The Feuerbach Point and Euler lines

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Abstract. Given a triangle, we construct three triangles associated its incircle whose Euler lines intersect on the Feuerbach point, the point of tangency of the incircle and the nine-point circle. By studying a generalization, we show that the Feuerbach point in the Euler reflection point of the intouch triangle, namely, the intersection of the reflections of the line joining the circumcenter and incenter in the sidelines of the intouch triangle.

1. A MONTHLY problem

Consider a triangle $ABC$ with incenter $I$, the incircle touching the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. Let $Y$ (respectively $Z$) be the intersection of $DF$ (respectively $DE$) and the line through $A$ parallel to $BC$. If $E'$ and $F'$ are the midpoints of $DZ$ and $DY$, then the six points $A$, $E$, $F$, $I$, $E'$, $F'$ are on the same circle. This is Problem 10710 of the American Mathematical Monthly with slightly different notations. See [3].

Here is an alternative solution. The circle in question is indeed the nine-point circle of triangle $DYZ$. In Figure 1, $\angle AZE = \angle CDE = \angle CED = \angle AEZ$. Therefore $AZ = AE$. Similarly, $AY = AF$. It follows that $AY = AF = AE = AZ$, and $A$ is the midpoint of $YZ$. The circle through $A$, $E'$, $F'$, the midpoints of the sides of triangle $DYZ$, is the nine-point circle of the triangle. Now, since $AY = AZ = AE$, the point $E$ is the foot of the altitude on $DZ$. Similarly, $F$
is the foot of the altitude on $DY$, and these two points are on the same nine-point circle. The intersection $H_a = EY \cap FZ$ is the orthocenter of triangle $DYZ$. Since $\angle H_a E D = \angle H_a F D$ are right angles, $H_a$ lies on the circle containing $D, E, F$, which is the incircle of triangle $ABC$, and has $DH_a$ as a diameter. It follows that $I$, being the midpoint of the segment $DH_a$, is also on the nine-point circle. At the same time, note that $H_a$ is the antipodal point of the $D$ on the incircle of triangle $ABC$.

2. The Feuerbach point on an Euler line

The center of the nine-point circle of $DYZ$ is the midpoint $M$ of $IA$. The line $MH_a$ is therefore the Euler line of triangle $DYZ$.

**Theorem 1.** The Euler line of triangle $DYZ$ contains the Feuerbach point of triangle $ABC$, the point of tangency of the incircle and the nine-point circle of the latter triangle.

**Proof.** Let $O$, $H$, and $N$ be respectively the circumcenter, orthocenter, and nine-point center of triangle $ABC$. It is well known that $N$ is the midpoint of $OH$. Denote by $\ell$ the Euler line $MH_a$ of triangle $DYZ$. We show that the parallel through $N$ to the line $IH_a$ intersects $\ell$ at a point $N'$ such that $NN' = \frac{R}{2}$, where $R$ is the circumradius of triangle $ABC$.

![Figure 2. The Euler line of $T_a$](image)
Clearly, the line $HA$ is parallel to $IH_a$. Since $M$ is the midpoint of $IA$, $AH$ intersects $\ell$ at a point $H'$ such that $AH' = H_aI = r$, the inradius of triangle $ABC$. See Figure 2. Let the line through $O$ parallel to $IH_a$ intersect $\ell$ at $O'$.

If $A'$ is the midpoint of $BC$, it is well known that $AH = 2 \cdot OA'$.

Consider the excircle $(I_a)$ on the side $BC$, with radius $r_a$. The midpoint of $I_a$ is also the midpoint $J$ of the arc $BC$ of the circumcircle (not containing the vertex $A$). Consider also the reflection $I'$ of $I$ in $O$, and the excircle $(I_a)$. It is well known that $I'I_a$ passes through the point of tangency $D'$ of $(I_a)$ and $BC$. We first show that $JO' = r_a$:

$$JO' = \frac{JM}{IM} \cdot IH_a = \frac{I_aA}{IA} \cdot r = \frac{2r_a}{2r} \cdot r = r_a.$$

Since $N$ is the midpoint of $OH$, and $O$ that of $I'I$, we have

$$2NN' = HH' + OO' = (HA - H'A) + (JO' - R) = 2 \cdot A'O - r + r_a - R = DI + D'I' + r_a - (R + r) = r + (2R - r_a) + r_a - (R + r) = R.$$

This means that $N'$ is a point on the nine-point circle of triangle $ABC$. Since $NN'$ and $IH_a$ are directly parallel, the lines $N'H_a$ and $NI$ intersect at the external center of similitude of the nine-point circle and the incircle. It is well known that the two circles are tangent internally at the Feuerbach point $F_e$, which is their external center of similitude. See Figure 3.

Remark. Since $DH_a$ is a diameter of the incircle, the Feuerbach point $F_e$ is indeed the pedal of $D$ on the Euler line of triangle $DYZ$.

Figure 3. The Euler line of $T_a$ passes through the Feuerbach point
Denote the triangle $DYZ$ by $T_a$. Analogous to $T_a$, we can also construct the triangles $T_b$ and $T_c$ (containing respectively $E$ with a side parallel to $CA$ and $F$ with a side parallel to $AB$). Theorem 1 also applies to these triangles.

**Corollary 2.** The Feuerbach point is the common point of the Euler lines of the three triangles $T_a$, $T_b$, and $T_c$.

3. The excircle case

If, in the construction of $T_a$, we replace the incircle by the $A$-excircle ($I_a$), we obtain another triangle $T'_a$. More precisely, if the excircle ($I_a$) touches $BC$ at $D'$, and $CA$, $AB$ at $E'$, $F'$ respectively, $T'_a$ is the triangle $DYZ$ bounded by the lines $D'E'$, $D'F'$, and the parallel through $A$ to $BC$. The method in §2 leads to the following conclusions.

![Figure 4. The Euler line of $T'_a$ passes through $S'_c = X_{442}$](image-url)
The lines $A = (0)$. On the other hand, the vertices of the intouch triangle are perspective with $H$, and $M$ are the centers of the two circles. K. L. Nguyen [2] has recently studied the line containing $M$ at the image of $S$. Recall that the Euler lines of triangles $ABC$ intersect at a point on the Euler line, the Schiffler point $S$, and the Euler lines of $CA$ and $AB$ under the same homothety. Recall that the Euler lines of triangles $IBC$, $ICA$, and $IAB$ intersect at a point on the Euler line, the Schiffler point $S$, which is the triangle center $X_{21}$ in [1]. From this we conclude that the Euler lines of $T_a$, $T_b$, $T_c$ concur at the image of $S$, under the homothety $h$. This, again, is a point on the Euler line of triangle $ABC$. It appears in [1] as the triangle center $X_{142}$.

4. A generalization

The concurrency of the Euler lines of $T_a$, $T_b$, $T_c$, can be paraphrased as the perspectiveivity of the “midway triangle” of $I$ with the triangle $H_bH_c$. Here, $H_a$, $H_b$, $H_c$ are the orthocenters of $T_a$, $T_b$, $T_c$ respectively. They are the antipodes of $D, E, F$ on the incircle. More generally, every homothetic image of $ABC$ in $I$ is perspective with $H_bH_c$. This is clearly equivalent to the following theorem.

**Theorem 3.** Every homothetic image of $ABC$ in $I$ is perspective with the intouch triangle $DEF$.

**Proof.** We work with homogeneous barycentric coordinates.

The image of $ABC$ under the homothety $h(I, t)$ has vertices

- $A_t = (a + t(b + c) : (1 - t)b : (1 - t)c),$
- $B_t = ((1 - t)a : b + t(c + a) : (1 - t)c),$
- $C_t = ((1 - t)a : (1 - t)b : c + t(a + b)).$

On the other hand, the vertices of the intouch triangle are

- $D = (0 : s - c : s - b),$
- $E = (s - c : 0 : s - a),$
- $F = (s - b : s - a : 0).$

The lines $A_tD, B_tE,$ and $C_tF$ have equations

- $(1 - t)(b - c)(s - a)x + (s - b)(a + b + c)y - (s - c)(a + b + c)z = 0,$
- $-(s - a)(b + c)x + (1 - t)(c - a)(s - b)y + (s - c)(b + c)z = 0,$
- $(s - a)(c + a)x - (s - b)(c + a)y + (1 - t)(a - b)(s - c)z = 0.$

These three lines intersect at the point

$$P_t = \left(\frac{(a + t(b + c))(b + c - a + 2at)}{b + c - a} : \cdots : \cdots\right).$$
Remark. More generally, for an arbitrary point $P$, every homothetic image of $ABC$ in $P = (u : v : w)$ is perspective with the cevian triangle of the isotomic conjugate of the superior of $P$, namely, the point $\left(\frac{1}{u+w-v} : \frac{1}{u+w-v} : \frac{1}{u+w-v}\right)$. With $P = I$, we get the cevian triangle of the Gergonne point which is the intouch triangle.

**Proposition 4.** The perspector of $A_t B_t C_t$ and $H_a H_b H_c$ is the reflection of $P_t$ in the incenter.

It is clear that the perspector $P_t$ traverses a conic $\Gamma$ as $t$ varies, since its coordinates are quadratic functions of $t$. The conic $\Gamma$ clearly contains $I$ and the Gergonne point, corresponding respectively to $t = 0$ and $t = 1$. Note also that $D = P_t$ for $t = -\frac{a}{b+c}$ or $-\frac{a+b}{b}$. Therefore, $\Gamma$ contains $D$, and similarly, $E$ and $F$. It is a circumconic of the intouch triangle $DEF$. Now, as $t = \infty$, the line $A_t D$ is parallel to the bisector of angle $A$, and is therefore perpendicular to $EF$. Similarly, $B_t E$ and $C_t F$ are perpendicular to $FD$ and $DE$ respectively. The perspector $P_{\infty}$ is therefore the orthocenter of triangle $DEF$, which is the triangle center $X_{65}$ in [1]. It follows that $\Gamma$ is a rectangular hyperbola. Since it contains also the circumcenter $I$ of $DEF$, $\Gamma$ is indeed the Jerabek hyperbola of the intouch triangle. Its center is the point

$$Q = \left(\frac{a(a^2(b+c) - 2a(b^2 + c^2) + (b^3 + c^3))}{b + c - a} : \cdots : \cdots\right).$$

![Figure 5. The Jerabak hyperbola of the intouch triangle](image)

The reflection of $\Gamma$ in the incenter is the conic $\Gamma'$ which is the locus of the perspectors of $H_a H_b H_c$ and homothetic images of $ABC$ in $I$.

Note that the fourth intersection of $\Gamma$ with the incircle is the isogonal conjugate, with respect to the intouch triangle, of the infinite point of its Euler line. Its antipode on the incircle is therefore the Euler reflection point of the intouch triangle.
This must also be the perspector of $H_aH_bH_c$ (the antipode of $DEF$ in the incircle) and a homothetic image of $ABC$. It must be the Feuerbach point on $\Gamma'$.

**Theorem 5.** The Feuerbach point is the Euler reflection point of the intouch triangle. This means that the reflections of $OI$ (the Euler line of the intouch triangle) concur at $F$.

![Figure 6. The Feuerbach point as the Euler reflection point of DEF](image)

**Remarks.** (1) The fourth intersection of $\Gamma$ with the incircle, being the antipode of the Feuerbach point, is the triangle center $X_{1317}$. The conic $\Gamma$ also contains $X_n$ for the following values of $n$: 145, 224, and 1537. (Note: $X_{145}$ is the reflection of the Nagel point in the incenter). These are the perspectors for the homothetic images of $ABC$ with ratios $t = -1$, $-\frac{R}{R+r}$, and $-\frac{r}{2(R-r)}$ respectively.

(2) The hyperbola $\Gamma'$ contains the following triangle centers apart from $I$ and $F_e$: $X_8$ and $X_{390}$ (which is the reflection of the Gergonne point in the incenter). These are the perspector for the homothetic images with ratio $+1$ and $-1$ respectively.

**References**


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