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The Feuerbach Point and Euler lines

Bogdan Suceavă and Paul Yiu

Abstract. Given a triangle, we construct three triangles associated its incircle whose Euler lines intersect on the Feuerbach point, the point of tangency of the incircle and the nine-point circle. By studying a generalization, we show that the Feuerbach point in the Euler reflection point of the intouch triangle, namely, the intersection of the reflections of the line joining the circumcenter and incenter in the sidelines of the intouch triangle.

1. A MONTHLY problem

Consider a triangle ABC with incenter I, the incircle touching the sides BC, CA, AB at D, E, F respectively. Let Y (respectively Z) be the intersection of DF (respectively DE) and the line through A parallel to BC. If E' and F' are the midpoints of DZ and DY, then the six points A, E, F, I, E', F' are on the same circle. This is Problem 10710 of the *American Mathematical Monthly* with slightly different notations. See [3].

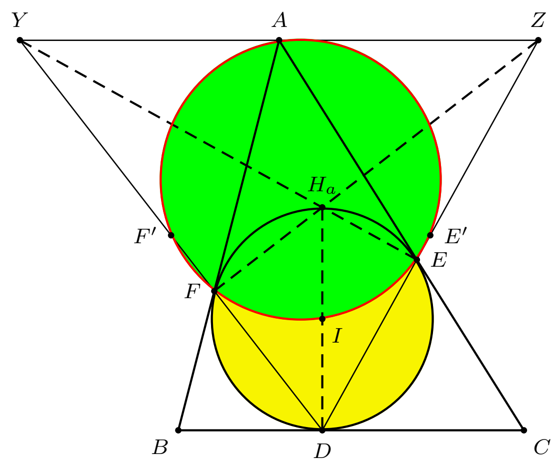

Figure 1. The triangle $\mathbf{T}_a$ and its orthocenter

Here is an alternative solution. The circle in question is indeed the nine-point circle of triangle DYZ. In Figure 1, $\angle AZE = \angle CDE = \angle CED = \angle AEZ$. Therefore $AZ = AE$. Similarly, $AY = AF$. It follows that $AY = AF = AE = AZ$, and A is the midpoint of YZ. The circle through A, E', F', the midpoints of the sides of triangle DYZ, is the nine-point circle of the triangle. Now, since $AY = AZ = AE$, the point E is the foot of the altitude on DZ. Similarly, F

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is the foot of the altitude on DY, and these two points are on the same nine-point circle. The intersection $H_a = EY \cap FZ$ is the orthocenter of triangle DYZ. Since $\angle H_aED = \angle H_aFD$ are right angles, H_a lies on the circle containing D, E, F, which is the incircle of triangle ABC, and has DH_a as a diameter. It follows that I, being the midpoint of the segment DH_a, is also on the nine-point circle. At the same time, note that H_a is the antipodal point of the D on the incircle of triangle ABC.

2. The Feuerbach point on an Euler line

The center of the nine-point circle of DYZ is the midpoint M of IA. The line MH_a is therefore the Euler line of triangle DYZ.

Theorem 1. *The Euler line of triangle DYZ contains the Feuerbach point of triangle ABC, the point of tangency of the incircle and the nine-point circle of the latter triangle.*

Proof. Let O, H, and N be respectively the circumcenter, orthocenter, and nine-point center of triangle ABC. It is well known that N is the midpoint of OH. Denote by ℓ the Euler line MH_a of triangle DYZ. We show that the parallel through N to the line IH_a intersects ℓ at a point N' such that $NN' = \frac{R}{2}$, where R is the circumradius of triangle ABC.

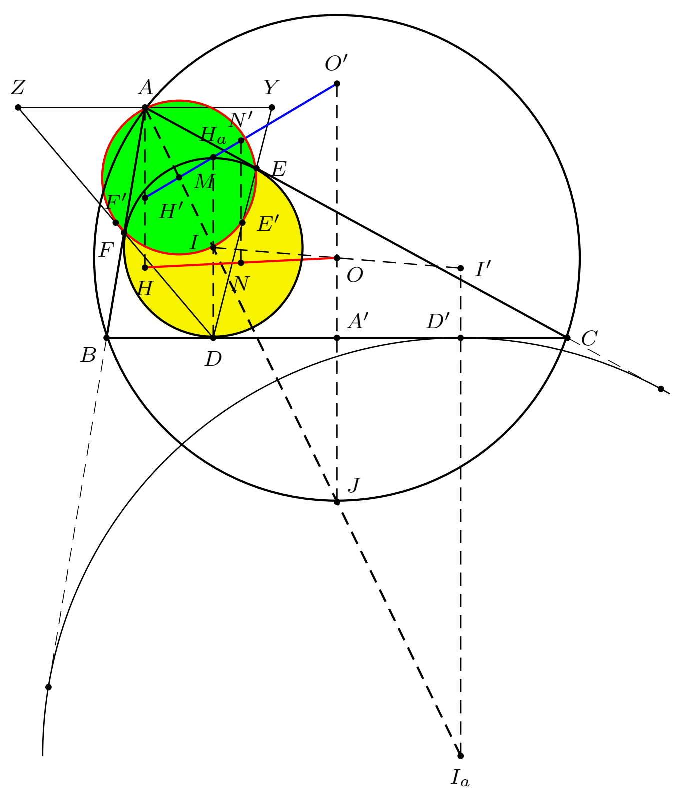

Figure 2. The Euler line of $\mathbf{T}_a$

Clearly, the line HA is parallel to IH_a. Since M is the midpoint of IA, AH intersects ℓ at a point H' such that $AH' = H_aI = r$, the inradius of triangle ABC. See Figure 2. Let the line through O parallel to IH_a intersect ℓ at O'.

If A' is the midpoint of BC, it is well known that $AH = 2 \cdot OA'$.

Consider the excircle (I_a) on the side BC, with radius r_a. The midpoint of II_a is also the midpoint J of the arc BC of the circumcircle (not containing the vertex A). Consider also the reflection I' of I in O, and the excircle (I_a). It is well known that $I'I_a$ passes through the point of tangency D' of (I_a) and BC. We first show that $JO' = r_a$:

$$JO' = \frac{JM}{IM} \cdot IH_a = \frac{I_aA}{IA} \cdot r = \frac{2r_a}{2r} \cdot r = r_a.$$

Since N is the midpoint of OH, and O that of II', we have

$$\begin{aligned}
2NN' =& HH' + OO' \\
=& (HA - H'A) + (JO' - R) \\
=& 2 \cdot A'O - r + r_a - R \\
=& DI + D'I' + r_a - (R + r) \\
=& r + (2R - r_a) + r_a - (R + r) \\
=& R.
\end{aligned}$$

This means that N' is a point on the nine-point circle of triangle ABC. Since NN' and IH_a are directly parallel, the lines $N'H_a$ and NI intersect at the external center of similitude of the nine-point circle and the incircle. It is well known that the two circles are tangent internally at the Feuerbach point F_e, which is their external center of similitude. See Figure 3. □

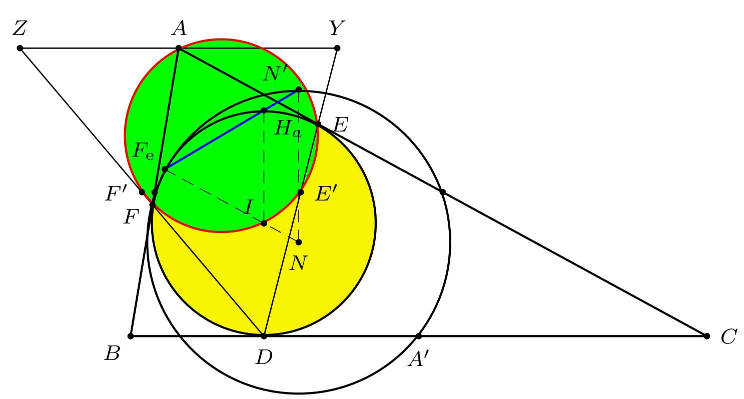

Figure 3. The Euler line of $\mathbf{T}_a$ passes through the Feuerbach point

Remark. Since DH_a is a diameter of the incircle, the Feuerbach point F_e is indeed the pedal of D on the Euler line of triangle DYZ.

Denote the triangle DYZ by $\mathbf{T}_a$. Analogous to $\mathbf{T}_a$, we can also construct the triangles $\mathbf{T}_b$ and $\mathbf{T}_c$ (containing respectively E with a side parallel to CA and F with a side parallel to AB). Theorem 1 also applies to these triangles.

Corollary 2. *The Feuerbach point is the common point of the Euler lines of the three triangles $\mathbf{T}_a$, $\mathbf{T}_b$, and $\mathbf{T}_c$.*

3. The excircle case

If, in the construction of $\mathbf{T}_a$, we replace the incircle by the A-excircle (I_a), we obtain another triangle $\mathbf{T}'_a$. More precisely, if the excircle (I_a) touches BC at D', and CA, AB at E', F' respectively, $\mathbf{T}'_a$ is the triangle DYZ bounded by the lines $D'E'$, $D'F'$, and the parallel through A to BC. The method in §2 leads to the following conclusions.

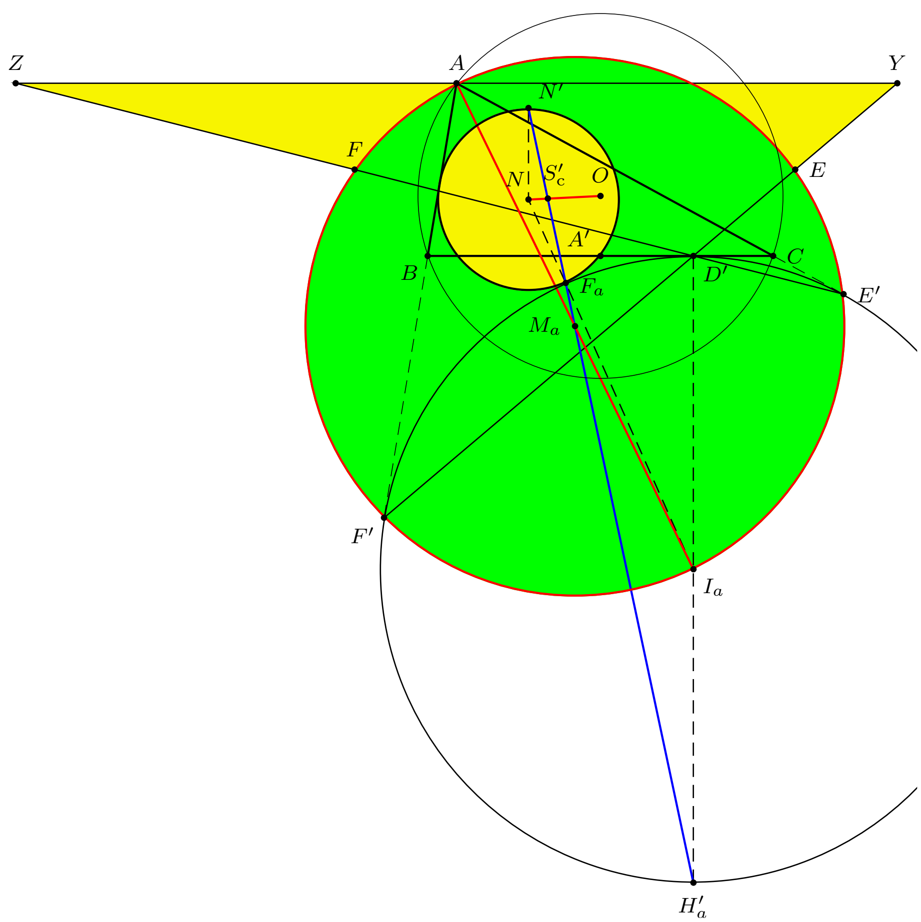

Figure 4. The Euler line of $\mathbf{T}'_a$ passes through $S'_c = X_{442}$

(1) The nine-point circle of $\mathbf{T}'_a$ contains the excenter I_a and the points E', F'; its center is the midpoint M_a of the segment AI_a.

(2) The orthocenter H'_a of $\mathbf{T}'_a$ is the antipode of D' on the excircle (I_a).

(3) The Euler line ℓ'_a of $\mathbf{T}'_a$ contains the point N'.

See Figure 4. Therefore, ℓ'_a also contains the internal center of similitude of the nine-point circle (N) and the excircle (I_a), which is the point of tangency F_a of these two circles. K. L. Nguyen [2] has recently studied the line containing F_a and M_a, and shown that it is the image of the Euler line of triangle IBC under the homothety $\mathsf{h} := \mathsf{h}\left(\mathsf{G}, -\frac{1}{2}\right)$. The same is true for the two analogous triangles $\mathbf{T}'_b$ and $\mathbf{T}'_c$. Their Euler lines are the images of the Euler lines of ICA and IAB under the same homothety. Recall that the Euler lines of triangles IBC, ICA, and IAB intersect at a point on the Euler line, the Schiffler point S_c, which is the triangle center X_{21} in [1]. From this we conclude that the Euler lines of $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$ concur at the image of S_c under the homothety h. This, again, is a point on the Euler line of triangle ABC. It appears in [1] as the triangle center X_{442}.

4. A generalization

The concurrency of the Euler lines of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$, can be paraphrased as the perspectivity of the "midway triangle" of I with the triangle $H_aH_bH_c$. Here, H_a, H_b, H_c are the orthocenters of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ respectively. They are the antipodes of D, E, F on the incircle. More generally, every homothetic image of ABC in I is perspective with $H_aH_bH_c$. This is clearly equivalent to the following theorem.

Theorem 3. *Every homothetic image of ABC in I is perspective with the intouch triangle DEF.*

Proof. We work with homogeneous barycentric coordinates.

The image of ABC under the homothety $\mathsf{h}(I,t)$ has vertices

$$\begin{aligned}A_t =&(a+t(b+c):(1-t)b:(1-t)c),\\ B_t =&((1-t)a:b+t(c+a):(1-t)c),\\ C_t =&((1-t)a:(1-t)b:c+t(a+b)).\end{aligned}$$

On the other hand, the vertices of the intouch triangle are

$$D=(0:s-c:s-b),\qquad E=(s-c:0:s-a),\qquad F=(s-b:s-a:0).$$

The lines A_tD, B_tE, and C_tF have equations

$$\begin{array}{rcrcrcl}(1-t)(b-c)(s-a)x &+& (s-b)(a+(b+c)t)y &-& (s-c)(a+(b+c)t)z &=& 0,\\ -(s-a)(b+(c+a)t)x &+& (1-t)(c-a)(s-b)y &+& (s-c)(b+(c+a)t)z &=& 0,\\ (s-a)(c+(a+b)t)x &-& (s-b)(c+(a+b)t)y &+& (1-t)(a-b)(s-c)z &=& 0.\end{array}$$

These three lines intersect at the point

$$P_t=\left(\frac{(a+t(b+c))(b+c-a+2at)}{b+c-a}:\cdots:\cdots\right).$$

□

Remark. More generally, for an arbitrary point P, every homothetic image of ABC in $P = (u : v : w)$ is perspective with the cevian triangle of the isotomic conjugate of the superior of P, namely, the point $\left(\frac{1}{v+w-u} : \frac{1}{w+u-v} : \frac{1}{u+v-w}\right)$. With $P = I$, we get the cevian triangle of the Gergonne point which is the intouch triangle.

Proposition 4. *The perspector of $A_tB_tC_t$ and $H_aH_bH_c$ is the reflection of P_{-t} in the incenter.*

It is clear that the perspector P_t traverses a conic Γ as t varies, since its coordinates are quadratic functions of t. The conic Γ clearly contains I and the Gergonne point, corresponding respectively to $t = 0$ and $t = 1$. Note also that $D = P_t$ for $t = -\frac{a}{b+c}$ or $-\frac{s-a}{a}$. Therefore, Γ contains D, and similarly, E and F. It is a cirumconic of the intouch triangle DEF. Now, as $t = \infty$, the line A_tD is parallel to the bisector of angle A, and is therefore perpendicular to EF. Simiarly, B_tE and C_tF are perpendicular to FD and DE respectively. The perspector P_∞ is therefore the orthocenter of triangle DEF, which is the triangle center X_{65} in [1]. It follows that Γ is a rectangular hyperbola. Since it contains also the circumcenter I of DEF, Γ is indeed the Jerabek hyperbola of the intouch triangle. Its center is the point

$$Q = \left(\frac{a(a^2(b+c) - 2a(b^2+c^2) + (b^3+c^3))}{b+c-a} : \cdots : \cdots\right).$$

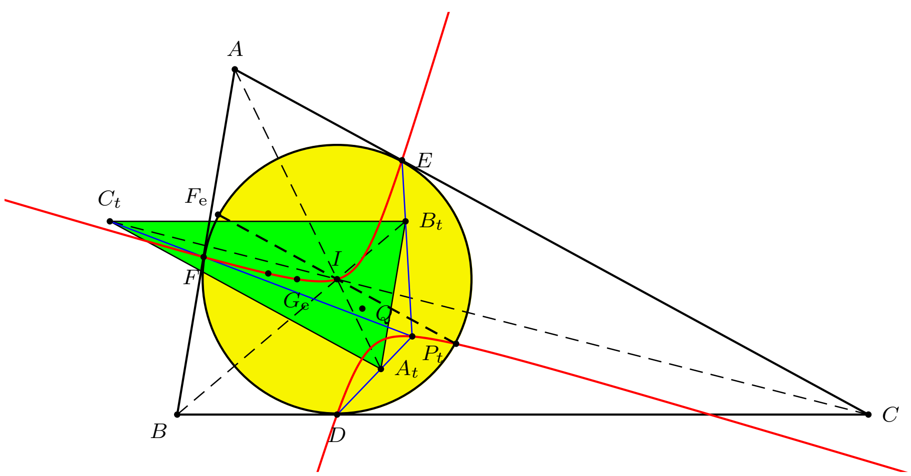

Figure 5. The Jerabak hyperbola of the intouch triangle

The reflection of Γ in the incenter is the conic Γ' which is the locus of the perspectors of $H_aH_bH_c$ and homothetic images of ABC in I.

Note that the fourth intersection of Γ with the incircle is the isogonal conjugate, with respect to the intouch triangle, of the infinite point of its Euler line. Its antipode on the incircle is therefore the Euler reflection point of the intouch triangle.

This must also be the perspector of $H_aH_bH_c$ (the antipode of DEF in the incircle) and a homothetic image of ABC. It must be the Feuerbach point on Γ'.

Theorem 5. *The Feuerbach point is the Euler reflection point of the intouch triangle. This means that the reflections of OI (the Euler line of the intouch triangle) concur at F.*

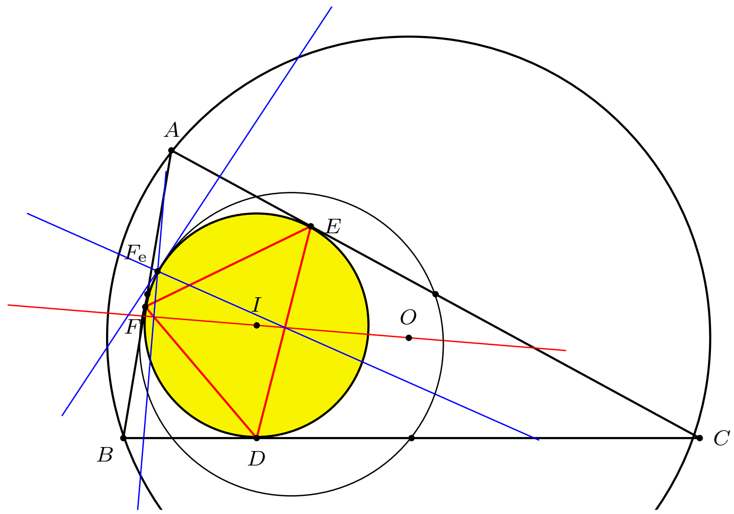

Figure 6. The Feuerbach point as the Euler reflection point of DEF

Remarks. (1) The fourth intersection of Γ with the incircle, being the antipode of the Feuerbach point, is the triangle cente X_{1317}. The conic Γ also contains X_n for the following values of n: 145, 224, and 1537. (Note: X_{145} is the reflection of the Nagel point in the incenter). These are the perspectors for the homothetic images of ABC with ratios $t = -1$, $-\frac{R}{R+r}$, and $-\frac{r}{2(R-r)}$ respectively.

(2) The hyperbola Γ' contains the following triangle centers apart from I and F_e: X_8 and X_{390} (which is the reflection of the Gergonne point in the incenter). These are the perspector for the homothetic images with ratio $+1$ and -1 respectively.

References

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Bogdan Suceavă: Department of Mathematics, California State University, Fullerton, CA 92834-6850, USA

E-mail address: bsuceava@fullerton.edu

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431-0991, USA

E-mail address: yiu@fau.edu