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# Pedals on Circumradii and the Jerabek Center

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**Abstract**. Given a triangle $ABC$, beginning with the orthogonal projections of the vertices on the circumradii $OA$, $OB$, $OC$, we construct two triangles each with circumcircle tangent to the nine-point circle at the center of the Jerabek hyperbola.

## 1. Introduction

Given a triangle $ABC$, with circumcenter $O$, let $A_b$ and $A_c$ be the pedals (orthogonal projections) of the vertex $A$ on the lines $OB$ and $OC$ respectively. Similarly, define $B_c$, $B_a$, $C_a$ and $C_b$. In this paper we prove some interesting results on triangles associated with these pedals.

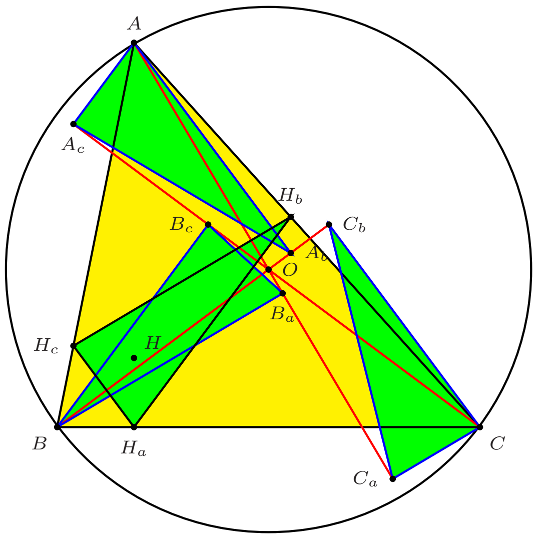


Figure 1

**Theorem 1.** *The triangles $AA_bA_c$, $B_aBB_c$ and $C_aC_bC$ are congruent to the orthic triangle $H_aH_bH_c$.* See Figure 1.

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**Theorem 2.** *The lines $B_cC_b$, $C_aA_c$ and $A_bB_a$ bound a triangle $\mathbf{T}_1$ homothetic to $ABC$. The circumcircle of $\mathbf{T}_1$ is tangent to the nine-point circle of $ABC$ at the Jerabek center.*

Recall that the Jerabek center $J$ is the center of the circum-hyperbola through the circumcenter $O$. This hyperbola is the isogonal conjugate of the Euler line. The Jerabek center $J$ is the triangle center $X_{125}$ in Kimberling's *Encyclopedia of Triangle Centers* [1].

**Theorem 3.** *The lines $A_bA_c$, $B_cB_a$ and $C_aC_b$ bound a triangle $\mathbf{T}_2$ whose circumcircle is tangent to the nine-point circle at the Jerabek center.*

Hence, the circumcircles of $\mathbf{T}_1$ and $\mathbf{T}_2$ are also tangent to each other at $J$. In this paper we work with homogeneous barycentric coordinates and adopt standard notations of triangle geometry. Basic results can be found in [2]. The Jerabek center $J$, for example, has coordinates

$$(S_A(S_B - S_C)^2 : \; S_B(S_C - S_A)^2 : \; S_C(S_A - S_B)^2). \tag{1}$$

The labeling of triangle centers, except for the common ones, follows [1].

**Proposition 4.** *The homogeneous barycentric coordinates of the pedals of the vertices of triangle $ABC$ on the circumradii are as follows.*

$$\begin{aligned}
A_b =&(S_A(S_B + S_C) : \; S_C(S_C - S_A) : \; S_C(S_A + S_B)),\\
A_c =&(S_A(S_B + S_C) : \; S_B(S_C + S_A) : \; S_C(S_B - S_A));\\
B_c =&(S_A(S_B + S_C) : \; S_B(S_C + S_A) : \; S_A(S_A - S_B)),\\
B_a =&(S_A(S_C - S_B) : \; S_B(S_C + S_A) : \; S_C(S_A + S_B));\\
C_a =&(S_A(S_B - S_C) : \; S_B(S_C + S_A) : \; S_C(S_A + S_B)),\\
C_b =&S_A((S_B + S_C) : \; S_B(S_A - S_C) : \; S_C(S_A + S_B)).
\end{aligned}$$

*Proof.* We verify that the point

$$P = (S_A(S_B + S_C) : \; S_C(S_C - S_A) : \; S_C(S_A + S_B))$$

is the pedal $A_b$ of $A$ on the line $OB$. Since

$$\begin{aligned}
&(S_A(S_B + S_C), \; S_C(S_C - S_A), \; S_C(S_A + S_B))\\
=&(S_A(S_B + S_C), \; S_B(S_C + S_A), \; S_C(S_A + S_B))\\
&+ (0, S_C(S_C - S_A) - S_B(S_C + S_A), \; 0),
\end{aligned}$$

this is a point on the line $OB$. The coordinate sum of $P$ being $(S_B+S_C)(S_C+S_A)$, the infinite point of the line $AP$ is

$$\begin{aligned}
&(S_A(S_B + S_C), \; S_C(S_C - S_A), \; S_C(S_A + S_B)) - ((S_C + S_A)(S_A + S_B), \; 0, \; 0)\\
=&S_C(-(S_B + S_C), \; (S_C - S_A), \; (S_A + S_B)).
\end{aligned}$$

The infinite point of $OB$ is

$$\begin{aligned}
&(S_A(S_B + S_C), \; S_B(S_C + S_A), \; S_C(S_A + S_B)) - (0, \; 2(S_{BC} + S_{CA} + S_{AB}), \; 0)\\
=&(S_A(S_B + S_C), \; -(S_{BC} + 2S_{CA} + S_{AB}), \; S_C(S_A + S_B)).
\end{aligned}$$

By the theorem in [2, §4.5], the two lines $AP$ and $OB$ are perpendicular since

$$\begin{aligned}&-S_A\cdot(S_B+S_C)\cdot S_A(S_B+S_C)\\&-S_B\cdot(S_C-S_A)\cdot(S_{BC}+2S_{CA}+S_{AB})\\&+S_C\cdot(S_A+S_B)\cdot S_C(S_A+S_B)\\&=0.\end{aligned}$$

□

## 2. Proof of Theorem 1

Note that the points $A_b$ and $A_c$ lie on the circle with diameter $OA$, so do the midpoints of $AC$ and $AB$. Therefore,

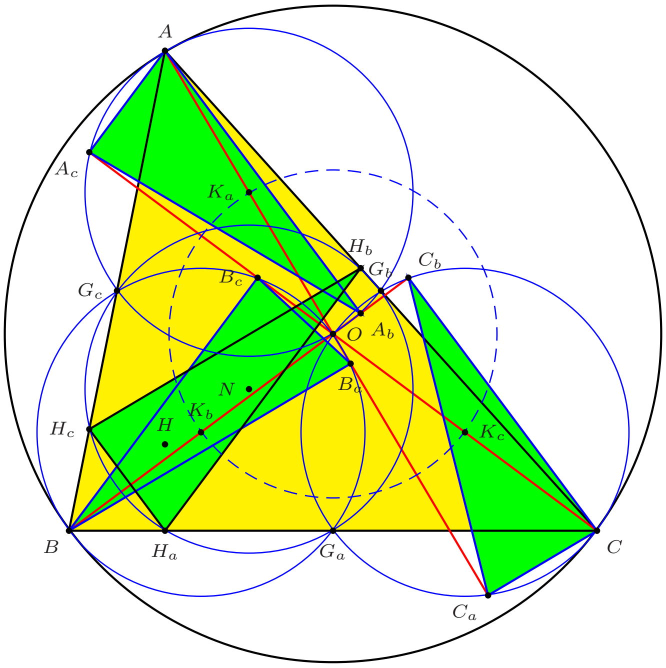


Figure 2

$$\begin{aligned}\angle A_cAA_b=&\pi-\angle A_bOA_c=\pi-\angle BOC=\pi-2A=\angle H_cH_aH_b,\\ \angle AA_bA_c=&\angle AOA_c=\pi-\angle COA=\pi-2B=\angle H_aH_bH_c,\\ \angle A_bA_cA=&\angle A_bOA=\pi-\angle AOB=\pi-2C=\angle H_bH_cH_a.\end{aligned}$$

Therefore the angles in triangles $AA_bA_c$ and $H_aH_bH_c$ are the same; similarly for triangles $B_aBB_c$ and $C_aC_bC$. Since these four triangles have equal circumradii $\frac{R}{2}$, they are congruent. This completes the proof of Theorem 1.

*Remarks.* (1) The side lengths of these triangles are $a\cos A$, $b\cos B$, and $c\cos C$ respectively.

(2) If $K_a$, $K_b$, $K_c$ are the midpoints of the circumradii $OA$, $OB$, $OC$, triangle $K_aK_bK_c$ is homothetic to
(i) $ABC$ at $O$, with ratio of homothety $\frac{1}{2}$, and
(ii) the medial triangle $G_aG_bG_c$ at $X_{140}$, the nine-point center of the medial triangle, with ratio of homothety $-1$.

(3) The circles $(K_b)$ and $(K_c)$ intersect at the circumcenter $O$ and the midpoint $G_a$ of $BC$; similarly for the other two pairs $(K_c)$, $(K_a)$ and $(K_a)$, $(K_b)$. The midpoints $G_a$, $G_b$, $G_c$ lie on the nine-point circle $(N)$ of triangle $ABC$. See Figure 2.

## 3. The triangle $\mathbf{T}_1$

We now consider the triangle $\mathbf{T}_1$ bounded by the lines $B_cC_b$, $C_aA_c$, and $A_bB_a$.

**Lemma 5.** *The quadrilateral $B_cC_bCB$ is an isosceles trapezoid.*

*Proof.* With reference to Figure 1, we have
(i) $\angle B_cBC = \frac{\pi}{2} - \angle OCB = \frac{\pi}{2} - \angle OBC = \angle C_bCB$,
(ii) $B_cB = C_bC$.
It follows that the quadrilateral $B_cC_bCB$ is an isosceles trapezoid. □

Therefore, the lines $B_cC_b$ and $BC$ are parallel. Similarly, the lines $C_aA_c$ and $CA$ are parallel, as are $A_bB_a$ and $AB$. The triangle $\mathbf{T}_1$ bounded by the lines $B_cC_b$, $C_aA_c$, $A_bB_a$ is homothetic to triangle $ABC$, and also to the medial triangle $G_aG_bG_c$.

**Proposition 6.** *Triangle $\mathbf{T}_1$ is homothetic to*
(i) *$ABC$ at the procircumcenter $(a^4S_A : b^4S_B : c^4S_C)$,* [1]
(ii) *the medial triangle $G_aG_bG_c$ at the Jerabek center $J$.*

*Proof.* The lines $B_cC_b$, $C_aA_c$, and $A_bB_a$ have equations

$$\begin{aligned}
-(S_{AA}+S_{BC})x + S_A(S_B+S_C)y + S_A(S_B+S_C)z =&0,\\
S_B(S_C+S_A)x - (S_{BB}+S_{CA})y + S_B(S_C+S_A)z =&0,\\
S_C(S_A+S_B)x + S_C(S_A+S_B)y - (S_{CC}+S_{AB})z =&0.
\end{aligned}$$

From these, we obtain the coordinates of the vertices of $\mathbf{T}_1$:

$$\begin{aligned}
A_1 =&(S_A(S_B-S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2),\\
B_1 =&(S_A(S_B+S_C)^2 : S_B(S_C-S_A)^2 : S_C(S_A+S_B)^2),\\
C_1 =&(S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A-S_B)^2).
\end{aligned}$$

From the coordinates of $A_1$, $B_1$, $C_1$, it is clear that the homothetic center of triangles $A_1B_1C_1$ and $ABC$ is the point

$$(S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2) = (a^4S_A : b^4S_B : c^4S_C).$$

[1] This is the triangle center $X_{184}$ in [1].

For (ii), the equations of the lines $G_aA_1$, $G_bB_1$, $G_cC_1$ are respectively

$$\begin{aligned}(S_{AA}-S_{BC})x-S_A(S_B-S_C)y+S_A(S_B-S_C)z=&0,\\ S_B(S_C-S_A)x+(S_{BB}-S_{CA})y-S_B(S_C-S_A)z=&0,\\ -S_C(S_A-S_B)x+S_C(S_A-S_B)y+(S_{CC}-S_{AB})z=&0.\end{aligned}$$

It is routine to check that this contains the Jerabek center $J$ whose coordinates are given in (1). $\square$

## 4. Proof of Theorem 2

Theorem 2 is now an immediate consequence of Proposition 6(ii). Since the homothetic center $J$ lies on the circumcircle of the medial triangle, it must also lie on the circumcircle of the other, and the two circumcircles are tangent at $J$.

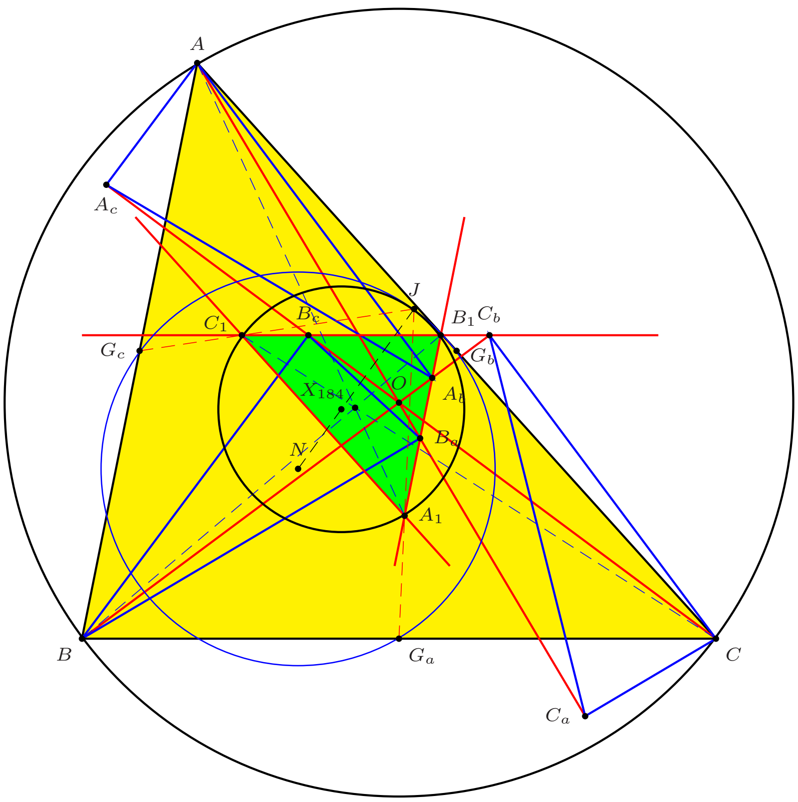


Figure 3

## 5. The triangle $\mathbf{T}_2$

From the coordinates of the pedals, we obtain the equations of the lines $A_bA_c$, $B_cB_a$, and $C_aC_b$:

$$\begin{aligned}-2S_{BC}x + (S^2 - S_{BB})y + (S^2 - S_{CC})z =&0,\\ (S^2 - S_{AA})x - 2S_{CA}y + (S^2 - S_{CC})z =&0,\\ (S^2 - S_{AA})x + (S^2 - S_{BB})y - 2S_{AB}z =&0.\end{aligned}$$

From these, the vertices of triangle $\mathbf{T}_2$ are the points

$$\begin{aligned}A_2 =&((S_B - S_C)^2 : 3S_{AB} + S_{BC} + S_{CA} - S_{CC} : 3S_{CA} + S_{AB} + S_{BC} - S_{BB}),\\ B_2 =&(3S_{AB} + S_{BC} + S_{CA} - S_{CC} : (S_C - S_A)^2 : 3S_{BC} + S_{CA} + S_{AB} - S_{AA}),\\ C_2 =&(3S_{CA} + S_{AB} + SBC - S_{CC} : 3S_{BC} + S_{CA} + S_{AB} - S_{AA} : (S_A - S_B)^2).\end{aligned}$$

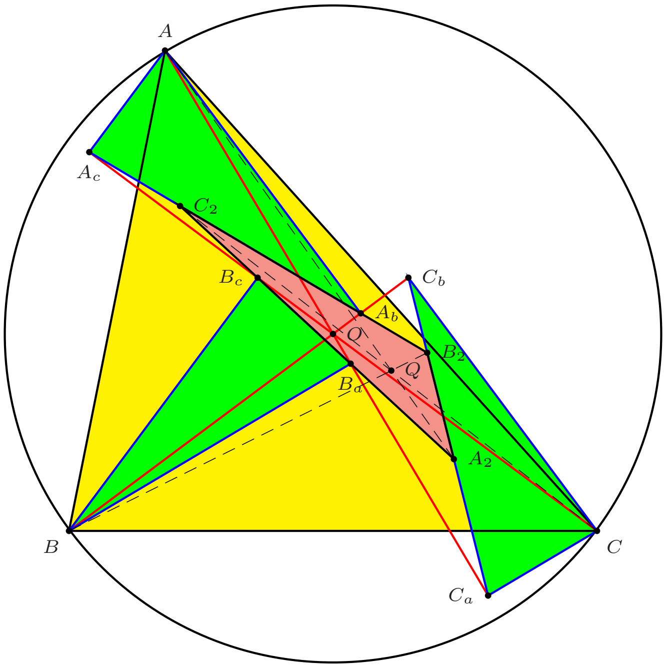


Figure 4

**Proposition 7.** *Triangles $ABC$ and $A_2B_2C_2$ are perspective at*

$$Q = \left(\frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - b^4 - c^4} : \frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - c^4 - a^4} : \frac{1}{a^2b^2 + b^2c^2 + c^2a^2 - a^4 - b^4}\right).$$

*Proof.* From the coordinates of $A_2$, $B_2$, $C_2$ given above, $\mathbf{T_2}$ is perspective with $ABC$ at

$$Q = \left(\frac{1}{3S_{BC} + S_{CA} + S_{AB} - S_{AA}} : \cdots : \cdots\right).$$

These are equivalent to those given above in terms of $a$, $b$, $c$. □

*Remark.* The triangle center $Q$ does not appear in [1].

## 6. Proof of Theorem 3

It is easier to work with the image of triangle $\mathbf{T}_2$ under the homothety $\mathsf{h}(H, 2)$. The images of the vertices are

$$\begin{aligned} A_2' =&(S_A(S_B + S_C)(S_{BB} - 4S_{BC} + S_{CC}) + S_{BC}(S_B - S_C)^2 \\ &: (S_C + S_A)(S_A(S_B + S_C)(3S_B - S_C) + S_{BC}(S_B - S_C)) \\ &: (S_A + S_B)(S_A(S_B + S_C)(S_B - 3S_C) + S_{BC}(S_B - S_C))), \end{aligned}$$

and $B_2'$, $C_2'$ whose coordinates are obtained by cyclic permutations of $S_A$, $S_B$, $S_C$. The circumcircle of $A_2'B_2'C_2'$ has equation

$$\begin{aligned} &8S^2 \cdot S_{ABC}((S_B + S_C)yz + (S_C + S_A)zx + (S_A + S_B)xy) \\ &+(x + y + z)\left(\sum_{\text{cyclic}}(S_A + S_B)(S_A + S_C)(S_{AB} + S_{CA} - 2S_{BC})^2x\right) = 0. \end{aligned}$$

To verify that this circle is tangent to the circumcircle

$$(S_B + S_C)yz + (S_C + S_A)zx + (S_A + S_B)xy = 0,$$

it is enough to consider the pedal of the circumcenter $O$ on the radical axis

$$\sum_{\text{cyclic}}(S_A + S_B)(S_A + S_C)(S_{AB} + S_{CA} - 2S_{BC})^2x = 0.$$

This is the point

$$Q' = \left(\frac{S_B + S_C}{S_{CA} + S_{AB} - 2S_{BC}} : \frac{S_C + S_A}{S_{AB} + S_{BC} - 2S_{CA}} : \frac{S_A + S_B}{S_{BC} + S_{CA} - 2S_{AB}}\right),$$

which is clearly on the circumcircle, and also on the Jerabek hyperbola

$$\frac{S_A(S_{BB} - S_{CC})}{x} + \frac{S_B(S_{CC} - S_{AA})}{y} + \frac{S_C(S_{AA} - S_{BB})}{z} = 0.$$

This shows that the circle $A_2'B_2'C_2'$ is tangent to the circumcircle at $Q'$. [2] Under the homothety $\mathsf{h}(H, 2)$, $Q'$ is the image of the midpoint of $HQ$, which is the center of the Jerabek hyperbola. Under the inverse homothety, the circumcircle of $\mathbf{T}_2$ is tangent to the nine-point circle at $J$. This completes the proof of Theorem 3.

[2] $Q'$ is the triangle center $X_{74}$

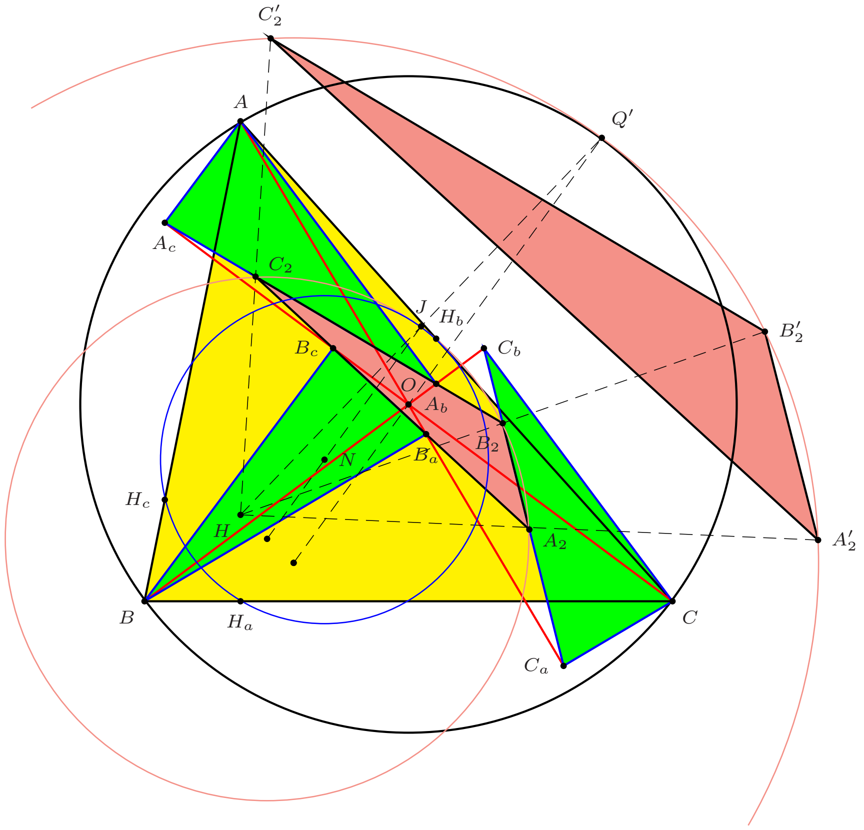


Figure 5

## References


[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.



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