Forum Geometricorum
Volume 6 (2006) 139–147.

FORUM GEOM
ISSN 1534-1178

Concurrent Medians of $(2n+1)$-gons

Matthew Hudelson

Abstract. We exhibit conditions that determine whether a set of $2n+1$ lines are the medians of a $(2n+1)$-sided polygon. We describe how to regard certain collections of sets of medians as a linear subspace of related collections of sets of lines, and as a consequence, we show that every set of $2n+1$ concurrent lines are the medians of some $(2n+1)$-sided polygon. Also, we derive conditions on $n+1$ points so that they can be consecutive vertices of a $(2n+1)$-sided polygon whose medians intersect at the origin. Each of these constructions demonstrates a procedure that generates $(2n+4)$-degree of freedom families of median-concurrent polygons. Furthermore, this number of degrees of freedom is maximal.

1. Motivation

It is well-known that the medians of a triangle intersect in a common point. We wish to explore which polygons in general have this property. Necessarily, such polygons must have an odd number of edges. One easy source of such polygons is to begin with a regular $(2n+1)$-gon centered at the origin and transform the vertices using an affine transformation. This exhausts the triangles as every triangle is an affine image of an origin-centered equilateral triangle. On the other hand, if we begin with either a regular pentagon or a regular pentagram, this process fails to exhaust the median-concurrent pentagons. Consider the pentagon with the sequence of vertices $v_0 = (0,1)$, $v_1 = (1,0)$, $v_2 = (2,1)$, $v_3 = (-2,-2)$, and $v_4 = (6,2)$, depicted in Figure 1.

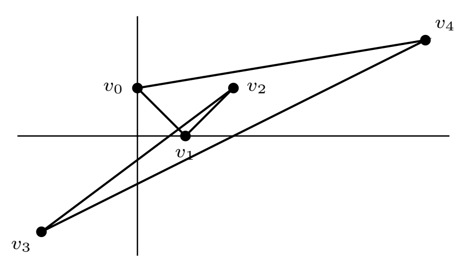

Figure 1. A non-affinely regular median concurrent pentagon

It is easy to check that for each i (all indices modulo 5), the line through v_i and $\frac{1}{2}(v_{i+2}+v_{i+3})$ contains the origin. Alternatively, it suffices to check that v_i and $v_{i+2}+v_{i+3}$ are scalar multiples. On the other hand, this pentagon has a single self-intersection whereas a regular pentagon has none and a regular pentagram has five,

Publication Date: April 10, 2006. Communicating Editor: Floor van Lamoen.

so this example cannot be the image under an affine map of a regular 5-gon. Thus, we seek alternative and more comprehensive means to construct median concurrent pentagons specifically and $(2n+1)$-gons in general.

We approach this problem by two different means. In the next section, we consider families of lines that serve as the medians of polygons and in the section afterwards, we begin with a collection of $n+1$ consecutive vertices and show how to "complete" the collection with the remaining n vertices; the result will also be a $(2n+1)$-gon whose medians intersect.

2. Families of polygons constructed by medians

2.1. *Oriented lines and the signed law of sines.* In this section, we will be working with oriented lines. Given a line ℓ' in $\mathbb{R}^2$, we associate with it a unit vector $\mathbf{v}$ that is parallel with ℓ'. The oriented line ℓ is defined as the pair $(\ell', \mathbf{v})$. Then given points A and B on ℓ, we can solve $\overrightarrow{AB} = t\mathbf{v}$ for t and say that t is the "signed length" from A to B along ℓ; this quantity is denoted $d_\ell(A, B)$. If $t > 0$, we will say that B is on the "positive side" of A along ℓ; if $t < 0$, we will say that B is on the "negative side" of A along ℓ. Switching the orientation of a line switches the sign of the signed length from one point to another on that line.

Let ℓ_1 and ℓ_2 be two non-parallel oriented lines and C be their intersection point. Let D_i be a point on the positive side of C along ℓ_i. The "signed angle" from ℓ_1 to ℓ_2, denoted θ_{12} is the angle whose magnitude (in the range $(0, \pi)$) is that of $\angle D_1 C D_2$ and whose sign is that of the cross product $\mathbf{v}_1 \times \mathbf{v}_2$, the vectors thought of as lying in the $z = 0$ plane of $\mathbb{R}^3$. The signed angle of two parallel lines with the same unit vector is 0, and with opposite unit vectors is π. Switching the orientation of a single line switches the sign of the signed angle between them.

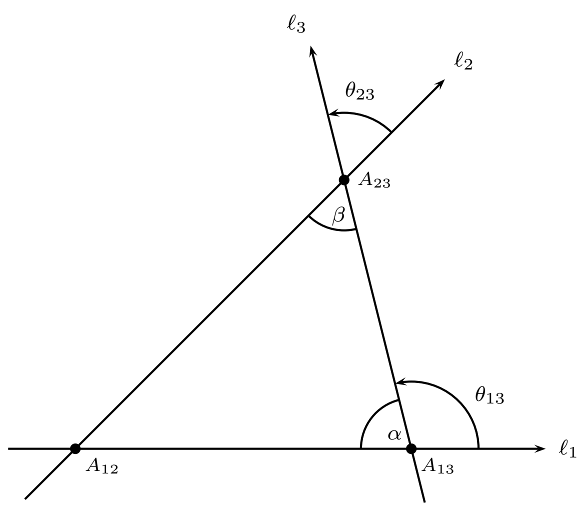

Figure 2. The signed law of sines

In Figure 2 we have three oriented lines ℓ_1, ℓ_2, ℓ_3. Point A_{ij} is at the intersection of ℓ_i and ℓ_j. As drawn, A_{13} is on the positive side of A_{12} along ℓ_1, A_{23} is on the positive side of A_{12} along ℓ_2, and A_{23} is on the positive side of A_{13} along ℓ_3. Letting $\alpha = \angle A_{23}A_{13}A_{12}$ and $\beta = \angle A_{12}A_{23}A_{13}$, we have $\sin\alpha = \sin\theta_{13}$ and $\sin\beta = \sin\theta_{23}$.

By the ordinary law of sines and the above comments about α and β, we have

$$\frac{d_{\ell_1}(A_{12}, A_{13})}{\sin\theta_{23}} = \frac{d_{\ell_2}(A_{12}, A_{23})}{\sin\theta_{13}}.$$

Note that if we switch the orientation of ℓ_1, then the numerator of the LHS and the denominator of the RHS change signs. Switching the orientation of ℓ_2 changes the signs of the numerator of the RHS and the denominator of the LHS. Switching the orientation of ℓ_3 changes the signs of both denominators. Any of these orientation switches leaves the LHS and RHS equal, and so the equation above is true for oriented lines and signed angles as well; we call this equation "the signed law of sines."

2.2. *Constructing polygons via medians.* Let $\ell_0, \ell_1, \ldots, \ell_{2n}$ be $2n+1$ oriented lines, no two parallel, in $\mathbb{R}^2$. Let $B_{i,j}$ be the point of intersection of ℓ_i and ℓ_j, and let δ_{i+1} be $d_{\ell_{i+1}}(B_{i,i+1}, B_{i+1,i+2})$. Finally, choose a point A_0 on ℓ_0 and let $S_0 = d_{\ell_0}(B_{0,1}, A_0)$.

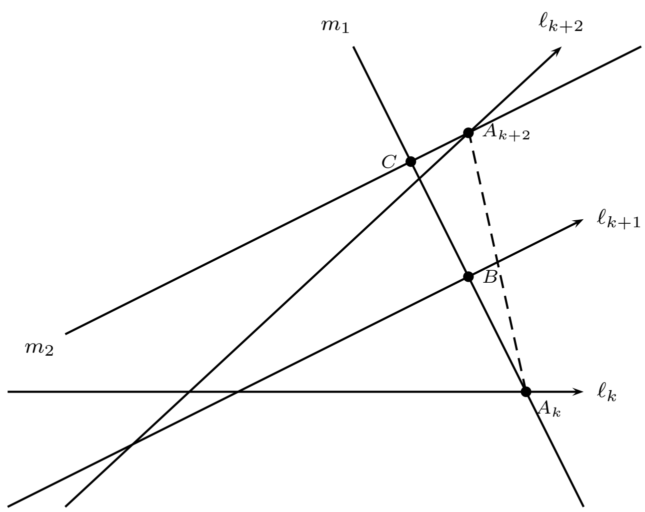

Figure 3. Constructing consecutive points via medians

Given a point A_k on ℓ_k, we construct the point A_{k+2} on ℓ_{k+2} (the indices of lines are modulo $2n+1$) as follows and as depicted in Figure 3.

Construction 1. *Construct line m_1 through A_k and perpendicular to ℓ_{k+1}. Let point B be the intersection of line ℓ_{k+1} and m_1. Construct point C on line m_1 so that segments $\overline{A_kB}$ and $\overline{BC}$ are congruent. Construct line m_2 through C and perpendicular to m_1. Let point A_{k+2} be the intersection of lines m_2 and ℓ_{k+2}.*

We note that line ℓ_{k+1} intersects segment $\overline{A_k, A_{k+2}}$ at its midpoint.

We define $S_k = d_{\ell_k}(B_{k,k+1}, A_k)$ and θ_{ij} to be the signed angle subtended from ℓ_i to ℓ_j. Letting m be the line through A_k and A_{k+2} and $\varphi = \theta_{\ell_{k+1},m}$, we have by the signed law of sines,

$$\frac{x}{\sin\theta_{k,k+1}} = \frac{S_k}{\sin(\theta_{k,k+1}+\varphi)}$$

and

$$\frac{x}{\sin\theta_{k+1,k+2}} = \frac{S_{k+2}+\delta_{k+1}}{\sin(\theta_{k,k+1}+\varphi)}.$$

Eliminating x, we have

$$S_{k+2} = \frac{\sin\theta_{k,k+1}}{\sin\theta_{k+1,k+2}} S_k - \delta_{k+1}.$$

Let

$$g_k = \frac{\sin\theta_{k,k+1}}{\sin\theta_{k+1,k+2}}.$$

Then we have the recurrence

$$S_{k+2} = g_k S_k - \delta_{k+1}.$$

This leads to

$$\begin{aligned} S_{k+4} &= g_{k+2} S_{k+2} - \delta_{k+3} \\ &= g_{k+2} g_k S_k - g_{k+2}\delta_{k+1} - \delta_{k+3}, \end{aligned}$$

and

$$\begin{aligned} S_{k+6} &= g_{k+4} S_{k+4} - \delta_{k+5} \\ &= g_{k+4} g_{k+2} g_k S_k - g_{k+4} g_{k+2}\delta_{k+1} - g_{k+4}\delta_{k+3} - \delta_{k+5}. \end{aligned}$$

In general, if we define

$$h_{k,p,m} = g_{k+2p}\, g_{k+2p+2}\, g_{k+2p+4} \cdots g_{k+2(m-1)},$$

we have

$$S_{k+2m} = h_{k,0,m} S_k - h_{k,1,m}\delta_{k+1} - h_{k,2,m}\delta_{k+3} - \cdots - \delta_{k+2m-1}.$$

We are interested in the case when we begin with a point A_0 on ℓ_0 and eventually construct the point $A_{2(2n+1)}$ which will also be on line ℓ_0. When $k = 0$ and $m = 2n+1$, we have

$$S_{2(2n+1)} = h_{0,0,2n+1} S_0 - h_{0,1,2n+1}\delta_1 - h_{0,2,2n+1}\delta_3 - \cdots - \delta_{2(2n+1)-1}.$$

We notice first that

$$h_{0,0,2n+1} = \prod_{k=0}^{2n} g_{2k} = \prod_{k=0}^{2n} \frac{\sin\theta_{2k,2k+1}}{\sin\theta_{2k+1,2k+2}} = \frac{\prod_{k=0}^{2n}\sin\theta_{2k,2k+1}}{\prod_{k=0}^{2n}\sin\theta_{2k+1,2k+2}}.$$

Since the subscripts in the latter products are all modulo $2n+1$, it follows that the terms in the numerator are a permutation of those in the denominator. This means that $h_{0,0,2n+1} = 1$. The second observation is that

$$S_{2(2n+1)} = S_0 + \text{ a linear combination of the } h_{0,i,2n+1} \text{ values}.$$

The coefficients of this linear combination are the δ's. The nullspace of the h values will in fact be a codimension 1 subspace of the space of all possible choices of $(\delta_0, \delta_1, \ldots, \delta_{2n})^T$. An immediate consequence of this is that if for all i we have $\delta_i = 0$ then $S_{4n+2} = S_0$ and so we have closed the polygon A_0, A_2, A_4, $\ldots$, $A_{4n+2} = A_0$. We have shown

Proposition 1. *Any set of $2n+1$ concurrent lines, no two parallel, in $\mathbb{R}^2$ are the medians of some $(2n+1)$-gon.*

Consider choosing a family of $2n+1$ concurrent lines. Each line can be chosen by choosing a unit vector, the choice of each being a single degree of freedom (for instance, the angle that vector makes with the vector $(1,0)^T$). Another degree of freedom is the choice of point A_0 on ℓ_0. Finally, there are two more degrees of freedom in the choice of the point of concurrency. This is a total of $2n+4$ degrees of freedom for constructing $(2n+1)$-gons with concurrent medians.

3. Families of median-concurrent polygons constructed by vertices

Suppose we have three points $(a,b), (c,d)$, and (e,f) in $\mathbb{R}^2$ such that $(a,b) \neq (-c,-d)$. We seek a fourth point (u,v) such that (u,v), $(a+c, b+d)$ and $(0,0)$ are collinear, and (a,b), $(e+u, f+v)$ and $(0,0)$ are also collinear.

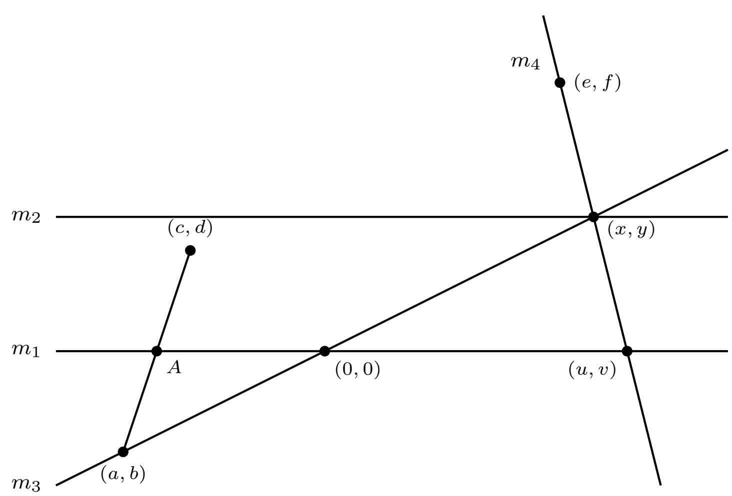

Figure 4. Constructing the fourth point

The point (u,v) can be constructed as follows:

Construction 2. *Let A be the midpoint of the segment joining (a,b) and (c,d) and m_1 be the line through A and the origin. Construct the line m_2 parallel to m_1 that is on the same side of, but half the distance from (e,f) as m_1. Let m_3 be the line through (a,b) and the origin, intersecting m_2 at (x,y), and let m_4 be the line through (e,f) and (x,y). The point (u,v) is the intersection of lines m_1 and m_4.*

It must be the case that $(u, v) = k(a + c, b + d)$ and also

$$u(b + d) = v(a + c)$$
$$b(e + u) = a(f + v).$$

Subtracting, we have

$$ud - be = vc - af$$

or

$$k(a + c)d - be = k(b + d)c - af.$$

Isolating k, we have

$$k = \frac{be - af}{ad - bc}.$$

Notice that the fourth point is uniquely determined by the other three, provided $ad - bc \neq 0$.

We formalize this in the following definition.

Definition 1. *Given point $A = (a, b), C = (c, d)$, and $E = (e, f)$, we define the point $F(A, C, E)$ by the formula*

$$F = F(A, C, E) = \frac{be - af}{ad - bc}(A + C).$$

This point satisfies the property that the lines $\overline{F, (A + C)}$ and $\overline{A, (E + F)}$ intersect at the origin.

Now, suppose we have $n+1$ points in $\mathbb{R}^2$, $x_i = (a_i, b_i)$, $0 \leq i \leq n$, and suppose that no line joining two consecutive points contains the origin. Starting at $i = 0$ we define $x_{i+n+1} = F(x_i, x_{i+1}, x_{i+n})$. We address what happens in the sequence $x_0, x_1, x_2, \ldots$.

We can recast our definition of the point x_{i+n+1} using the following definition.

Definition 2. *For $0 \leq j, k$, $\Delta_{j,k} = a_j b_k - b_j a_k$.*

Armed with this, we have

$$x_{i+n+1} = \frac{\Delta_{i+n,i}}{\Delta_{i,i+1}}(x_i + x_{i+1}).$$

Also, we use induction to prove:

Proposition 2. *For all $j \geq 0$, $\Delta_{j+n,j} = \Delta_{n,0}$.*

Proof. The case $j = 0$ is obvious. Suppose the result is true for $j = k$. Then

$$\begin{aligned}
\Delta_{k+1+n,k+1} &= a_{k+1+n} b_{k+1} - b_{k+1+n} a_{k+1} \\
&= \frac{\Delta_{k+n,k}}{\Delta_{k,k+1}}((a_k + a_{k+1})b_{k+1} - (b_k + b_{k+1})a_{k+1}) \\
&= \frac{\Delta_{k+n,k}}{\Delta_{k,k+1}}(a_k b_{k+1} - b_k a_{k+1}) \\
&= \Delta_{k+n,k} \\
&= \Delta_{n,0}
\end{aligned}$$

which completes the induction. $\square$

As a consequence, we have

$$x_{i+n+1} = \frac{\Delta_{n,0}}{\Delta_{i,i+1}}(x_i + x_{i+1}).$$

We verify a useful property of $\Delta_{j,k}$:

Proposition 3. *For all* j, k, ℓ,

$$\Delta_{j,k} x_\ell + \Delta_{\ell,j} x_k = \Delta_{\ell,k} x_j.$$

Proof. We work component-by-component:

$$\begin{aligned}
\Delta_{j,k} a_\ell + \Delta_{\ell,j} a_k &= (a_j b_k - b_j a_k) a_\ell + (a_\ell b_j - b_\ell a_j) a_k \\
&= a_j b_k a_\ell - b_j a_k a_\ell + a\ell b_j a_k - b_\ell a_j a_k \\
&= (a_\ell b_k - b_\ell a_k) a_j \\
&= \Delta_{\ell,k} a_j,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{j,k} b_\ell + \Delta_{\ell,j} b_k &= (a_j b_k - b_j a_k) b_\ell + (a_\ell b_j - b_\ell a_j) b_k \\
&= a_j b_k b_\ell - b_j a_k b_\ell + a\ell b_j b_k - b_\ell a_j b_k \\
&= (a_\ell b_k - b_\ell a_k) b_j \\
&= \Delta_{\ell,k} b_j.
\end{aligned}$$

□

We can now prove the following:

Proposition 4. *For all* $0 \leq i \leq 2n$, *there is a* k_i *such that* $x_{i-1} + x_i = k_i x_{n+i}$ (*all subscripts modulo* $2n+1$).

Proof. For $i = 0$, we calculate

$$\begin{aligned}
x_{2n} + x_0 &= \frac{\Delta_{n,0}}{\Delta_{n-1,n}}(x_{n-1} + x_n) + x_0 \\
&= \frac{1}{\Delta_{n-1,n}}(\Delta_{n,0} x_{n-1} + \Delta_{n,0} x_n + \Delta_{n-1,n} x_0) \\
&= \frac{1}{\Delta_{n-1,n}}((\Delta_{n,0} x_{n-1} + \Delta_{n-1,n} x_0) + \Delta_{n,0} x_n) \\
&= \frac{1}{\Delta_{n-1,n}}(\Delta_{n-1,0} x_n + \Delta_{n,0} x_n) \\
&= \frac{\Delta_{n-1,0} + \Delta_{n,0}}{\Delta_{n-1,n}} x_n
\end{aligned}$$

and so

$$k_0 = \frac{\Delta_{n-1,0} + \Delta_{n,0}}{\Delta_{n-1,n}}.$$

If $1 \leq i \leq n$, then by the definition of x_{n+i}, $k_i = \Delta_{i-1,i}/\Delta_{n,0}$.

To handle the case when $i = n + 1$, we calculate

$$\begin{aligned} x_n + x_{n+1} &= x_n + \frac{\Delta_{n,0}}{\Delta_{0,1}}(x_0 + x_1) \\ &= \frac{1}{\Delta_{0,1}}(\Delta_{0,1}x_n + \Delta_{n,0}(x_0 + x_1)) \\ &= \frac{1}{\Delta_{0,1}}(\Delta_{0,1}x_n + \Delta_{n,0}x_1 + \Delta_{n,0}x_0) \\ &= \frac{1}{\Delta_{0,1}}(\Delta_{n,1}x_0 + \Delta_{n,0}x_0) \\ &= \frac{\Delta_{n,1} + \Delta_{n,0}}{\Delta_{0,1}}x_0 \end{aligned}$$

and so

$$k_{n+1} = \frac{\Delta_{n,1} + \Delta_{n,0}}{\Delta_{0,1}}.$$

For the values of i, $n + 2 \le i \le 2n$, we set $m = i - n$ and we have, using the symbol $\delta = \Delta_{n,0}/(\Delta_{m-2,m-1}\Delta_{m-1,m})$,

$$\begin{aligned} x_{i-1} + x_i &= x_{n+m-1} + x_{n+m} \\ &= \frac{\Delta_{n,0}}{\Delta_{m-2,m-1}}(x_{m-2} + x_{m-1}) + \frac{\Delta_{n,0}}{\Delta_{m-1,m}}(x_{m-1} + x_m) \\ &= \delta(\Delta_{m-1,m}(x_{m-2} + x_{m-1}) + \Delta_{m-2,m-1}(x_{m-1} + x_m) \\ &= \delta((\Delta_{m-1,m}x_{m-2} + \Delta_{m-2,m-1}x_m) + (\Delta_{m-1,m} + \Delta_{m-2,m-1})x_{m-1}) \\ &= \delta(\Delta_{m-2,m} + \Delta_{m-1,m} + \Delta_{m-2,m-1})x_{m-1} \\ &= \delta(\Delta_{m-2,m} + \Delta_{m-1,m} + \Delta_{m-2,m-1})x_{i+n} \end{aligned}$$

noting that $m - 1 = i - n - 1 \equiv i + n$ modulo $2n + 1$, and so for $n + 2 \le i \le 2n$, we have

$$k_i = \frac{\Delta_{n,0}(\Delta_{i-n-2,i-n} + \Delta_{i-n-1,i-n} + \Delta_{i-n-2,i-n-1})}{\Delta_{i-n-2,i-n-1}\Delta_{i-n-1,i-n}}.$$

□

What this proposition says, geometrically, is that the points $x_{i-1} + x_i$, x_{i+n} and the origin are collinear. Alternatively, setting $i = j + n + 1$, we find that the points x_j, $x_{j+n} + x_{j+n+1}$ and the origin are collinear. But this means that $\frac{1}{2}(x_{j+n} + x_{j+n+1})$ is also on the same line, and so the line connecting x_j and the midpoint of the segment joining x_{j+n} and x_{j+n+1} contains the origin.

As a direct consequence, we obtain the following result:

Proposition 5. *Given any sequence of $n + 1$ points, $x_0, x_1, \ldots, x_n$ such that the origin is not on any line $\overline{x_i, x_{i+1}}$ or $\overline{x_n, x_0}$, then these points are $n + 1$ vertices in sequence of a unique $(2n + 1)$-gon whose medians intersect in the origin.*

Here, we can choose $n + 1$ points to determine a $(2n + 1)$-gon whose medians intersect at the origin. Each point contributes two degrees of freedom for a total of

$2n+2$ degrees of freedom. Two more degrees of freedom are obtained for the point of concurrency, for a total of $(2n + 4)$ degrees of freedom. This echoes the final result from the previous section. That we cannot obtain further degrees of freedom follows from the previous section as well. There, *any* set of $2n+1$ concurrent lines (in general position) produced a concurrent-median $(2n+1)$-gon. We cannot hope for more freedom than this.

Matthew Hudelson: Department of Mathematics, Washington State University, Pullman, Washington, 99164-3113, USA

E-mail address: hudelson@math.wsu.edu