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Pedals on Circumradii and the Jerabek Center

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Abstract. Given a triangle ABC, beginning with the orthogonal projections of the vertices on the circumradii OA, OB, OC, we construct two triangles each with circumcircle tangent to the nine-point circle at the center of the Jerabek hyperbola.

1. Introduction

Given a triangle ABC, with circumcenter O, let A_b and A_c be the pedals (orthogonal projections) of the vertex A on the lines OB and OC respectively. Similarly, define B_c, B_a, C_a and C_b. In this paper we prove some interesting results on triangles associated with these pedals.

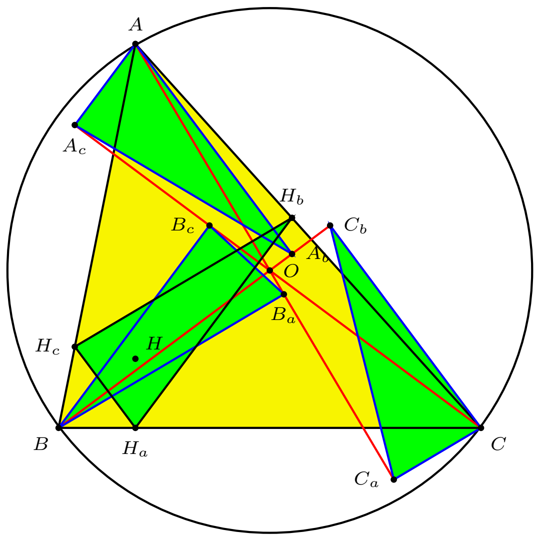

Figure 1

Theorem 1. *The triangles AA_bA_c, B_aBB_c and C_aC_bC are congruent to the orthic triangle $H_aH_bH_c$.* See Figure 1.

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Theorem 2. *The lines B_cC_b, C_aA_c and A_bB_a bound a triangle $\mathbf{T}_1$ homothetic to ABC. The circumcircle of $\mathbf{T}_1$ is tangent to the nine-point circle of ABC at the Jerabek center.*

Recall that the Jerabek center J is the center of the circum-hyperbola through the circumcenter O. This hyperbola is the isogonal conjugate of the Euler line. The Jerabek center J is the triangle center X_{125} in Kimberling's *Encyclopedia of Triangle Centers* [1].

Theorem 3. *The lines A_bA_c, B_cB_a and C_aC_b bound a triangle $\mathbf{T}_2$ whose circumcircle is tangent to the nine-point circle at the Jerabek center.*

Hence, the circumcircles of $\mathbf{T}_1$ and $\mathbf{T}_2$ are also tangent to each other at J. In this paper we work with homogeneous barycentric coordinates and adopt standard notations of triangle geometry. Basic results can be found in [2]. The Jerabek center J, for example, has coordinates

$$(S_A(S_B - S_C)^2 : \ S_B(S_C - S_A)^2 : \ S_C(S_A - S_B)^2). \tag{1}$$

The labeling of triangle centers, except for the common ones, follows [1].

Proposition 4. *The homogeneous barycentric coordinates of the pedals of the vertices of triangle ABC on the circumradii are as follows.*

$$\begin{aligned}
A_b =&(S_A(S_B + S_C) : \ S_C(S_C - S_A) : \ S_C(S_A + S_B)),\\
A_c =&(S_A(S_B + S_C) : \ S_B(S_C + S_A) : \ S_C(S_B - S_A));\\
B_c =&(S_A(S_B + S_C) : \ S_B(S_C + S_A) : \ S_A(S_A - S_B)),\\
B_a =&(S_A(S_C - S_B) : \ S_B(S_C + S_A) : \ S_C(S_A + S_B));\\
C_a =&(S_A(S_B - S_C) : \ S_B(S_C + S_A) : \ S_C(S_A + S_B)),\\
C_b =&S_A((S_B + S_C) : \ S_B(S_A - S_C) : \ S_C(S_A + S_B)).
\end{aligned}$$

Proof. We verify that the point

$$P = (S_A(S_B + S_C) : \ S_C(S_C - S_A) : \ S_C(S_A + S_B))$$

is the pedal A_b of A on the line OB. Since

$$\begin{aligned}
&(S_A(S_B + S_C), \ S_C(S_C - S_A), \ S_C(S_A + S_B))\\
=&(S_A(S_B + S_C), \ S_B(S_C + S_A), \ S_C(S_A + S_B))\\
&+ (0, S_C(S_C - S_A) - S_B(S_C + S_A), \ 0),
\end{aligned}$$

this is a point on the line OB. The coordinate sum of P being $(S_B+S_C)(S_C+S_A)$, the infinite point of the line AP is

$$\begin{aligned}
&(S_A(S_B + S_C), \ S_C(S_C - S_A), \ S_C(S_A + S_B)) - ((S_C + S_A)(S_A + S_B), \ 0, \ 0)\\
=&S_C(-(S_B + S_C), \ (S_C - S_A), \ (S_A + S_B)).
\end{aligned}$$

The infinite point of OB is

$$\begin{aligned}
&(S_A(S_B + S_C), \ S_B(S_C + S_A), \ S_C(S_A + S_B)) - (0, \ 2(S_{BC} + S_{CA} + S_{AB}), \ 0)\\
=&(S_A(S_B + S_C), \ -(S_{BC} + 2S_{CA} + S_{AB}), \ S_C(S_A + S_B)).
\end{aligned}$$

By the theorem in [2, §4.5], the two lines AP and OB are perpendicular since

$$\begin{aligned}
&-S_A \cdot (S_B + S_C) \cdot S_A(S_B + S_C) \\
&-S_B \cdot (S_C - S_A) \cdot (S_{BC} + 2S_{CA} + S_{AB}) \\
&+S_C \cdot (S_A + S_B) \cdot S_C(S_A + S_B) \\
=&0.
\end{aligned}$$

□

2. Proof of Theorem 1

Note that the points A_b and A_c lie on the circle with diameter OA, so do the midpoints of AC and AB. Therefore,

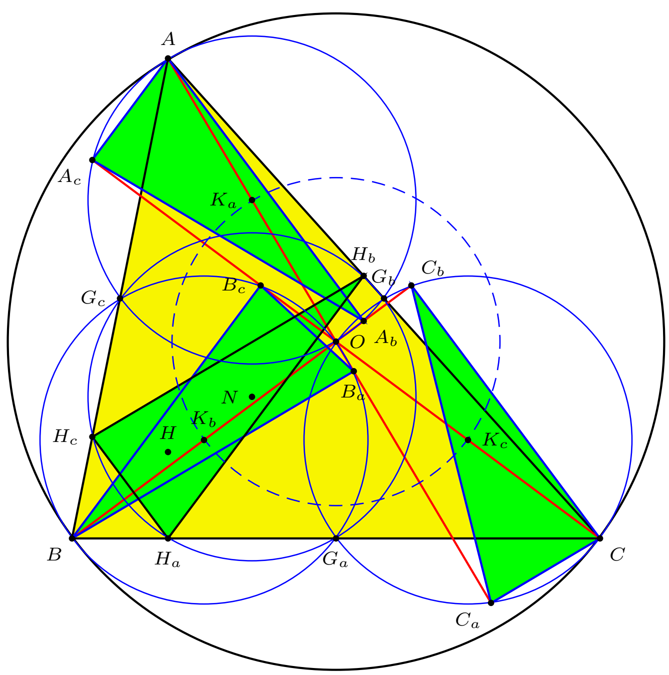

Figure 2

$$\begin{aligned}
\angle A_cAA_b =&\pi - \angle A_bOA_c = \pi - \angle BOC = \pi - 2A = \angle H_cH_aH_b, \\
\angle AA_bA_c =&\angle AOA_c = \pi - \angle COA = \pi - 2B = \angle H_aH_bH_c, \\
\angle A_bA_cA =&\angle A_bOA = \pi - \angle AOB = \pi - 2C = \angle H_bH_cH_a.
\end{aligned}$$

Therefore the angles in triangles AA_bA_c and $H_aH_bH_c$ are the same; similarly for triangles B_aBB_c and C_aC_bC. Since these four triangles have equal circumradii $\frac{R}{2}$, they are congruent. This completes the proof of Theorem 1.

Remarks. (1) The side lengths of these triangles are $a\cos A$, $b\cos B$, and $c\cos C$ respectively.

(2) If K_a, K_b, K_c are the midpoints of the circumradii OA, OB, OC, triangle $K_aK_bK_c$ is homothetic to
(i) ABC at O, with ratio of homothety $\frac{1}{2}$, and
(ii) the medial triangle $G_aG_bG_c$ at X_{140}, the nine-point center of the medial triangle, with ratio of homothety -1.

(3) The circles (K_b) and (K_c) intersect at the circumcenter O and the midpoint G_a of BC; similarly for the other two pairs (K_c), (K_a) and (K_a), (K_b). The midpoints G_a, G_b, G_c lie on the nine-point circle (N) of triangle ABC. See Figure 2.

3. The triangle $\mathbf{T}_1$

We now consider the triangle $\mathbf{T}_1$ bounded by the lines B_cC_b, C_aA_c, and A_bB_a.

Lemma 5. *The quadrilateral B_cC_bCB is an isosceles trapezoid.*

Proof. With reference to Figure 1, we have
(i) $\angle B_cBC = \frac{\pi}{2} - \angle OCB = \frac{\pi}{2} - \angle OBC = \angle C_bCB$,
(ii) $B_cB = C_bC$.
It follows that the quadrilateral B_cC_bCB is an isosceles trapezoid. □

Therefore, the lines B_cC_b and BC are parallel. Similarly, the lines C_aA_c and CA are parallel, as are A_bB_a and AB. The triangle $\mathbf{T}_1$ bounded by the lines B_cC_b, C_aA_c, A_bB_a is homothetic to triangle ABC, and also to the medial triangle $G_aG_bG_c$.

Proposition 6. *Triangle $\mathbf{T}_1$ is homothetic to*
(i) *ABC at the procircumcenter $(a^4S_A : b^4S_B : c^4S_C)$,* [1]
(ii) *the medial triangle $G_aG_bG_c$ at the Jerabek center J.*

Proof. The lines B_cC_b, C_aA_c, and A_bB_a have equations

$$\begin{aligned}
-(S_{AA}+S_{BC})x + S_A(S_B+S_C)y + S_A(S_B+S_C)z =& 0,\\
S_B(S_C+S_A)x - (S_{BB}+S_{CA})y + S_B(S_C+S_A)z =& 0,\\
S_C(S_A+S_B)x + S_C(S_A+S_B)y - (S_{CC}+S_{AB})z =& 0.
\end{aligned}$$

From these, we obtain the coordinates of the vertices of $\mathbf{T}_1$:

$$\begin{aligned}
A_1 =& (S_A(S_B-S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2),\\
B_1 =& (S_A(S_B+S_C)^2 : S_B(S_C-S_A)^2 : S_C(S_A+S_B)^2),\\
C_1 =& (S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A-S_B)^2).
\end{aligned}$$

From the coordinates of A_1, B_1, C_1, it is clear that the homothetic center of triangles $A_1B_1C_1$ and ABC is the point

$$(S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2) = (a^4S_A : b^4S_B : c^4S_C).$$

[1]This is the triangle center X_{184} in [1].

For (ii), the equations of the lines G_aA_1, G_bB_1, G_cC_1 are respectively

$$\begin{aligned}(S_{AA}-S_{BC})x-S_A(S_B-S_C)y+S_A(S_B-S_C)z=&0,\\ S_B(S_C-S_A)x+(S_{BB}-S_{CA})y-S_B(S_C-S_A)z=&0,\\ -S_C(S_A-S_B)x+S_C(S_A-S_B)y+(S_{CC}-S_{AB})z=&0.\end{aligned}$$

It is routine to check that this contains the Jerabek center J whose coordinates are given in (1). □

4. Proof of Theorem 2

Theorem 2 is now an immediate consequence of Proposition 6(ii). Since the homothetic center J lies on the circumcircle of the medial triangle, it must also lie on the circumcircle of the other, and the two circumcircles are tangent at J.

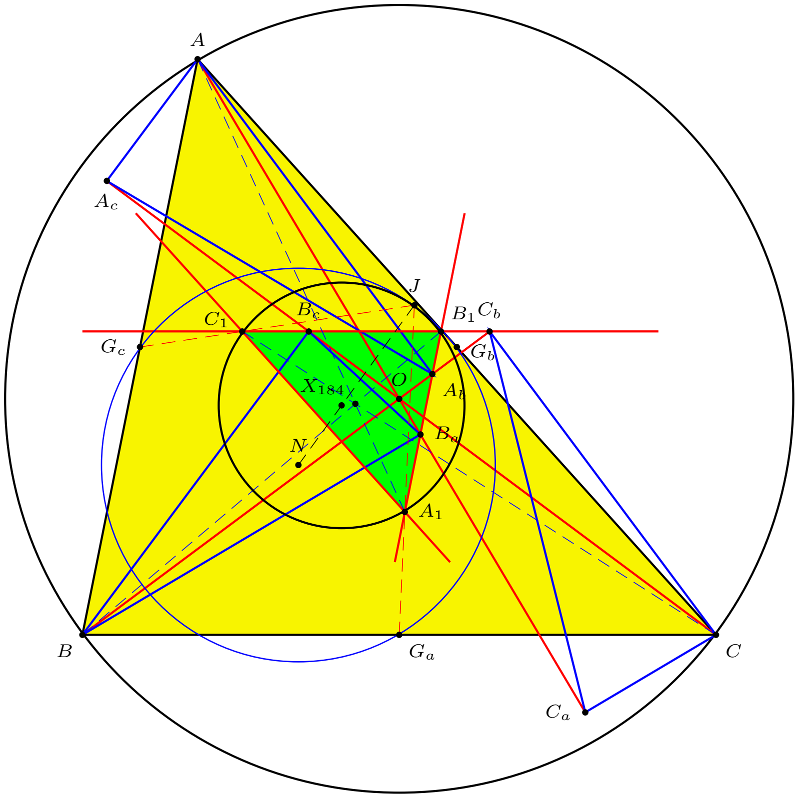

Figure 3

5. The triangle $\mathbf{T}_2$

From the coordinates of the pedals, we obtain the equations of the lines A_bA_c, B_cB_a, and C_aC_b:

$$\begin{aligned} -2S_{BC}x + (S^2 - S_{BB})y + (S^2 - S_{CC})z =& 0, \\ (S^2 - S_{AA})x - 2S_{CA}y + (S^2 - S_{CC})z =& 0, \\ (S^2 - S_{AA})x + (S^2 - S_{BB})y - 2S_{AB}z =& 0. \end{aligned}$$

From these, the vertices of triangle $\mathbf{T}_2$ are the points

$$\begin{aligned} A_2 =& ((S_B - S_C)^2 : \ 3S_{AB} + S_{BC} + S_{CA} - S_{CC} : \ 3S_{CA} + S_{AB} + S_{BC} - S_{BB}), \\ B_2 =& (3S_{AB} + S_{BC} + S_{CA} - S_{CC} : \ (S_C - S_A)^2 : \ 3S_{BC} + S_{CA} + S_{AB} - S_{AA}), \\ C_2 =& (3S_{CA} + S_{AB} + SBC - S_{CC} : \ 3S_{BC} + S_{CA} + S_{AB} - S_{AA} : \ (S_A - S_B)^2). \end{aligned}$$

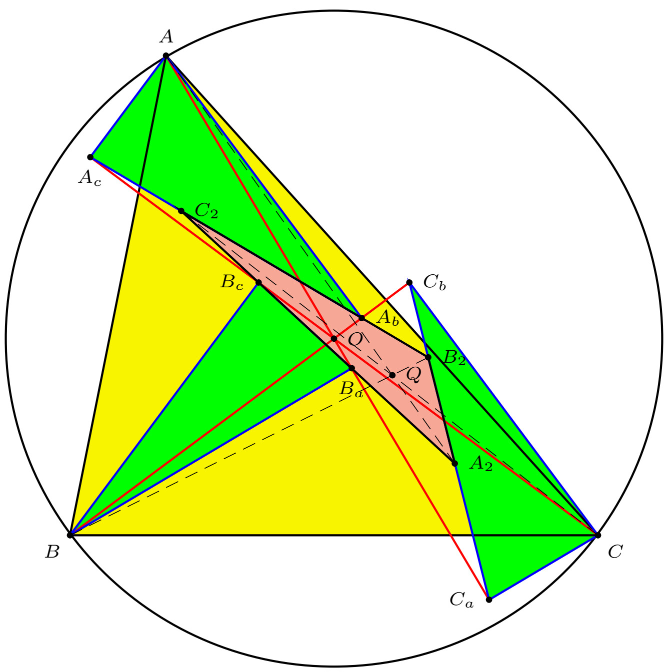

Figure 4

Proposition 7. *Triangles ABC and $A_2B_2C_2$ are perspective at*

$$Q = \left(\frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - b^4 - c^4} : \frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - c^4 - a^4} : \frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - a^4 - b^4} \right).$$

Proof. From the coordinates of A_2, B_2, C_2 given above, $\mathbf{T_2}$ is perspective with ABC at

$$Q = \left(\frac{1}{3S_{BC} + S_{CA} + S_{AB} - S_{AA}} : \cdots : \cdots\right).$$

These are equivalent to those given above in terms of a, b, c. □

Remark. The triangle center Q does not appear in [1].

6. Proof of Theorem 3

It is easier to work with the image of triangle $\mathbf{T}_2$ under the homothety $\mathsf{h}(H, 2)$. The images of the vertices are

$$\begin{aligned} A_2' =&(S_A(S_B + S_C)(S_{BB} - 4S_{BC} + S_{CC}) + S_{BC}(S_B - S_C)^2 \\ &: (S_C + S_A)(S_A(S_B + S_C)(3S_B - S_C) + S_{BC}(S_B - S_C)) \\ &: (S_A + S_B)(S_A(S_B + S_C)(S_B - 3S_C) + S_{BC}(S_B - S_C))), \end{aligned}$$

and B_2', C_2' whose coordinates are obtained by cyclic permutations of S_A, S_B, S_C. The circumcircle of $A_2'B_2'C_2'$ has equation

$$\begin{aligned} &8S^2 \cdot S_{ABC}((S_B + S_C)yz + (S_C + S_A)zx + (S_A + S_B)xy) \\ +&(x + y + z)\left(\sum_{\text{cyclic}} (S_A + S_B)(S_A + S_C)(S_{AB} + S_{CA} - 2S_{BC})^2 x\right) = 0. \end{aligned}$$

To verify that this circle is tangent to the circumcircle

$$(S_B + S_C)yz + (S_C + S_A)zx + (S_A + S_B)xy = 0,$$

it is enough to consider the pedal of the circumcenter O on the radical axis

$$\sum_{\text{cyclic}} (S_A + S_B)(S_A + S_C)(S_{AB} + S_{CA} - 2S_{BC})^2 x = 0.$$

This is the point

$$Q' = \left(\frac{S_B + S_C}{S_{CA} + S_{AB} - 2S_{BC}} : \frac{S_C + S_A}{S_{AB} + S_{BC} - 2S_{CA}} : \frac{S_A + S_B}{S_{BC} + S_{CA} - 2S_{AB}}\right),$$

which is clearly on the circumcircle, and also on the Jerabek hyperbola

$$\frac{S_A(S_{BB} - S_{CC})}{x} + \frac{S_B(S_{CC} - S_{AA})}{y} + \frac{S_C(S_{AA} - S_{BB})}{z} = 0.$$

This shows that the circle $A_2'B_2'C_2'$ is tangent to the circumcircle at Q'. [2] Under the homothety $\mathsf{h}(H, 2)$, Q' is the image of the midpoint of HQ, which is the center of the Jerabek hyperbola. Under the inverse homothety, the circumcircle of $\mathbf{T}_2$ is tangent to the nine-point circle at J. This completes the proof of Theorem 3.

[2] Q' is the triangle center X_{74}

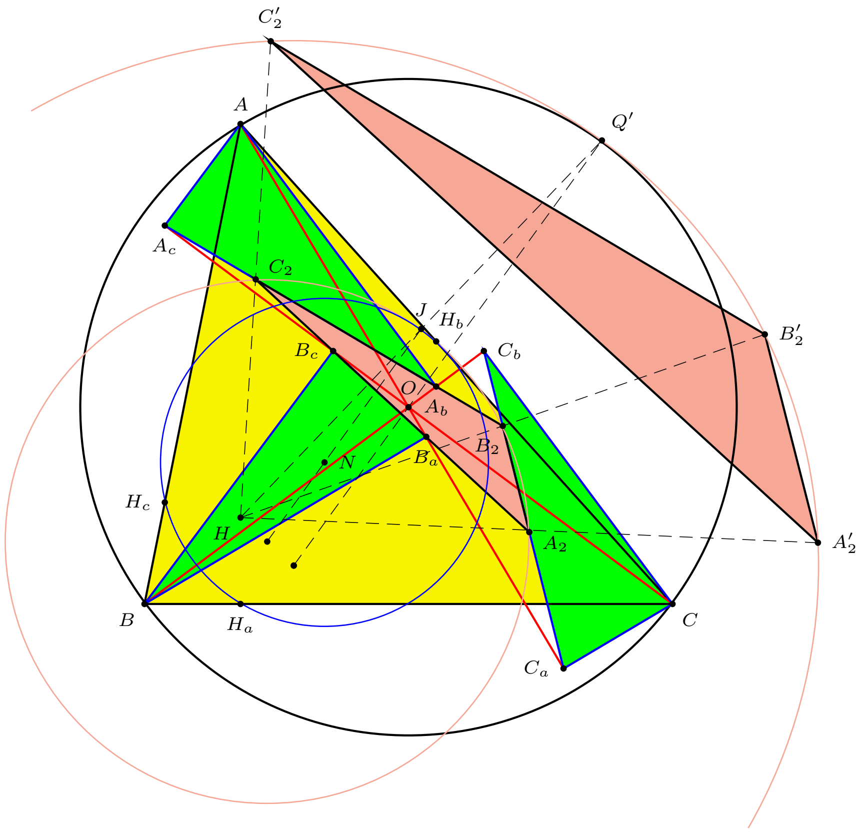

Figure 5

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

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