

A Synthetic Proof and Generalization of Bellavitis' Theorem

Nikolaos Dergiades

Abstract. In this note we give a synthetic proof of Bellavitis' theorem and then generalizing this theorem, for not only convex quadrilaterals, we give a synthetic geometric proof for both theorems direct and converse, as Eisso Atzema proved, by trigonometry, for the convex case [1]. From this approach evolves clearly the connection between hypothesis and conclusion.

1. Bellavitis' theorem

Eisso J. Atzema has recently given a trigonometric proof of Bellavitis' theorem [1]. We present a synthetic proof here. Inside a convex quadrilateral $ABCD$, let the diagonal AC form with one pair of opposite sides angles w_1, w_3 . Similarly let the angles inside the quadrilateral that the other diagonal BD forms with the remaining pair of opposite sides be w_2, w_4 .

Theorem 1 (Bellavitis, 1854). *If the side lengths of a convex quadrilateral $ABCD$ satisfy $AB \cdot CD = BC \cdot DA$, then $w_1 + w_2 + w_3 + w_4 = 180^\circ$*

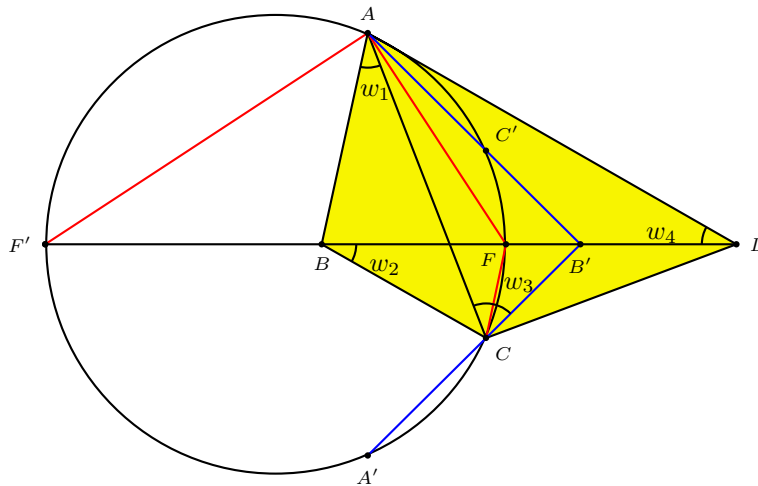


Figure 1

Proof. If $AB = AD$ then $BC = CD$ and AC is the perpendicular bisector of BD . Hence $ABCD$ is a kite, and it is obvious that $w_1 + w_2 + w_3 + w_4 = 180^\circ$.

If $ABCD$ is not a kite, then from $AB \cdot CD = BC \cdot DA$, we have $\frac{AB}{AD} = \frac{CB}{CD}$. Hence, C lies on the A -Apollonius circle of triangle ABD . See Figure 1. This

circle has diameter FF' , where AF and AF' are the internal and external bisectors of angle BAD and CF is the bisector of angle BCD . The reflection of AC in AF meets the Apollonius circle at C' . Since arc $CF = arc FC'$, the point C' is the reflection of C in BD . Similarly the reflection of AC in CF meets the Apollonius circle at A' that is the reflection of A in BD . Hence the lines AC' , CA' are reflections of each other in BD and are met at a point B' on BD . So we have

$$w_2 + w_3 = w_2 + \angle BCB' = \angle CB'D = \angle AB'D \tag{1}$$

$$w_1 + w_4 = \angle B'AD + w_4 = \angle BB'A. \tag{2}$$

From (1) and (2) we get

$$w_1 + w_2 + w_3 + w_4 = \angle BB'A + \angle AB'D = 180^\circ.$$

□

2. A generalization

There is actually no need for $ABCD$ to be a convex quadrilateral. Since it is clear that $w_1 + w_2 + w_3 + w_4 = 180^\circ$ for a cyclic quadrilateral, we consider non-cyclic quadrilaterals below. We make use of oriented angles and arcs. Denote by $\theta(XY, XZ)$ the oriented angle from XY to XZ . We continue to use the notation

$$\begin{aligned} w_1 &= \theta(AB, AC), & w_3 &= \theta(CD, CA), \\ w_2 &= \theta(BC, BD), & w_4 &= \theta(DA, DB). \end{aligned}$$

Theorem 2. *In an arbitrary noncyclic quadrilateral $ABCD$, the side lengths satisfy the equality $AB \cdot CD = BC \cdot DA$ if and only if*

$$w_1 + w_2 + w_3 + w_4 = \pm 180^\circ.$$

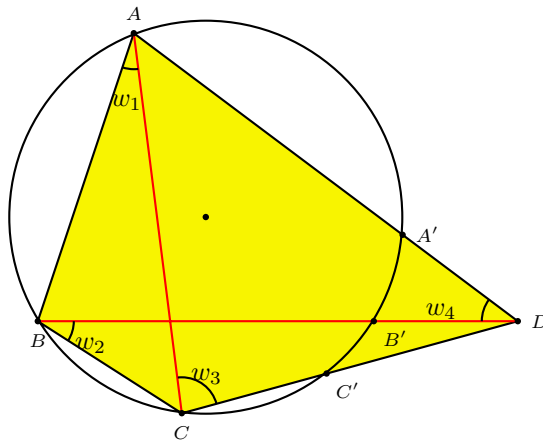


Figure 2

Proof. Since $ABCD$ is not a cyclic quadrilateral the lines DA , DB , DC meet the circumcircle of triangle ABC at the distinct points A' , B' , C' . The triangle $A'B'C'$ is the circumcevian triangle of D relative to ABC . Note that

$$\begin{aligned} 2w_1 &= \text{arc } BC, \\ 2w_2 &= \text{arc } CC' + \text{arc } C'B', \\ 2w_3 &= \text{arc } C'A, \\ 2w_4 &= \text{arc } AB + \text{arc } A'B'. \end{aligned}$$

From these, $w_1 + w_2 + w_3 + w_4 = \pm 180^\circ$ if and only if

$$(\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB) + \text{arc } C'B' + \text{arc } A'B' = \pm 360^\circ.$$

Since $\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB = \pm 360^\circ$, the above condition holds if and only if $\text{arc } C'B' = \text{arc } B'A'$. This means that the circumcevian triangle of D is isosceles, *i.e.*,

$$B'A' = B'C'. \quad (3)$$

It is well known that $A'B'C'$ is similar to with the pedal triangle $A''B''C''$ of D . See [2, §7.18] The condition (3) is equivalent to

$$B''A'' = B''C''.$$

This, in turn, is equivalent to the fact that D lies on the B -Apollonius circle of ABC because for a pedal triangle we know that

$$B''A'' = DC \cdot \sin C = B''C'' = DA \cdot \sin A$$

or

$$\frac{DC}{DA} = \frac{\sin A}{\sin C} = \frac{BC}{BA}.$$

From this we have $AB \cdot CD = BC \cdot DA$. □

References

- [1] E. J. Atzema, A theorem by Giusto Bellavitis on a class of quadrilaterals, *Forum Geom.*, 6 (2006) 181–185.
- [2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr