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A Projectivity Characterized by the Pythagorean Relation

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Abstract. We study an interesting configuration that gives an example of an elliptic projectivity characterized by the Pythagorean relation.

1. A Romanian Olympiad problem

It is known that any projectivity relating two ranges on one line with more than two invariant points is the identity transformation of the line onto itself. Depending on whether the number of invariant points is 0, 1, or 2 the projectivity would be called *elliptic*, *parabolic*, or *hyperbolic* (see Coxeter [1, p.45], or [3, pp.41-43]). This note will point out an interesting and unusual configuration that gives an example of projectivity characterized by a Pythagorean relation.

The configuration appears in a problem introduced in the National Olympiad 2001, in Romania, by Mircea Fianu. The statement of the problem is the following: *Consider the right isosceles triangle ABC and the points M, N on the hypothenuse BC in the order B, M, N, C such that $BM^2 + NC^2 = MN^2$. Prove that $\angle MAN = \frac{\pi}{4}$.*

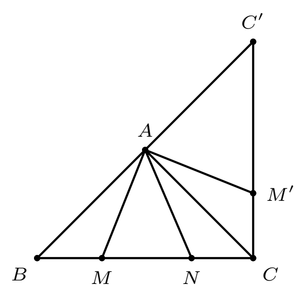

Figure 1

We present first an elementary solution for this problem. Consider a counterclockwise rotation around A of angle $\frac{\pi}{2}$. By applying this rotation, ΔABC becomes $\Delta ACC'$ (see Figure 1) and AM becomes AM'; thus, the angle $\angle MAM'$ is right. The equality $BM^2 + NC^2 = MN^2$ transforms into $CM'^2 + NC^2 = MN^2 = M'N^2$, since $\Delta CNM'$ is right in C. Therefore, $\Delta MAN \equiv \Delta NAM'$ (SSS case), and this means $\angle MAN \equiv \angle NAM'$. Since $\angle MAM' = \frac{\pi}{2}$, we get $\angle MAN = \frac{\pi}{4}$, which is what we wanted to prove.

We shall show that the metric relation introduced in the problem above, similar to the Pythagorean relation, is hiding an elliptic projectivity of focus A. Actually, this is what makes this problem and this geometric structure so special and deserving of our attention. First, we would like to recall a few facts of projective geometry.

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2. Projectivities

Let A, B, C, and D be four points, in this order, on the line $\mathcal{L}$ in the Euclidean plane. Consider a system of coordinates on $\mathcal{L}$ such that A, B, C, and D correspond to x_1, x_2, x_3, and x_4, respectively. The cross ratio of four ordered points A, B, C, D on $\mathcal{L}$, is by definition (see for example [5, p.248]):

$$(ABCD) = \frac{AC}{BC} \div \frac{AD}{BD} = \frac{x_3 - x_1}{x_3 - x_2} \div \frac{x_4 - x_1}{x_4 - x_2}. \qquad (1)$$

This definition may be extended to a pencil consisting of four ordered lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$. By definition, the cross ratio of four ordered lines is the cross ratio determined by the points of intersection with a line $\mathcal{L}$. Therefore,

$$(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4) = \frac{A_1A_3}{A_2A_3} \div \frac{A_1A_4}{A_2A_4}$$

where $\{A_i\} = \mathcal{L} \cap \mathcal{L}_i$. The law of sines shows us that the above definition is independent on $\mathcal{L}$.

We call a projectivity on a line $\mathcal{L}$ a map $f : \mathcal{L} \to \mathcal{L}$ with the property that the cross ratio of any four points is preserved, that is

$$(A_1A_2A_3A_4) = (B_1B_2B_3B_4)$$

where $B_i = f(A_i)$, $i = 1, 2, 3, 4$. The points A_i and B_i are called homologous points of the projectivity on $\mathcal{L}$, and the relation $B_i = f(A_i)$ is denoted $A_i \to B_i$.

The following result is presented in many references (see for example [3], Theorem 4.12, p.34).

Theorem 1. *A projectivity on $\mathcal{L}$ is determined by three pairs of homologous points.*

A consequence of this theorem is that two projectivities which have three common pairs of homologous points must coincide. Actually, we will use this consequence in the proof we present below. In fact, the coordinates x and y of the homologous points under a projectivity are related by

$$y = \frac{mx + n}{px + q}, \quad mq - np \neq 0,$$

where $m, n, p, q \in \mathbb{R}$.

In formula (1), it is possible that $(ABCD)$ takes the value -1, as for example in the case of the feet of interior and exterior bisectors associated to the side BD of a triangle MBD (see Figure 2).

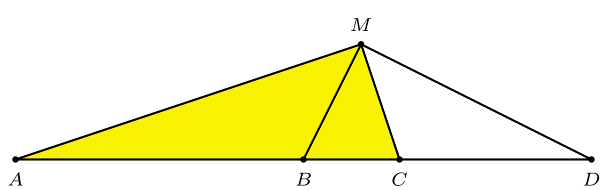

Figure 2

Observe that if C is the midpoint of the segment BD, then point A is not on the line determined by points B and D, since MA becomes parallel to BD. Indeed, for any C on the straight line BD there exist the point M in the plane (not necessarily unique) such that MC is the interior bisector of $\angle BMD$. The point A with the property $(ABCD) = -1$ can be found at the intersection between the external bisector of $\angle BMD$ and the straight line BD. In the particular case when C is the midpoint of BD, we have that ΔMBD is isosceles and the external bisector MA is parallel to BD. To extend the bijectivity of the projectivity presented above, we will say that the homologous of the point C is the point at infinity, denoted ∞, which we attach to the line d. We shall also accept the convention

$$\frac{\infty C}{\infty D} \div \frac{BC}{BD} = -1.$$

For our result, we need the following.

Lemma 2. *A moving angle with vertex in the fixed point A in the plane intersects a fixed line $\mathcal{L}$, A not on $\mathcal{L}$, in a pair of points related by a projectivity.*

Proof. As mentioned in the statement, let A be a fixed point and $\mathcal{L}$ a fixed line such that A is not on $\mathcal{L}$. Consider the rays h and k with origin in A, the moving angle $\angle hk$ with the vertex in A and of constant measure α. Denote by $\{M\} = h \cap \mathcal{L}$ and $\{N\} = k \cap \mathcal{L}$. We have to prove that $f : \mathcal{L} \to \mathcal{L}$ defined by $f(M) = N$ is a projectivity on the line $\mathcal{L}$ determined by the rotation of $\angle hk$. Consider four positions of the angle $\angle hk$, denoted consecutively $\angle h_1 k_1$, $\angle h_2 k_2$, $\angle h_3 k_3$, $\angle h_4 k_4$. Their intersections with the line $\mathcal{L}$ yield the points $M_1, N_1; M_2, N_2; M_3, N_3; M_4, N_4$, respectively. It is sufficient to prove that the cross ratio $[M_1 M_2 M_3 M_4]$ and $[N_1 N_2 N_3 N_4]$ are equal. The rotation of the moving angle $\angle hk$ yields, for the pencil of rays h_1, h_2, h_3, h_4 and k_1, k_2, k_3, k_4, respectively, the pairs of equal angles:

$$\begin{aligned}\angle M_1 A M_2 =& \angle N_1 A N_2 = \beta_1,\\ \angle M_2 A M_3 =& \angle N_2 A N_3 = \beta_2,\\ \angle M_3 A M_4 =& \angle N_3 A N_4 = \beta_3.\end{aligned}$$

By the law of sines we get that the two cross ratios are equal, both of them having the value

$$\frac{\sin(\beta_1 + \beta_2)}{\sin \beta_2} \div \frac{\sin(\beta_1 + \beta_2 + \beta_3)}{\sin(\beta_2 + \beta_3)}.$$

This proves the claim that f is a projectivity on $\mathcal{L}$ in which the homologous points are M and N. □

3. A projective solution to Romanian Olympiad problem

With these preparations, we are ready to give a projective solution to the initial problem.

Consider a system of coordinates in which the vertices of the right isosceles triangle are $A(0, a)$, $B(-a, 0)$, and $C(a, 0)$. See Figure 3. We consider also $M(x, 0)$ and $N(y, 0)$. The relation $BM^2 + NC^2 = MN^2$ becomes

$$(x + a)^2 + (a - y)^2 = (y - x)^2,$$

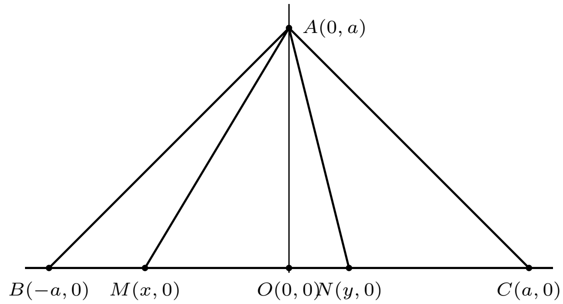

Figure 3

or, solving for y,

$$y = \frac{ax + a^2}{a - x}. \tag{2}$$

This is the equation of a projectivity on the line BC, represented by the homologous points $M \to N$.

Consider now another projectivity on BC determined by the rotation about A by $\frac{\pi}{4}$ (see Lemma 2). This projectivity is completely determined by three pairs of homologous points. First, we see that $B \to O$, since $\angle BAO = \frac{\pi}{4}$. We also have $O \to C$, since $\angle CAO = \frac{\pi}{4}$. Finally, $C \to \infty$, since $\angle CA\infty = \frac{\pi}{4}$.

On the other hand, $B \to O$, since by replacing the $x-$coordinate of B in (2) we get 0, i.e. the $x-$coordinate of O. Similarly, $0 \to a$ and $a \to \infty$ express that $O \to C$ and, respectively, $C \to \infty$. Since a projectivity is completely determined by a triple set of homologous points, the two projectivities must coincide. Therefore, the pair $M \to N$ has the property $\angle MAN = \frac{\pi}{4}$. □

This concludes the proof and the geometric interpretation: the Pythagorean-like metric relation from the original problem reveals a projectivity, which makes this geometric structure remarkable. Furthermore, this solution shows that M and N can be anywhere on the line determined by the points B and C.

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