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On the Generating Motions and the Convexity of a Well-Known Curve in Hyperbolic Geometry

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Abstract. In Euclidean geometry the vertices P of those angles $\angle APB$ of size α that pass through the endpoints A, B of a given segment trace the arc of a circle. In hyperbolic geometry on the other hand a set of equivalently defined points P determines a different kind of curve. In this paper the most basic property of the curve, its convexity, is established. No straight-forward proof could be found. The argument rests on a comparison of the rigid motions that map one of the angles $\angle APB$ into other ones.

1. Introduction

In the hyperbolic plane let AB be a segment and H one of the halfplanes with respect to the line through A and B. What will be established here is the convexity of the locus Ω of the point P which lies in H and which determines together with A and B an angle $\angle APB$ of a given fixed size. In Euclidean geometry this locus is well-known to be an arc of the circle through A and B whose center C determines the (oriented) angle $\angle ACB = 2 \cdot \angle APB$. In hyperbolic geometry, on the other hand, one obtains a wider, flatter curve (see Figure 1; [2, p.79, Exercise 4], [1], and also [6, Section 50], [7, Section 2]). The evidently greater complexity of the non-Euclidean version of this locus shows itself most clearly when one considers the (direct) motion that carries a defining angle $\angle APB$ into another defining angle $\angle AP'B$. Whereas in Euclidean geometry it has to be a rotation, it can in hyperbolic geometry also be a horocyclic rotation about an improper center, or, surprisingly, even a translation. For our convexity proof it appears to be practical to consider the given angle as fixed and the given segment as moving. Then, as will be shown in the *Main Lemma*, the relative position of the centers or axes of our motions can be described in a very simple fashion, with the sought-after convexity proof as an easy consequence. As to proving the Lemma itself, one has to take into account that the motions involved can be rotations, horocyclic rotations, or translations, and it seems that a distinction of cases is the only way to proceed. Still, it would be desirable if the possibility of an overarching but nonetheless elementary argument would be investigated further.

The fact of the convexity of our curve yields at least one often used by-product:

Theorem. *Let AB be a segment, H a halfplane with respect to the line through A and B, and ℓ a line which has points in common with H but avoids segment AB.*

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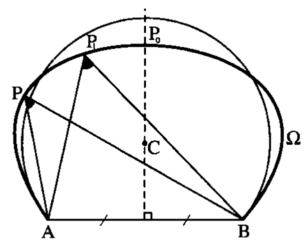

Figure 1

Then the point X, when running through ℓ in H, determines angles $\angle AXB$ that first monotonely increase, and thereafter monotonely decrease in size.

Our approach will be strictly axiomatic and elementary, based on Hilbert's axiom system of Bolyai-Lobachevskian geometry (see [3, Appendix III]). The application of Archimedes' axiom in particular is excluded. Beyond the initial concepts of hyperbolic plane geometry we will only rely on the facts about angle sum, defect, and area of polygons (see e.g. [2, 5, 6, 8]), and on the basic properties of isometries. To facilitate the reading of our presentation we precede it with a list of frequently used abbreviations.

1.1. *Abbreviations.*

1.1.1. $[A_1A_2 \dots A_h \dots A_i \dots A_k \dots A_n]$ for an n-tuple of points with A_i *between* A_h and A_k for $1 \leq h < i < k \leq n$.

1.1.2. $AB, CD, \dots$ for *segments*, and $(AB), (CD), \dots$ for the related *open intervals* $AB - \{A, B\}$, $CD - \{C, D\}, \dots$; $\overrightarrow{AB}, \overrightarrow{CD}, \dots$ for the *rays* from A through B, from C through $D, \dots$, and $\overrightarrow{(A)B}, \overrightarrow{(C)D}, \dots$ for the related *halflines* $\overrightarrow{AB} - \{A\}$, $\overrightarrow{CD} - \{C\}, \dots$; $\ell(AB), \ell(CD), \dots$ for the *lines* through A and B, C and $D, \dots$.

1.1.3. $a, b, c, \dots$ are general abbreviations for lines, $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \dots$ for rays in those lines, and $(\overrightarrow{a}), (\overrightarrow{b}), (\overrightarrow{c}), \dots$ for the related halflines.

1.1.4. $H(a, B)$, where the point B is not on line a, for the *halfplane with respect to a which contains B*, and $\overline{H}(a, B)$ for the *halfplane with respect to a which does not contain B*. The improper ends of rays which enter halfplane H through a are considered as belonging to H.

1.1.5. perp(a, B) for the line which is *perpendicular* to a and incident with B; proj(S, ℓ) for the *orthogonal projection* of the point or pointset S to ℓ.

1.1.6. $ABC\underline{D}$ for the *Lambert quadrilateral* with right angles at A, B, C and an acute angle at D.

1.1.7. $\mathbf{R}$ for the size of a right angle.

1.1.8. $a \not\wedge b, \ a \not\wedge \overrightarrow{p}, \ldots$ for the *intersection point* of the lines a and b, of the line a and the ray $\overrightarrow{p}, \ldots$.

1.1.9. $\cdot, \cdot\cdot, \circ$ (in figures) for specific acute angles with $\cdot\cdot$ denoting a smaller angle than $\cdot$.

Remark. In the figures of Section 3, lines and metric are distorted to better exhibit the betweenness features.

2. Segments that join the legs of an angle

In this section we compile a number of facts about segments whose endpoints move along the legs of a given angle. All statements hold in Euclidean and hyperbolic geometry alike; the easy absolute proofs are for the most part left to the reader.

Let $\angle(\overrightarrow{a}, \overrightarrow{b})$ be an angle with vertex P, and $\mathcal{C}$ be the class of segments $A_\nu B_\nu$ of length s that have endpoint A_ν on leg $(\overrightarrow{a})$ and endpoint B_ν on leg $(\overrightarrow{b})$ of this angle, and satisfy the equivalent conditions

$$\text{(1a)} \quad \angle PA_\nu B_\nu \geq \angle PB_\nu A_\nu, \qquad \text{(1b)} \quad PA_\nu \leq PB_\nu,$$

(see Figures 2a, b). We will always draw $\overrightarrow{a}, \overrightarrow{b}$ as rays that are *directed downwards* and, to simplify expression, refer to P as the *summit* of $\angle(\overrightarrow{a}, \overrightarrow{b})$. As a result of (1a) the segments $A_\nu B_\nu$ are uniquely determined by their endpoints on $(\overrightarrow{a})$, and $\mathcal{C}$ can be generated by sliding downwards through the points on $(\overrightarrow{a})$ and finding the related points on $(\overrightarrow{b})$.

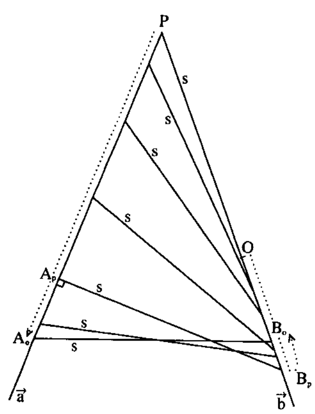

Figure 2a

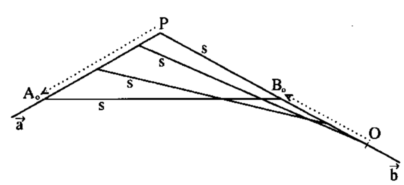

Figure 2b

It is easy to see that during this downwards movement $\angle PA_\nu B_\nu$ decreases and $\angle PB_\nu A_\nu$ increases in size. Due to (1a) the segment A_0B_0 which satisfies $\angle PA_0B_0 \equiv \angle PB_0A_0, \ PA_0 \equiv PB_0$ is the lowest of class $\mathcal{C}$.

If $\angle(\overrightarrow{a}, \overrightarrow{b}) < \mathbf{R}$ then the class $\mathcal{C}$ contains a segment A_pB_p such that $\angle PA_pB_p = \mathbf{R}$. Note that when A_ν moves downwards from P to A_p, B_ν moves in tandem down from the point O s units below P to B_p, but that when A_ν moves on downwards from A_p to A_0, B_0 moves back upwards from B_p to B_0 (see Figure 2a). If $\angle(\overrightarrow{a}, \overrightarrow{b}) \geq \mathbf{R}$ no perpendicular line to $(\overrightarrow{a})$ meets $(\overrightarrow{b})$ and the points B_ν move invariably upwards when the points A_ν move downwards (see Figure 2b).

Now consider three segments $AB, A_1B_1, A_2B_2 \in \mathcal{C}$ whose endpoints on $(\overrightarrow{a})$ satisfy the order relation $[AA_1A_2P]$, and the direct motions that carry segment AB to segment A_1B_1 and to segment A_2B_2. These motions belong to the inverses of the ones described above and may carry B first downwards and then upwards. As a result there are seven conceivable situations as far as the order of the points B, B_1 and B_2 is concerned (see Figure 3):

(I)	$[B_2B_1BP]$,	(II)	$[B_1BP],\ B_2 = B_1$
(III)	$[B_1B_2BP]$,	(IV)	$[B_1BP],\ B_2 = B$,
(V)	$[B_1BB_2P]$,	(VI)	$[BB_2P],\ B_1 = B$, and
(VII)	$[BB_1B_2P]$.		

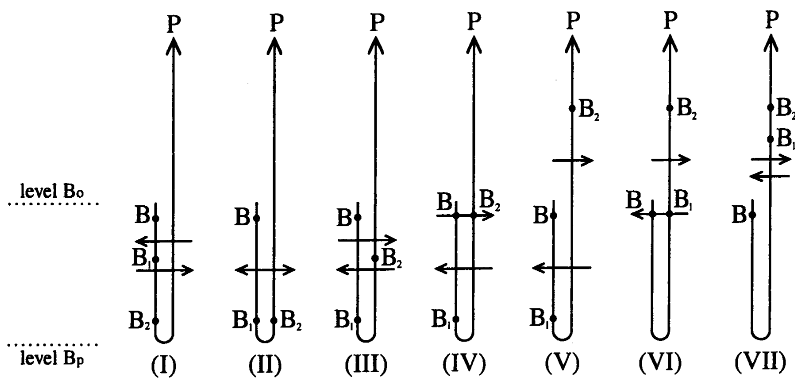

Figure 3

In case $\angle(\overrightarrow{a}, \overrightarrow{b}) \geq \mathbf{R}$ the point B moves solely downwards (see Figure 2b) and we find ourselves automatically in situation (I). On the other hand if $\angle(\overrightarrow{a}, \overrightarrow{b}) < \mathbf{R}$ and A lies on or above A_p both endpoints of segment AB move simultaneously upwards, first to A_1B_1 and then on to A_2B_2 (see Figure 2a); this means that we are dealing with situation (VII).

In Figure 3 the level of the midpoint N_1 of BB_1 is indicated by an arrow to the left, and the level of the midpoint N_2 of BB_2 by an arrow to the right. We recognize at once that we can use N_1 and N_2 instead of B_1 and B_2 to characterize the above seven situations. Set forth explicitly, a triple of segments $AB, A_1B_1, A_2B_2 \in \mathcal{C}$ with $[AA_1A_2P]$ can be classified according to the following conditions on the midpoints N_1, N_2 of BB_1, BB_2:

(2)

(I)	$[N_2N_1BP]$,	(II)	$[N_1BP], N_2 = N_1$,
(III)	$[N_1N_2BP]$,	(IV)	$[N_1BP],\ N_2 = B$,
(V)	$[N_1BN_2P]$,	(VI)	$[BN_2P],\ N_1 = B$, and
(VII)	$[BN_1N_2P]$.		

Note that always $N_1N_2 = \dfrac{1}{2}B_1B_2$. The midpoints M_1, M_2 of AA_1, AA_2 similarly satisfy $M_1M_2 = \dfrac{1}{2}A_1A_2$; here the direction $M_1 \to M_2$ like the direction $A_1 \to A_2$ points invariably upwards.

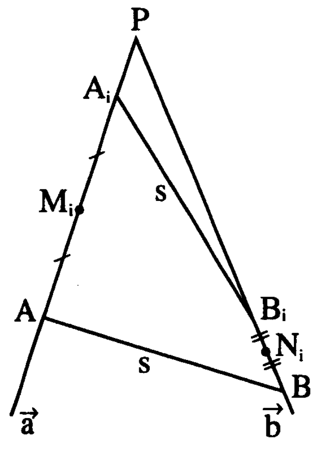

Figure 4

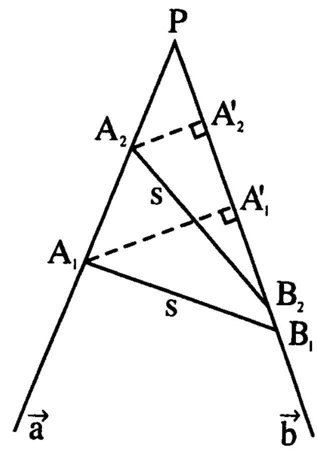

Figure 5

In closing this section we deduce two important inequalities which involve the points M_1, M_2, N_1 and N_2.

From $PA \leq PB,\ PA_i < PB_i$ (see (1b)) follows

(3) $$PM_i < PN_i \quad (i = 1, 2),$$

(see Figure 4). In addition, for situations (III) - (VII) in which B_2 lies above B_1 and N_2 above N_1 we can establish this. In the right triangles $\triangle A_1'A_1B_1$, $\triangle A_2'A_2B_2$ where $A_1' = \text{proj}(A_1, b)$, $A_2' = \text{proj}(A_2, b)$, $A_1'A_1 > A_2'A_2$, $A_1B_1 \equiv A_2B_2\ (= s)$, and as a result $A_1'B_1 < A_2'B_2$ (see Figure 5). So $A_1'A_2' > B_1B_2$, and because $A_1A_2 > A_1'A_2'$, $A_1A_2 > B_1B_2$. Noting what was said above we therefore have:

(4) If N_2 lies above N_1 then $M_1M_2 > N_1N_2$.

3. The centers of two key segment motions

In this section we locate the centers of the segment motions described above. Our setting is the hyperbolic plane in which (as is well-known) three kinds of direct motions have to be considered. The Euclidean case could be subsumed with few modifications under the heading of rotations.

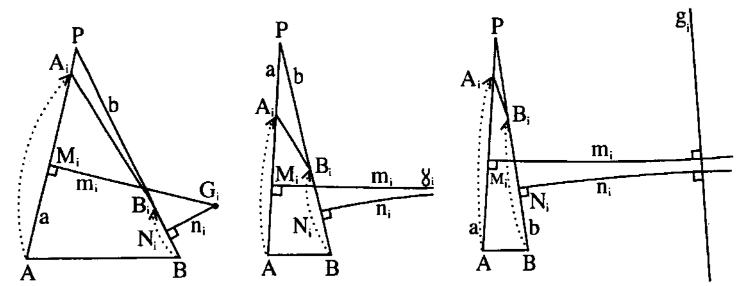

Figure 6

Let μ_i be the rigid, direct motion that carries the segment $AB \in \mathcal{C}$ onto the segment $A_iB_i \in \mathcal{C}$ $(i = 1, 2)$ where A_i lies above A, and let $m_i = \text{perp}(a, M_i)$, $n_i = \text{perp}(b, N_i)$. If lines m_i and n_i meet, μ_i is a *rotation* about their intersection point G_i, if they are boundary parallel, μ_i is an *improper* (horocyclic) *rotation* about their common end γ_i, and if they are hyperparallel, μ_i is a *translation* along their common perpendicular g_i (see e.g. [4, p. 455, Satz 13; Figure 6]). We call G_i, γ_i or g_i the *center* $[G_i]$ of the motion μ_i. For any point X disjoint from the center, $\ell(X[G_i])$ denotes the line joining X to the center of μ_i, namely $\ell(XG_i)$, $\ell(X\gamma_i)$, or $\text{perp}(g_i, X)$. The ray from X contained in this line and in the direction of $[G_i]$ will be referred to as the *ray* $\overrightarrow{X[G_i]}$ *from* X *towards the center of* μ_i; specifically for $X = P, M_i, N_i$ we define $\overrightarrow{p_i} = \overrightarrow{P[G_i]}$, $\overrightarrow{m_i} = \overrightarrow{M_i[G_i]}$ and (if it exists) $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$.

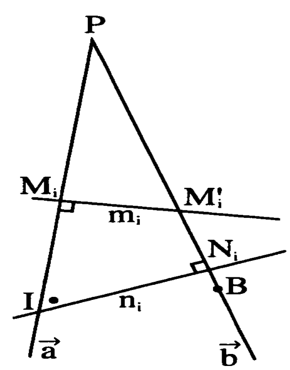

Figure 7

We now show that the center $[G_i]$ of motion μ_i must lie in $H(a, B)$.

If n_i does not intersect $(\overrightarrow{a})$ this is clear; if n_i meets a in a point I (see Figure 7) we verify the statement as follows. Segment PI as the hypothenuse of $\triangle PIN_i$ is larger than PN_i and so (see (3)) larger than PM_i. Consequently the angle $\angle PIN_i = \angle M_i IN_i$ is acute, which indicates that n_i when entering $H(a, B)$ at I, approaches m_i. As a result $[G_i]$ must lie in $H(a, B)$.

Some additional consequences are implied by the fact that the center $[G_i]$ of either motion μ_i is determined by a pair of perpendiculars m_i, n_i to lines a and b (see again Figure 6). If $[G_i] = \gamma_i$ is a common end of m_i, n_i and thus the center of a horocyclic rotation, it cannot be the end of ray $\overrightarrow{b}$. Similarly, if $[G_i] = g_i$ is the common perpendicular of m_i and n_i, and thus the translation axis, it is hyperparallel to both of the intersecting lines a, b and, as a result, has no point in common with either; furthermore, a and b, being connected, must belong to the same halfplane with respect to g_i. On the other hand, if $[G_i] = G_i$ is the common point of m_i and n_i, and thus the rotation center, it is indeed possible that it lies on $(\overrightarrow{b})$. The point G_i then is collinear with B and with its image B_i which means that for $B \neq B_i$ the rotation is a half-turn and G_i coincides with the midpoint N_i of BB_i; in addition G_i should be the midpoint M_i of AA_i which is impossible. So $B = B_i = N_i = G_i$; conversely, one establishes easily that if any two of the three points B, B_i, N_i coincide, μ_i is a rotation with center G_i equal to all three.

We now assume that our plane is furnished with an orientation (see [3, Section 20]), and that without loss of generality P lies to the left of ray $\overrightarrow{AB}$. This ray enters $H(a, B)$ at the point A of $(\overrightarrow{a})$ and $\overline{H}(b, A)$ at the point B of $(\overrightarrow{b})$. Also $\overrightarrow{m_i} = \overrightarrow{M_i[G_i]}$ enters $H(a, B)$ at a point of $(\overrightarrow{a})$ and so has P on its left hand side as well (see Figure 8). As to the ray $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$ which (if existing, i.e. for $[G_i] \neq N_i$) originates at the point N_i of $(\overrightarrow{b})$, it has P on its left hand side if and only if it enters $\overline{H}(b, A)$, i.e. if and only if $[G_i]$ belongs to $\overline{H}(b, A)$.

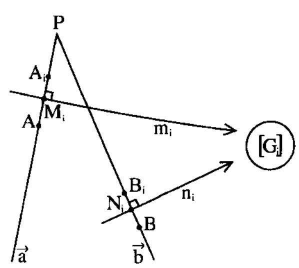

Figure 8a

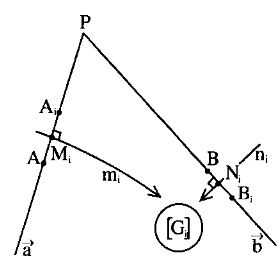

Figure 8b

Because the motion μ_i carries A across $\overrightarrow{m_i}$ to A_i on the side of P, A_i lies to the left and A to the right of $\overrightarrow{m_i}$. Being a direct motion, μ_i consequently also moves B (if $B \neq N_i$) from the right hand side of $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$ to B_i on the left hand side which is the side of P iff $[G_i]$ belongs to $\overline{H}(b, A)$. In short, motion μ_i carries B upwards on $(\overrightarrow{b})$ iff $[G_i]$ lies in $\overline{H}(b, A)$.

We gather from the previous two paragraphs that

- $[G_i]$ belongs to $\overline{H}(b, A)$ if B_i and N_i lie above B on $(\overrightarrow{b})$,

- to $(\overrightarrow{b})$ if $B_i = N_i = B$, and
- to $H(b, A)$ if B_i and N_i lie below B on $(\overrightarrow{b})$.

Considering the motions μ_1, μ_2 again together we can tell in each of the seven situations listed in (2) where the two motion centers $[G_1], [G_2]$ (which both belong to $H(a, B)$) lie with respect to b. As we shall see, the relative positions of $[G_1], [G_2]$ can be described in a way that covers all seven situations: rotating ray $\overrightarrow{a} = \overrightarrow{PA}$ about P into $H(a, B)$ we always pass ray $\overrightarrow{P[G_1]}$ first, and ray $\overrightarrow{P[G_2]}$ second. More concisely,

MAIN LEMMA (ML). *Ray* $\overrightarrow{p_1} = \overrightarrow{P[G_1]}$ *always enters* $\angle(\overrightarrow{a}, \overrightarrow{p_2}) = \angle AP[G_2]$.

Proof. (The essential steps of the proof are outlined at the end.)

From (2) and the previous paragraph follows that $[G_1]$ lies in $H(b, A)$ in situations (I)-(V), on $\overrightarrow{(b)}$ in situation (VI) and in $\overline{H}(b, A)$ in situation (VII); $[G_2]$ lies in $H(b, A)$ in situations (I)-(III), on $(\overrightarrow{b})$ in situation (IV) and in $\overline{H}(b, A)$ in situations (V)-(VII), (see Figure 9). As a result the Lemma follows trivially for situations (IV)-(VI). The other situations are more complex, and their proofs require that the nature of the motion centers $[G_i]$, $(i = 1, 2)$, which can be a point G_i, end γ_i or axis g_i be taken into account. Thus a pair of motion centers $[G_1], [G_2]$ can be equal to G_1, G_2; G_1, γ_2; G_1, g_2; γ_1, G_2; γ_1, γ_2; γ_1, g_2; g_1, G_2; g_1, γ_2; g_1, g_2.

The arguments to be presented are dependent on the mutual position of P, M_1, M_2 on $\overrightarrow{a}$ and of P, N_1, N_2 on $\overrightarrow{b}$, and are best followed through Figure 9.

We first consider situations (I)-(III) in which $\angle(\overrightarrow{a}, \overrightarrow{b})$ includes $(\overrightarrow{p_1})$ and $(\overrightarrow{p_2})$. To verify (ML) we have to show that $\overrightarrow{p_2}$ does not enter $\angle(\overrightarrow{a}, \overrightarrow{p_1})$, or equivalently that $\overrightarrow{p_1}$ does not enter $\angle(\overrightarrow{b}, \overrightarrow{p_2})$. (This assumes $\overrightarrow{p_1} \neq \overrightarrow{p_2}$ which either follows automatically or as an easy consequence of the arguments below.)

We begin with the special case that $\overrightarrow{p_1}$ meets m_2 in a point I. In this case statement (ML) holds if $\overrightarrow{p_2}$ does not intersect line m_2 at I or in a point between M_2 and I. Obviously this is so if $[G_2] = \gamma_2$ or g_2 because then $\overrightarrow{p_2}$ and m_2 do not intersect. If $[G_2] = G_2$, $\overrightarrow{p_2}$ and m_2 do intersect and we have to show that the intersection point, which is G_2, does not coincide with I or lie between M_2 and I. We first note that line n_1 does not intersect ray $\overrightarrow{p_1}$ in I or between I and P because the intersection point would have to be G_1 and so lie on m_1, a line entirely below m_2. As a consequence I, P, M_2, and, if it would lie between M_2 and I, also G_2, would all belong to the same halfplane with respect to n_1, namely $H(n_1, P)$. However this would entail that line n_2 which runs through G_2 would belong to this halfplane, which is not the case in situations (I) and (II). Thus we have established for those situations that $G_2 \neq I$, and $[M_2G_2I]$ does not hold, which means (ML) is true. We will present the proof of the same in situation (III) later on.

Due to the Axiom of Pasch the point I always exists if $\triangle PM_1[G_1]$ is a proper or improper triangle, i.e. if $[G_1] = G_1$, or γ_1. This means that we have so far proved (ML) for the cases G_1, γ_2; G_1, g_2; γ_1, γ_2; γ_1, g_2 and in addition for G_1, G_2; γ_1, G_2 in situations (I) and (II).

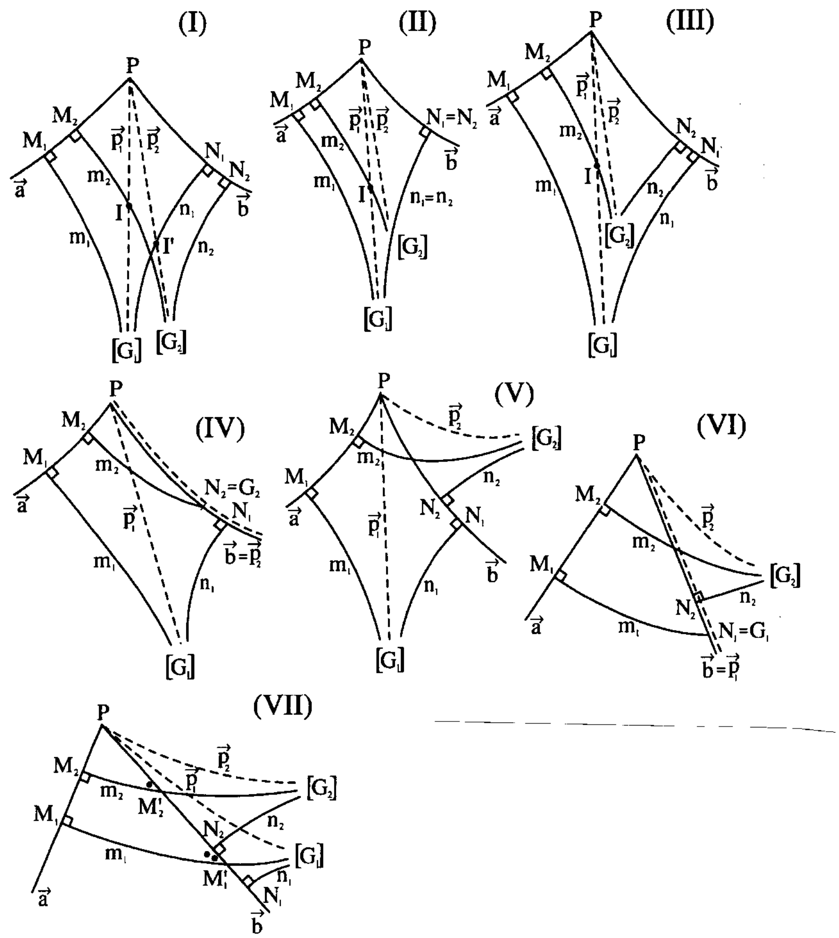

Figure 9

In the opposite special case that $\overrightarrow{p_2}$ meets n_1 in a point I' we can analogously show that for $[G_1] = \gamma_1$ or g_1 ray $\overrightarrow{p_1}$ does not enter $\angle(\overrightarrow{b}, \overrightarrow{p_2})$ and (ML) holds. In fact it is useful to mention here that this statement and its proof can be extended to include configurations in which $\overrightarrow{p_2}$ meets n_1 in an improper point ι'.

In situation (I) the point I' always exists if $[G_2] = G_2$ or γ_2 due to the Axiom of Pasch. In situation (II) with $n_1 = n_2$ I' exists for $[G_2] = G_2$ and ι' exists for $[G_2] = \gamma_2$ because in the first case $G_2 = I'$ and in the second case $\gamma_2 = \iota'$. This means that we have proved (ML) also for g_1, G_2; g_1, γ_2 in situations (I) and (II).

The proofs of the remaining cases, namely g_1, g_2 in situations (I)-(III), and G_1, G_2; γ_1, G_2; g_1, G_2; g_1, γ_2 in situation (III) require metric considerations and will be presented later.

Remark. Taking into account that we have already established (ML) in the case in which $\overrightarrow{p_1}$ and m_2 meet in a point I and $[G_2] = \gamma_2$ or g_2 we will assume when proving (ML) for g_1, g_2 and γ_1, g_2 that $\overrightarrow{p_1}$ and m_2 do not meet. At the same time, taking into account that we have already established (ML) in the case that $\overrightarrow{p_2}$ and n_1 meet in a point I' and $[G_1] = \gamma_1$ or g_1 we will assume that $\overrightarrow{p_2}$ and n_1 do not meet.

Turning to situation (VII) we observe that each of the rays $\overrightarrow{m_i}$ $(i = 1, 2)$ intersects $(\overrightarrow{b})$ in a point M_i' and approaches ray $\overrightarrow{n_i}$ in $\overline{H}(b, A)$, thus causing $\angle N_i M_i'[G_i]$ to be acute. Angle $\angle PM_i'M_i$ of the right triangle $\triangle PM_iM_i'$ is also acute with P above m_i, which means $\angle N_i M_i'[G_i]$ is its vertically opposite angle and N_i lies below m_i. As to the rays $\overrightarrow{p_i} = \overrightarrow{P[G_i]}$ they both enter $\overline{H}(b, A)$ at P which means that the angles $\angle(\overrightarrow{a}, \overrightarrow{p_i})$ have halfline $(\overrightarrow{b})$ in their interior.

If $\overrightarrow{p_2}$ does not intersect m_2, i.e. for $[G_2] = \gamma_2, g_2$, angle $\angle(\overrightarrow{a}, \overrightarrow{p_2})$ includes $(\overrightarrow{m_2}), (\overrightarrow{n_2})$ together with $(\overrightarrow{b})$. So, if in addition $[G_1] = G_1$ or γ_1, halfline $(\overrightarrow{p_1})$ crosses $(\overrightarrow{m_2})$ in order to meet $(\overrightarrow{m_1})$, i.e. runs in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$. Lemma (ML) thus is fulfilled for G_1, γ_2; G_1, g_2; γ_1, γ_2; γ_1, g_2.

The remaining cases of (VII) depend on two metric properties. From $N_1N_2 < M_1M_2$ and $M_1M_2 = \text{proj}(M_1'M_2', a) < M_1'M_2'$ (see (4) and Figure 9, VII) follows $N_1N_2 < M_1'M_2'$ and so

$$N_1M_1' = N_1M_2' - M_1'M_2' < N_1M_2' - N_1N_2 = N_2M_2'. \tag{5}$$

In addition, from the fact that $\triangle PM_2'M_2$ has the smaller area (larger defect) than $\triangle PM_1'M_1$ follows $\angle PM_2'M_2 > \angle PM_1'M_1$, and so

$$\angle N_1M_1'[G_1] < \angle N_2M_2'[G_2]. \tag{6}$$

From (5) and (6) it is clear that if m_2 intersects or is boundary parallel to n_2 then m_1 must intersect n_1, i.e. that the cases $\gamma_1, G_2; g_1, G_2; g_1, \gamma_2$ cannot occur. Also, from (5) and (6) follows that if m_1, n_1 intersect in G_1 and m_2, n_2 intersect in G_2 then side N_1G_1 of $\triangle N_1M_1'G_1$ is shorter than side N_2G_2 of $\triangle N_2M_2'G_2$. This and $PN_1 > PN_2$ applied to $\triangle PN_1G_1, \triangle PN_2G_2$ implies $\angle(\overrightarrow{b}, \overrightarrow{p_1}) < \angle(\overrightarrow{b}, \overrightarrow{p_2})$, and so settles (ML) in the case of G_1, G_2.

The main case left is that of g_1, g_2, both in situation (VII) and situations (I) - (III). For use in the following we define $\text{proj}(M_i, g_i) = R_i$, $\text{proj}(N_i, g_i) = S_i$, $\text{proj}(P, g_i) = P_i$, and, assuming the points exist, $m_2 \wedge g_1 = U, n_2 \wedge g_1 = V, p_2 \wedge g_1 = W$.

If in situation (VII) (in which $\overrightarrow{p_2}$ lies above m_2, see Figure 10) the point W does not exist $\overrightarrow{n_1}$ lies with $(\overrightarrow{b})$, g_1 with $\overrightarrow{n_1}$ and $(\overrightarrow{p_1})$ with g_1 in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$ thus fulfilling (ML). If W exists, line n_2 which runs between the lines n_1, m_2 and so avoids $\overrightarrow{p_2}$, enters quadrilateral PN_1S_1W and leaves it, defining V, between S_1 and W.

From (6) follows that Lambert quadrilateral $N_1S_1R_1\underline{M_1'}$ has the smaller angle sum and so the larger area than $N_2S_2R_2\underline{M_2'}$, which because of (5) requires that

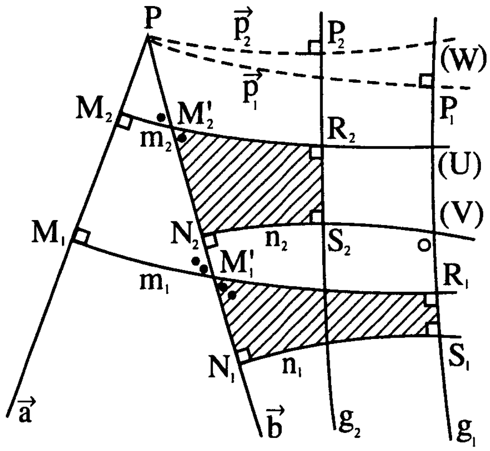

Figure 10

$N_1S_1 > N_2S_2$. As a result V, W on ray $\overrightarrow{S_1R_1}$ satisfy $[N_2S_2V]$, $[PP_2W]$ respectively. As $\angle S_1VN_2 = \angle S_1VS_2$ of $N_2N_1S_1\underline{V}$ is acute, $\angle V$ in P_2S_2VW is obtuse and $\angle P_2 + \angle S_2 + \angle V > 3\mathbf{R}$. This means that $\angle W = \angle PWV$ must be acute and identical with $\angle PWP_1$; consequently $(\overrightarrow{p_1}) = (\overrightarrow{PP_1})$ must lie with V, N_2 in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$, again confirming (ML).

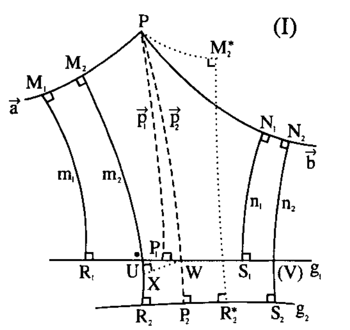

Figure 11a

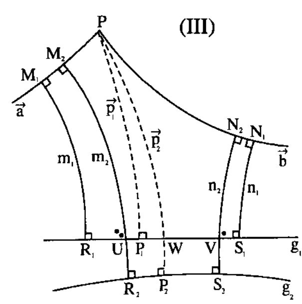

Figure 11b

Each of the Figures 11a, b relating to situations (I), (III) contains two pentagons $PM_iR_iS_iN_i$ $(i = 1, 2)$ with interior altitude PP_i. Adding the images M_i^*, R_i^* of M_i, R_i under reflection in PP_i (as illustrated for $i = 2$ in Figure 11a) we note that $P_iR_i^* < P_iS_i$ because otherwise we would have $PM_i \equiv PM_i^* \geq PN_i$ in contradiction to (3). Moreover $\angle P_iPM_i > \angle P_iPN_i$ as $\angle P_iPM_i \equiv \angle P_iPM_i^* \leq \angle P_iPN_i$ together with $P_iR_i^* < P_iS_i$ would imply that $PM_i^*R_i^*P_i$ would be a part polygon of $PN_iS_iP_i$ while not having a larger angle sum (i.e. smaller defect). So

$$(7a)\quad P_iR_i < P_iS_i, \quad (7b)\quad \angle P_iPM_i > \angle P_iPN_i, \quad (7c)\quad R_iM_i > S_iN_i.$$

In view of an earlier *Remark* we assume that the point U exists and that it satisfies $[R_1UP_1]$; together with $[R_1P_1S_1]$ this extends to $[R_1UP_1S_1]$. In situation (I) we can similarly assume that $\overrightarrow{p_2}$ and n_1 do not meet which means that the point W exists and that it satisfies $[UWS_1]$, a relation that can be extended to $[R_1UWS_1]$. In situation (III) we automatically have V such that $[UVS_1]$ and W such that $[UWV]$ is fulfilled, altogether therefore $[R_1UWVS_1]$.

In both situations m_2 and $\overrightarrow{UR_1}$ include an acute angle which coincides with the fourth angle $\angle R_1UM_2$ of Lambert quadrilateral $M_2M_1R_1\underline{U}$ and so lies on the upper side of g_1. It is congruent to the vertically opposite angle between m_2 and $\overrightarrow{UW}$ which thus lies on the lower side of g_1. In situation (III), for similar reasons, n_2 and $\overrightarrow{VW}$ include an acute angle which is congruent to $\angle N_2VS_1$ and lies on the lower side of g_1. As a result of all this in situation (III) the closest connection R_2S_2 between m_2 and n_2 lies below g_1, and so does the auxiliary point $X = \text{proj}(W, m_2)$ in situation (I).

Statement (ML) holds in both situations if $[UP_1W]$ is fulfilled i.e. if $P_1 = \text{proj}(P, g_1)$ belongs to leg $\overrightarrow{(W)U}$ of $\angle PWU$. We note that this is the case iff $\angle PWU$ is acute.

Now, if in situation (I) R_2, P_2 and S_2 lie below g_1 then the intersection point V of n_2 and g_1 exists and lies between N_2 and S_2, $\angle N_2VS_1$ is acute, $\angle S_2VS_1 = \angle S_2VW$ therefore obtuse and in quadrilateral P_2S_2VW $\angle P_2 + \angle S_2 + \angle V > 3\mathbf{R}$; as a consequence $\angle W = \angle P_2WV$ is acute and so is its vertically opposite angle, $\angle PWU$. This, as we mentioned, proves (ML). If R_2P_2 lies below XW and ray $\overrightarrow{R_2P_2}$ intersects g_1 in a point Y, angle $\angle P_2WY$ in triangle $\triangle P_2WY$ is acute, which leads to the same conclusion. If $R_2P_2 = XW$ then $\angle PWU < \angle PWX = \angle PP_2R_2 = \mathbf{R}$. Finally, if R_2P_2 lies above XW, $\angle PWX$ as the fourth angle of Lambert quadrilateral $XR_2P_2\underline{W}$ is acute, and because $\angle PWU < \angle PWX$, $\angle PWU < \mathbf{R}$. This concludes the proof of (ML) in situation (I).

In situation (III) we have area $M_2M_1R_1\underline{U} > N_2N_1S_1\underline{V}$ because of (4), (7c). Consequently $\angle M_2UR_1 < \angle N_2VS_1$, and so $\angle WUR_2 < \angle WVS_2$ on the other side of g_1. If we also had $\angle P_2WU \leq \angle P_2WV$ then quadrilateral R_2P_2WU would have a smaller angle sum and larger defect than S_2P_2WV. At the same time (7a) and this angle inequality would imply that the former quadrilateral would fit into the latter, i.e. have the smaller area. Since this is contradictory $\angle P_2WU$ must be larger than the adjacent angle $\angle P_2WV$; as a result $\angle P_2WV < \mathbf{R}$ and vertically opposite, $\angle PWU < \mathbf{R}$ which establishes (ML) for g_1, g_2 in situation (III).

The proof of (ML) in situation (III) can be extended with only very minor changes to situation (II). Also closely related is the case of g_1, γ_2 in situation (III). If here, in addition to $\angle WU\gamma_2 < \angle WV\gamma_2$, the inequality $\angle\gamma_2WU \leq \angle\gamma_2WV$ were to hold then line m_2 would have to run farther away from line p_2 than line n_2 in contradiction to (3). An analogous argument applies to the case g_1, G_2 in situation (III) when G_2 lies below g_1. Note that if line m_2 intersects g_1 in a point U between

R_1 and P_1 rather than $\overrightarrow{p_1}$ in a point I between P and P_1 then G must lie below g_1 (see Figure 11b). This is so because according to (4) and (7c) the existence of a Lambert quadrilateral $M_2M_1R_1\underline{U}$ with $R_1U < R_1P_1$ implies the existence of $N_2N_1S_1\underline{V}$ with $S_1V < R_1U < R_1P_1$, and so due to (7a) with $S_1V < S_1P_1$; the point P_1 thus lies between U and V, and M_2 and N_2 meet below g_1.

To conclude the proof of (ML) we still have to settle the cases G_1, G_2; γ_1, G_2 and g_1, G_2 (this with $I = m_2 \between \overrightarrow{p_1}$ on or above P_1) of situation (III). We present here the last case (Figure 12b) which is easy and representative also for the proofs of the other two cases (Figure 12a).

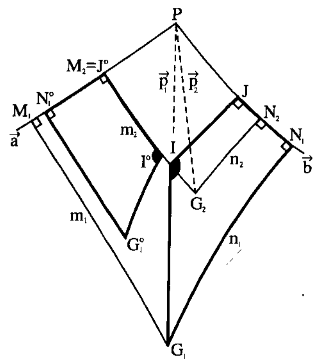

Figure 12a

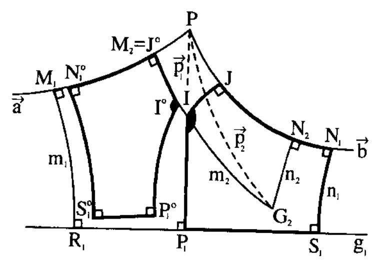

Figure 12b

Call $J = \text{proj}(I, b)$ and note that $\angle IPM_2 > \angle IPJ$ (7b) implies *(i)* $M_2I > JI$, and *(ii)* $\angle PIM_2 < \angle PIJ$, $\angle P_1IM_2 > \angle P_1IJ$. If G_2 would lie in $H(p_1, M_2)$ then the point $N_2 = \text{proj}(G_2, b)$ would determine a segment $N_1N_2 > N_1J$, and due to (4) the inequality *(iii)* $M_1M_2 > N_1J$ would result.

We now carry the pentagon $\mathcal{P}_b = JN_1S_1P_1I$ by an indirect motion to $\mathcal{P}_b^0 = J^0N_1^0S_1^0P_1^0I^0$ where $J_0 = M_2$, N_1^0 lies on $\overrightarrow{M_2M_1}$ and I^0 on $\overrightarrow{M_2I}$. Assuming that G_2 belongs to $H(p_1, M_2)$ we have according to *(i)*, *(iii)* that I^0 lies between M_2 and I, and N_1^0 between M_2 and M_1. Due to (7c) ray $\overrightarrow{S_1^0P_1^0}$ lies in the interior of $\angle M_1R_1P_1$, and due to *(ii)* halfline $\overrightarrow{(I^0)P^0}$ lies in the interior of $\angle P_1IM_2$ which implies that $\mathcal{P}_b^0$ is a proper part of polygon $\mathcal{P}_a = M_2M_1R_1P_1I$ in contradiction to the fact that $\mathcal{P}_b^0$ has the smaller angle sum, i.e. the larger defect. So G_2 and $\overrightarrow{p_2}$ do not lie in $\angle(\overrightarrow{a}, \overrightarrow{p_1})$ and the proof of (ML) is complete. □

Summary of the Proof.
(1) Situations (IV) - (VI) are trivial.
(2) In situations (I) - (III), (ML) holds if $\overrightarrow{p_1} \between m_2 = I$ with $[G_2] \neq G_2$, and in situations (I), (II) also with $[G_2] = G_2$.
$\longrightarrow G_1, \gamma_2; G_1, g_2; \gamma_1, \gamma_2; \gamma_1, g_2$ of (I) - (III), $G_1, G_2; \gamma_1, G_2$ of (I), (II).
(3) In situations (I) - (III), (ML) holds if $\overrightarrow{p_2} \between n_1 = I'$ (ι') with $[G_1] \neq G_1$.

$\longrightarrow g_1, G_2; g_1, \gamma_2$ of (I), (II).
(4) In situation (VII) a direct comparison of $\triangle N_1 M_1'[G_1]$, $\triangle N_2 M_2'[G_2]$ reveals the relative position of $[G_1], [G_2]$ in all but one case.
$\longrightarrow$ all cases of (VII) except g_1, g_2.
(5) In situation (VII) the area comparison of $N_1 S_1 R_1 \underline{M_1'}$, $N_2 S_2 R_2 \underline{M_2'}$ helps to solve the remaining case.
$\longrightarrow g_1, g_2$ of (VII).
(6) In situations (I), (III) the area comparison of $PM_1R_1S_1N_1$, $PM_2R_2S_2N_2$ helps to solve the same case as in 5.
$\longrightarrow g_1, g_2$ of (I), (III).
(7) The arguments of 6. can be extended to three more cases.
$\longrightarrow g_1, g_2$ of (II); g_1, γ_2 of (III); g_1, G_2 of (III) for G_2 below g_1.
(8) The area comparison between a part polygon of $N_1 S_1 P_1 \underline{P}$, and one of $M_1 R_1 P_1 \underline{P}$, together with two similar comparisons, settle the remaining cases of (III).
$\longrightarrow g_1, G_2$ with G_2 above g_1; G_1, G_2; γ_1, G_2 of (III).

4. Reinterpretation and solution of the posed problem

In the following we formulate, re-formulate and prove a statement which essentially contains the convexity claim of Section 1. Subsequently we discuss the details which make the convexity proof complete.

Theorem 1. *Let AB be a fixed segment and P_2^-, P_1^- and P three points in the same halfplane with respect to the line through A and B such that*

$$\angle AP_2^- B \equiv \angle AP_1^- B \equiv \angle APB \tag{8}$$

and

$$\angle BAP_2^- > \angle BAP_1^- > \angle BAP \geq \angle ABP. \tag{9}$$

Then the line r which joins P_2^- and P separates the point P_1^- from the segment AB (see Figure 13a).

For the purpose of re-formulating this theorem we carry the points A, B, P_1^-, P and the line r of this configuration by a rigid, direct motion μ_1 into the points A_1, B_1, P, P_1 and the line r_1 respectively such that A_1 lies on $\overrightarrow{PA}$ and B_1 on $\overrightarrow{PB}$ (see Figure 13b). This allows us to substitute the following equivalent theorem for Theorem 1.

Theorem 2. *In the configuration of the points A, B, P, A_1, B_1, P_1 and the line r_1 as defined above, the line r_1 separates the point P from segment AB.*

Remark. Note that Theorem 1 amounts to the statement that the intersection point C_1^- of ray $\overrightarrow{AP_1^-}$ and line r lies between A and P_1^-, and Theorem 2 to the statement that the intersection point C_1 of $\overrightarrow{A_1 P}$ and r_1 (i.e. the image of C_1^- under the motion μ_1) lies between A_1 and P.

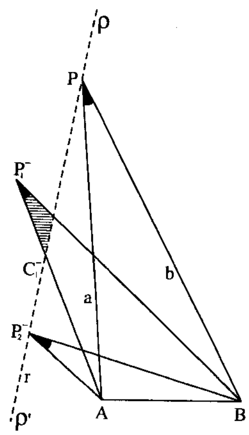

Figure 13a

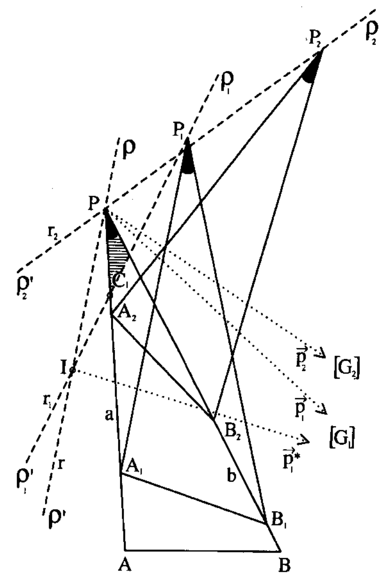

Figure 13b

Proof of Theorem 2. We first augment our configuration by the images of another rigid, direct motion μ_2 which carries the points A, B, P, P_2^- and the line r into A_2, B_2, P_2, P and r_2 respectively where A_2 lies on $\overrightarrow{PA}$ and B_2 on $\overrightarrow{PB}$. We note that because r joins P_2^- and P, r_2 joins P and P_2. From $\angle BAP_2^- > \angle BAP_1^-$ (see (9)) follows $\angle B_2A_2P = \mu_2(\angle BAP_2^-) > \mu_1(\angle BAP_1^-) = \angle B_1A_1P$ and so $[AA_1A_2P]$ according to Section 2. In the following we denote the ends of r by ρ and ρ' (with ρ' on the same side of $a = \ell(AP)$ as C_1^-) and their images on r_1 and r_2 by ρ_1, ρ_1' resp. ρ_2, ρ_2'. Since ρ' lies on the left (right) side of $\overrightarrow{AP_2^-}$ and of $\overrightarrow{AP_1^-}$ if and only if it lies on the left (right) side of $\overrightarrow{AP}$, and since $\overrightarrow{A_2P} = \mu_2(\overrightarrow{AP_2^-})$, $\overrightarrow{A_1P} = \mu_1(\overrightarrow{AP_1^-})$ and $\overrightarrow{AP}$ are equally directed, ρ_2', ρ_1' and ρ' lie together with C_1^- in $\overline{H}(a, B)$. We note that as an exterior angle of $\triangle AP_2^-C_1^-$, $\angle AP_2^-\rho' > \angle AC_1^-\rho'$, and that as an exterior angle of $\triangle AC_1^-P$, $\angle AC_1^-\rho' > \angle AP\rho'$. Applying μ_2 and μ_1 on the two sides of the first and μ_1 on the left hand side of the second inequality we obtain $\angle A_2P\rho_2' > \angle A_1C_1\rho_1'$ and $\angle A_1C_1\rho_1' > \angle AP\rho'$. The supplementary angles consequently satisfy

$$(10) \qquad \angle AP\rho > \angle AC_1\rho_1 > \angle AP\rho_2, \qquad \rho, \rho_1, \rho_2 \in H(a, B).$$

From (10) follows that ρ_2 lies on the same side of line $r = \ell(P\rho)$ as A, and (because ray $\overrightarrow{PA}$ does not enter $\angle\rho P\rho_2$) $\overrightarrow{PA}$ enters $\angle\rho' P\rho_2$.

At this point we augment our figure further by the rays $\overrightarrow{p_1}, \overrightarrow{p_2}$ which connect P to the centers $[G_1], [G_2]$ of the motions μ_1, μ_2 and, if r, r_1 have a point I in common, by the ray $\overrightarrow{p_1}^*$ connecting I to $[G_1]$. Because μ_2 maps r and ρ to r_2 and ρ_2, while μ_1 maps r and ρ to r_1 and ρ_1, the ray $\overrightarrow{p_2}$ is the bisector of angle $\angle\rho' P\rho_2$, and (if existing) the ray $\overrightarrow{p_1}^*$ is the bisector of angle $\angle\rho' I\rho_1$. Since $(\overrightarrow{p_2})$ lies together with ρ_2 in $H(a, B)$ (see Section 3) whereas ρ' lies in $\overline{H}(a, B)$, the ray $\overrightarrow{PA}$ enters $\angle\rho' P[G_2]$.

We now show by indirect proof that C_1 cannot lie on or above P on a.

For $C_1 = P$ (see Figure 14a) formula (10) reads: $\angle AP\rho > \angle AP\rho_1 > \angle AP\rho_2$, $\rho, \rho_1, \rho_2 \in H(a, B)$, and we can add to the sentence following (10) that also ρ_1 and A lie on the same side of r. Thus $\angle\rho' P\rho_1 = \angle\rho' PA + \angle AP\rho_1 > \angle\rho' PA + \angle AP\rho_2 = \angle\rho' P\rho_2$, and $\angle\rho' P[G_1] = \frac{1}{2}\angle\rho' P\rho_1 > \frac{1}{2}\angle\rho' P\rho_2 = \angle\rho' P[G_2]$. This means that $\overrightarrow{p_1}$ does not enter $\angle\rho' P[G_2]$ and so does not enter $\angle AP[G_2]$ in contradiction to Lemma (ML).

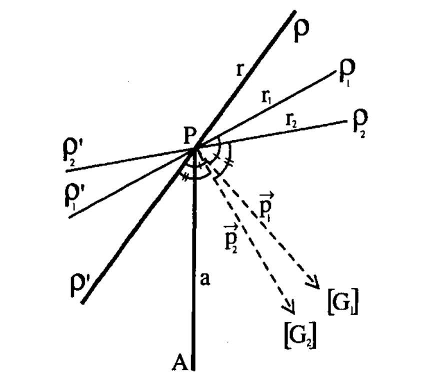

Figure 14a

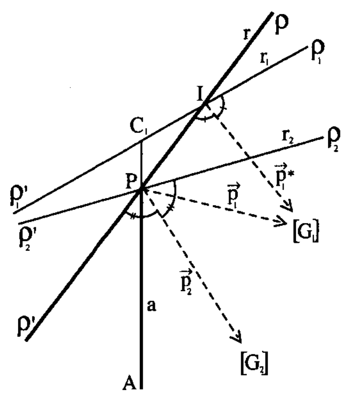

Figure 14b

If C_1 were to lie above P, on a, ray $\overrightarrow{P\rho}$ of r would, according to (10), approach ray $\overrightarrow{C_1\rho_1}$ when entering $H(a, B)$. This means $\overrightarrow{P\rho}$ and $\overrightarrow{P\rho_1}$ either have a point I or the ends ρ, ρ_1 in common, or $r = \ell(P\rho)$ and $r_1 = \ell(P\rho_1)$ share a perpendicular line whose intersection point with r lies in $H(a, B)$. Let us first assume that $\overrightarrow{P\rho}, \overrightarrow{P\rho_1}$ meet in I (see Figure 14b).

In this case line r intersects both segment C_1A and ray $\overrightarrow{C_1\rho_1}$ which means that A and ρ_1 lie on the same side of r. Note that $\angle\rho' I\rho_1$ is equal to the exterior angle $\angle C_1 I\rho$ of $\triangle PC_1 I$ and so satisfies $\angle\rho' I\rho_1 > \angle PC_1 I + \angle C_1 PI$. Because $\angle PC_1 I$ $(= \angle AC_1\rho_1) > \angle AP\rho_2$ (see (10)) and because $\angle C_1 PI \equiv \angle\rho' PA$ we have $\angle\rho' I\rho_1 > \angle AP\rho_2 + \angle\rho' PA = \angle\rho' P\rho_2$. The lower halves of the compared angles consequently satisfy $\angle\rho' I[G_1] > \angle\rho' P[G_2]$ which means that neither $\overrightarrow{p_1}^*$ nor the boundary parallel ray $\overrightarrow{p_1}$ would enter $\angle\rho' P[G_2]$, again in contradiction to Lemma (ML).

Should $\rho = \rho_1$ (see Figure 14c) then this common end is at the same time the center $[G_1]$ of motion μ_1. As a result the ray $\overrightarrow{p_1} = \overrightarrow{P\rho}$, and again fails to enter $\angle AP[G_2]$.

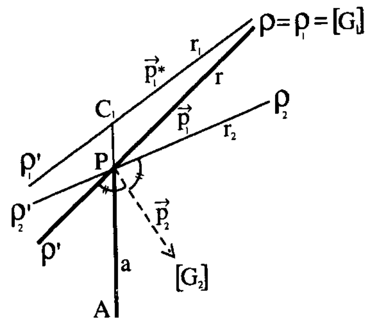

Figure 14c

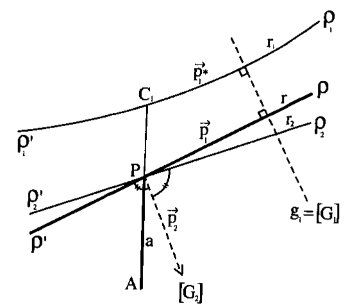

Figure 14d

Finally, if r and r_1 have the perpendicular line g_1 in common (see Figure 14d) then μ_1 which maps r, ρ to r_1, ρ_1 is a translation with axis g_1. As a result $\overrightarrow{p_1} = \overrightarrow{P[G_1]}$ must coincide with $\overrightarrow{P\rho}$ which means it does not enter $\angle\rho' P\rho_2$ and so does not enter $\angle\rho' P[G_2]$ and $\angle AP[G_2]$ in contradiction to Lemma (ML). This completes the proof of Theorem 2 and of Theorem 1.

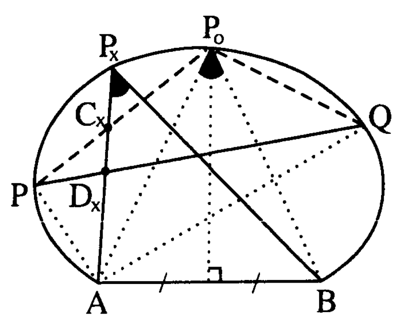

Figure 15

It should be noted that Theorem 1 contains the assumption (9), that P_2^-, P_1^-, P belong to the half-arc of our locus from A to the point P_0 on the perpendicular bisector of segment AB (see Figure 15)). By symmetry the Theorem also shows the convexity of the half-arc from B to P_0. In order to establish the convexity of the whole arc we need to confirm the additional fact that for P a point on the first half-arc and Q a point on the second, line $\ell(PQ)$ separates the points of arc $(\overset{\frown}{PQ})$ from those of segment AB. To do so we choose a point P_X, without loss of generality on half-arc $(\widehat{PP}_0)$, and establish that segment AP_X meets segment PQ between A and P_X. Obviously ray $\overrightarrow{AP_X}$, which enters $\angle PAQ$, meets PQ in a point D_X. Also, by Theorem 1 segment AP_X has a point C_X in common with segment P_0P, which means that our claim follows from $[AD_XC_X]$, a relation which is fulfilled if P_0, and so $(P_0P), (P_0Q)$ belong to $\overline{H}(PQ, A)$. This however is a consequence of the fact that P_0 has a greater distance from $\ell(AB)$ than P and Q, a fact of absolute geometry for which there are many easy proofs.

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