti circles of a triangle ABC. These are three circles that are mutually externally tangent with each circle being tangent to two sides of the triangle.

Corollary 8. *Let (S) be the circle circumscribing the three Malfatti circles,* i.e., *internally tangent to each of them.* (Figure 8a)*. Let the points of tangency of (S) with the Malfatti circles be X, Y, and Z. Then AX, BY, and CZ are concurrent.*

Corollary 9. *Let (S) be the circle inscribed in the curvilinear triangle bounded by the three Malfatti circles of triangle ABC* (Figure 8b)*. Let the points of tangency of (S) with the Malfatti circles be X, Y, and Z. Then AX, BY, and CZ are concurrent.*

[2] This is the triangle center X_{55} in Kimberling's list [5]; see also [4, p.75].

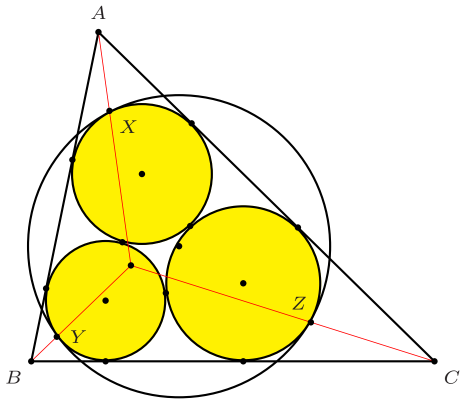

Figure 8a. Malfatti circumcircle

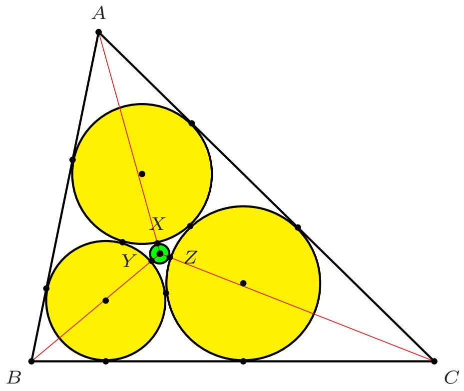

Figure 8b. Malfatti incircle

3.3. *Excircles.*

Corollary 10 (Kimberling [3]). *Let (S) be the circle circumscribing the three excircles of triangle ABC,* i.e., *internally tangent to each of them. Let the points of tangency of (S) with the excircles be X, Y, and Z.* (Figure 9). *Then AX, BY, and CZ are concurrent.*

The point of concurrence is known as the Apollonius point of the triangle. It is X_{181} in [5]. See also [4, p.102].

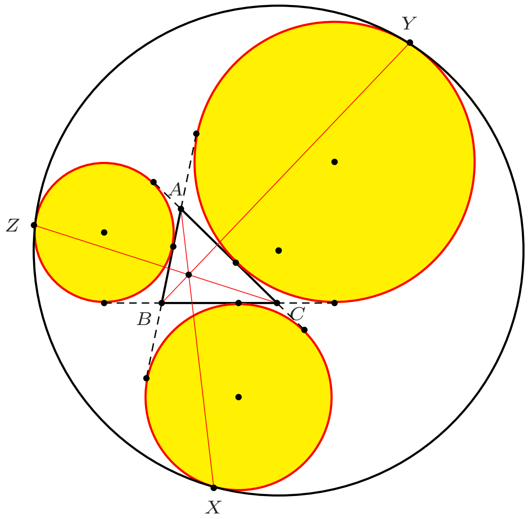

Figure 9. Excircles

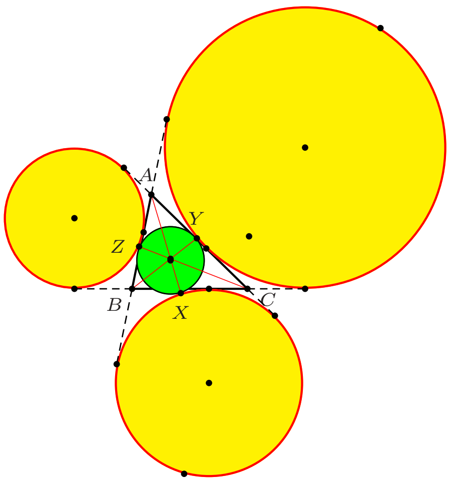

Figure 10. Nine-point circle

If we look at the circle externally tangent to the three excircles, we know by Feuerbach's Theorem, that this circle is the nine-point circle of $\triangle ABC$ (the circle that passes through the midpoints of the sides of the triangle).

Corollary 11 ([4, p.158])**.** *If the nine point circle of $\triangle ABC$ touches the excircles at points X, Y, and Z* (Figure 10)*, then AX, BY, and CZ are concurrent.*

4. Generalizations

In Theorem 2, we required that the three circles (S_a), (S_b), and (S_c) be inscribed in the angles of the triangle. In that case, lines AS_a, BS_b, and CS_c concur at the incenter of the triangle. We can use the exact same proof to handle the case where the three lines AS_a, BS_b, and CS_c meet at an excenter of the triangle. We get the following result.

Theorem 12. *Let (S) be any circle in the plane of $\triangle ABC$. Suppose that there are three circles, (S_a), (S_b), and (S_c), each tangent internally* (*respectively externally*) *to (S). Furthermore, suppose (S_a) is tangent to lines AB and AC; (S_b) is tangent to lines BC and BA; and (S_c) is tangent to lines CA and CB. Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Suppose lines AS_a, BS_b, and CS_c meet at the point J, one of the excenters of $\triangle ABC$* (Figure 11)*. Furthermore, assume that sides BA and BC of the triangle are the external common tangents between excircle (J) and circle (S_a); similarly for circles (S_b), and (S_c). Then AX, BY, and CZ are concurrent at a point P. The point P is the external* (*respectively internal*) *center of similitude of circles (J) and (S). The points J, P, and S are collinear.*

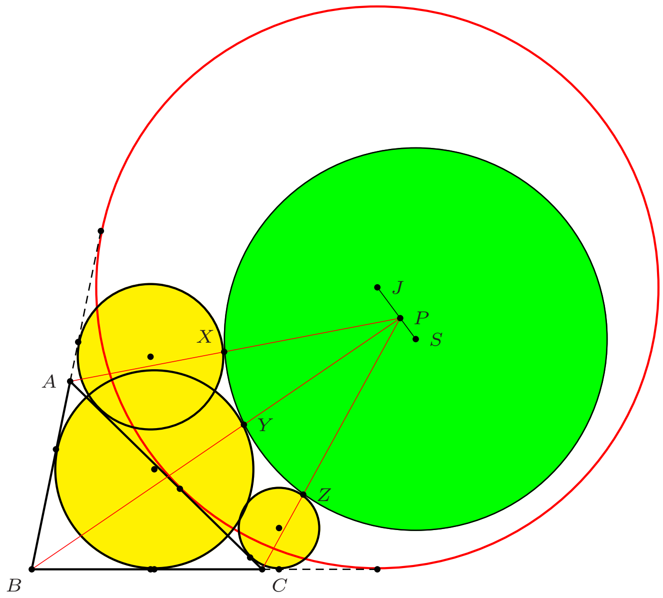

Figure 11. Pseudo-excircles

Figure 12 illustrates the case when (S) is tangent internally to each of (S_a), (S_b), (S_c).

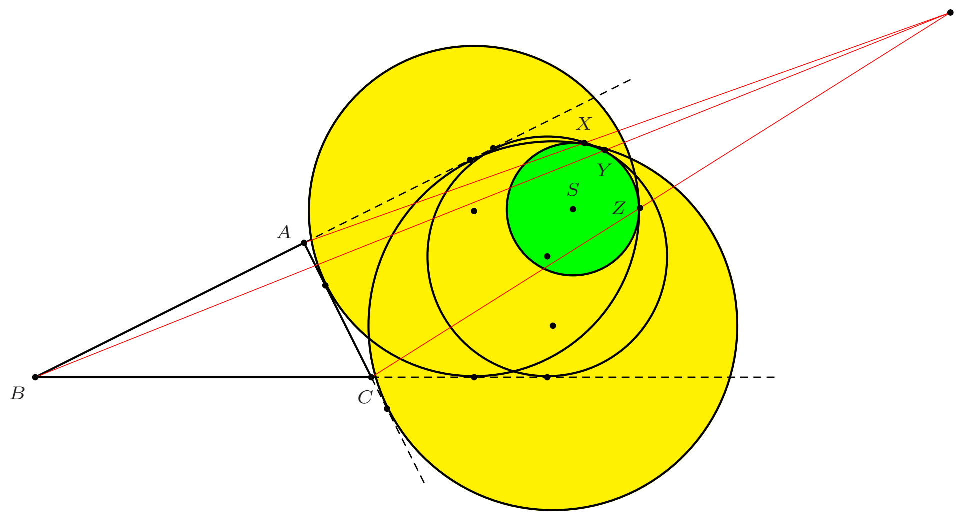

Figure 12. Pseudo-incircles

Examining the proof of Theorem 2, we note that at each vertex of the triangle, we have found that the line from that vertex to the point of contact of the circle inscribed in that angle and circle (S) passes through a fixed point (one of the centers of similitude of (S) and the incircle of the triangle). We note that the result and proof would be the same for a polygon provided that the polygon had an incircle. This gives us the following result.

Theorem 13. *Let $A_1A_2A_3\ldots A_n$ be a convex n-gon circumscribed about a circle (J). Let (S) be any circle in the interior of this n-gon. Suppose there are n circles, (S_1), (S_2), $\ldots$, (S_n) each tangent externally to (S) such that for $i = 1, 2, \ldots, n$, circle (S_i) is also inscribed in angle $A_{i-1}A_iA_{i+1}$ (where $A_0 = A_n$ and $A_{n+1} = A_1$). Let X_i be the point of tangency of circles (S_i) and (S)* (Figure 13). *Then the lines A_iX_i, $i = 1, 2, \ldots, n$ are concurrent. The point of concurrence is the internal center of similitude of the circles (S) and (J).*

In order to generalize Theorem 2 to three dimensions, we need to first note that Proposition 5 generalizes to 3 dimensions.

Theorem 14. *The six centers of similitude of three spheres taken in pairs lie by threes on four straight lines. In particular, the three external centers of similitude are collinear; and any two internal centers of similitude are collinear with the third external one.*

Proof. Consider the plane through the centers of the three spheres. This plane passes through all 6 centers of similitude. The plane cuts each sphere in a circle. Thus, on this plane, Proposition 5 applies, thus proving that the result holds for the spheres as well. □

Theorem 15. *Let $T = A_1A_2A_3A_4$ be a tetrahedron. Let (S) be any sphere in the interior of T (or let (S) be any sphere surrounding T). Suppose there are four*

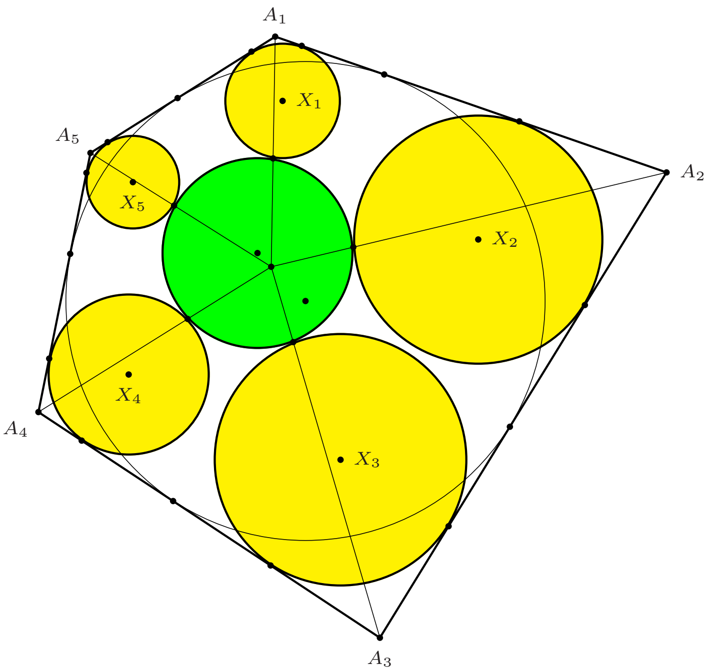

Figure 13. Pseudo-excircles in a pentagon

spheres, (S_1)*,* (S_2)*,* (S_3)*, and* (S_4) *each tangent internally* (*respectively externally*) *to* (S) *such that sphere* (S_i) *is also inscribed in the trihedral angle at vertex* A_i*. Let* X_i *be the point of tangency of spheres* (S_i) *and* (S)*. Then the lines* A_iX_i*,* $i = 1, 2, 3, 4$*, are concurrent. The point of concurrence is the external* (*respectively internal*) *center of similitude of sphere* (S) *and the sphere inscribed in* T*.*

The proof of this theorem is exactly the same as the proof of Theorem 2, replacing the reference to Proposition 5 by Theorem 14.

It is also clear that this result generalizes to E^n.

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