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A Very Short and Simple Proof of "The Most Elementary Theorem" of Euclidean Geometry

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Abstract. We give a very short and simple proof of the fact that if ABB' and $AC'C$ are straight lines with BC and $B'C'$ intersecting at D, then $AB+BD = AC' + C'D$ if and only if $AB' + B'D = AC + CD$. The "only if" part is attributed to Urquhart, and is referred to by Dan Pedoe as "the most elementary theorem of Euclidean geometry".

The theorem referred to in the title states that *if ABB' and $AC'C$ are straight lines with BC and $B'C'$ intersecting at D and if $AB + BD = AC' + C'D$, then $AB' + B'D = AC + CD$* ; see Figure 1. The origin and some history of this theorem are discussed in [9], where Professor Pedoe attributes the theorem to

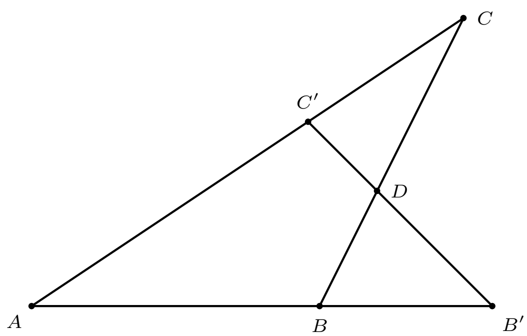

Figure 1

the late L. M. Urquhart (1902-1966) who *discovered it when considering some of the fundamental concepts of the theory of special relativity*, and where Professor Pedoe asserts that *the proof by purely geometric methods is not elementary*. Pedoe calls it *the most "elementary" theorem of Euclidean geometry* and gives variants and equivalent forms of the theorem and cites references where proofs can be found. Unaware of most of the existing proofs of this theorem (e.g., in [3], [4], [13], [14], [8], [10], [11] and [7, Problem 73, pages 23 and 128-129]), the author of this note has published a yet another proof in [5]. In view of all of this, it is interesting to know that De Morgan had published a proof of Urquhart's Theorem in 1841 and that Urquhart's Theorem may be viewed as a limiting case of a result due to Chasles that dates back to 1860; see [2] and [1].

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In this note, we give a much shorter proof based on a very simple and elegant lemma that Robert Breusch had designed for solving a 1961 MONTHLY problem. However, we make no claims that our proof meets the standards set by Professor Pedoe who hoped for a circle-free proof. Clearly our proof does not qualify since it rests heavily on properties of *circular* functions. Breusch's lemma [12] states that if $A_jB_jC_j$ ($j = 1, 2$), are triangles with angles $A_j = 2\alpha_j$, $B_j = 2\beta_j$, $C_j = 2\gamma_j$, and if $B_1C_1 = B_2C_2$, then the perimeter $p(A_1B_1C_1)$ of $A_1B_1C_1$ is equal to or greater than the perimeter $p(A_2B_2C_2)$ of $A_2B_2C_2$ according as $\tan\beta_1 \tan\gamma_1$ is equal to or greater than $\tan\beta_2 \tan\gamma_2$. This lemma follows immediately from the following sequence of simplifications, where we work with one of the triangles after dropping indices, and where we use the law of sines and the addition formulas for the sine and cosine functions.

$$\begin{aligned}\frac{p(ABC)}{BC} &= 1 + \frac{AB + AC}{BC} = 1 + \frac{\sin 2\gamma + \sin 2\beta}{\sin 2\alpha} = 1 + \frac{\sin 2\gamma + \sin 2\beta}{\sin(2\gamma + 2\beta)}\\ &= 1 + \frac{2\sin(\gamma+\beta)\cos(\gamma-\beta)}{2\sin(\gamma+\beta)\cos(\gamma+\beta)} = 1 + \frac{\cos\gamma\cos\beta + \sin\gamma\sin\beta}{\cos\gamma\cos\beta - \sin\gamma\sin\beta}\\ &= \frac{2\cos\gamma\cos\beta}{\cos\gamma\cos\beta - \sin\gamma\sin\beta} = \frac{2}{1 - \tan\gamma\tan\beta}.\end{aligned}$$

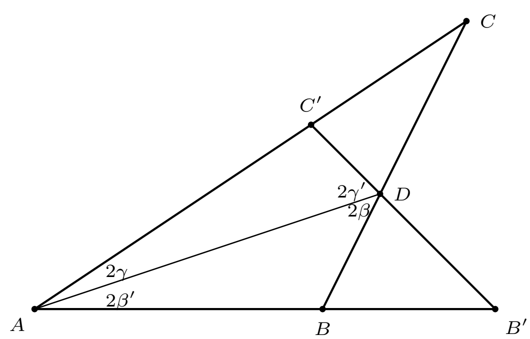

Figure 2

Urquhart's Theorem mentioned at the beginning of this note follows, together with its converse, immediately. Referring to Figure 1, and letting $\angle B'AD = 2\beta'$, $\angle CAD = 2\gamma$, $\angle BDA = 2\beta$, and $\angle C'DA = 2\gamma'$, as shown in Figure 2, we see from Breusch's Lemma that

$$\begin{aligned}p(AB'D) = p(ACD) &\Longleftrightarrow \tan\beta'\tan(90^\circ - \gamma') = \tan\gamma\tan(90^\circ - \beta)\\ &\Longleftrightarrow \tan\beta'\cot\gamma' = \tan\gamma\cot\beta\\ &\Longleftrightarrow \tan\beta'\tan\beta = \tan\gamma\tan\gamma'\\ &\Longleftrightarrow p(ABD) = p(AC'D),\end{aligned}$$

as desired.

The MONTHLY problem that Breusch's lemma was designed to solve appeared also as a conjecture in [6, page 78]. It states that if D, E, and F are points on the sides BC, CA, and AB, respectively, of a triangle ABC, then $p(DEF) \leq \min\{p(AFE), p(BDF), p(CED)\}$ if and only if D, E, and F are the midpoints of the respective sides, in which case the four perimeters are equal. In contrast with the analogous problem obtained by replacing perimeters by areas and the rich literature that this area version has generated, Breusch's solution of the perimeter version is essentially the only solution that the author was able to trace in the literature.

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