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A Synthetic Proof and Generalization of Bellavitis' Theorem

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Abstract. In this note we give a synthetic proof of Bellavitis' theorem and then generalizing this theorem, for not only convex quadrilaterals, we give a synthetic geometric proof for both theorems direct and converse, as Eisso Atzema proved, by trigonometry, for the convex case [1]. From this approach evolves clearly the connection between hypothesis and conclusion.

1. Bellavitis' theorem

Eisso J. Atzema has recently given a trigonometric proof of Bellavitis' theorem [1]. We present a synthetic proof here. Inside a convex quadrilateral $ABCD$, let the diagonal AC form with one pair of opposite sides angles w_1, w_3. Similarly let the angles inside the quadrilateral that the other diagonal BD forms with the remaining pair of opposite sides be w_2, w_4.

Theorem 1 (Bellavitis, 1854)**.** *If the side lengths of a convex quadrilateral $ABCD$ satisfy $AB \cdot CD = BC \cdot DA$, then $w_1 + w_2 + w_3 + w_4 = 180°$*

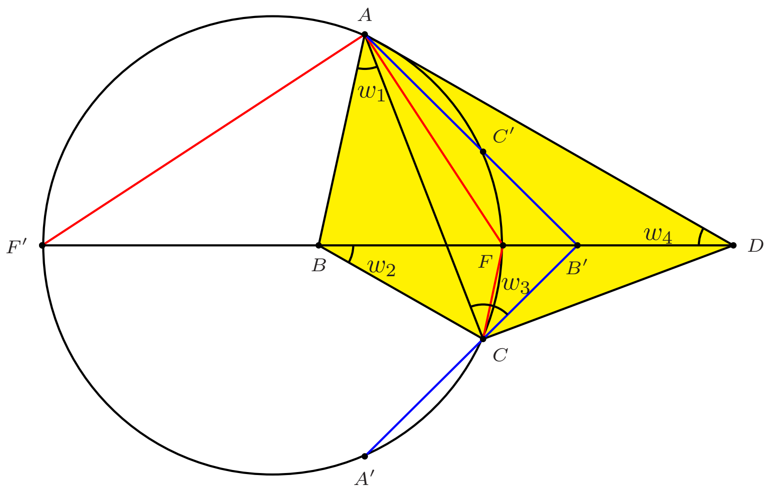

Figure 1

Proof. If $AB = AD$ then $BC = CD$ and AC is the perpendicular bisector of BD. Hence $ABCD$ is a kite, and it is obvious that $w_1 + w_2 + w_3 + w_4 = 180°$.

If $ABCD$ is not a kite, then from $AB \cdot CD = BC \cdot DA$, we have $\frac{AB}{AD} = \frac{CB}{CD}$. Hence, C lies on the A-Apollonius circle of triangle ABD. See Figure 1. This

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circle has diameter FF', where AF and AF' are the internal and external bisectors of angle BAD and CF is the bisector of angle BCD. The reflection of AC in AF meets the Apollonius circle at C'. Since arc CF = arc FC', the point C' is the reflection of C in BD. Similarly the reflection of AC in CF meets the Apollonius circle at A' that is the reflection of A in BD. Hence the lines AC', CA' are reflections of each other in BD and are met at a point B' on BD. So we have

$$w_2 + w_3 = w_2 + \angle BCB' = \angle CB'D = \angle AB'D \tag{1}$$

$$w_1 + w_4 = \angle B'AD + w_4 = \angle BB'A. \tag{2}$$

From (1) and (2) we get

$$w_1 + w_2 + w_3 + w_4 = \angle BB'A + \angle AB'D = 180^\circ.$$

□

2. A generalization

There is actually no need for $ABCD$ to be a convex quadrilateral. Since it is clear that $w_1 + w_2 + w_3 + w_4 = 180^\circ$ for a cyclic quadrilateral, we consider noncyclic quadrilaterals below. We make use of oriented angles and arcs. Denote by $\theta(XY, XZ)$ the oriented angle from XY to XZ. We continue to use the notation

$$w_1 = \theta(AB, AC), \qquad w_3 = \theta(CD, CA),$$
$$w_2 = \theta(BC, BD), \qquad w_4 = \theta(DA, DB).$$

Theorem 2. *In an arbitrary noncyclic quadrilateral $ABCD$, the side lengths satisfy the equality $AB \cdot CD = BC \cdot DA$ if and only if*

$$w_1 + w_2 + w_3 + w_4 = \pm 180^\circ.$$

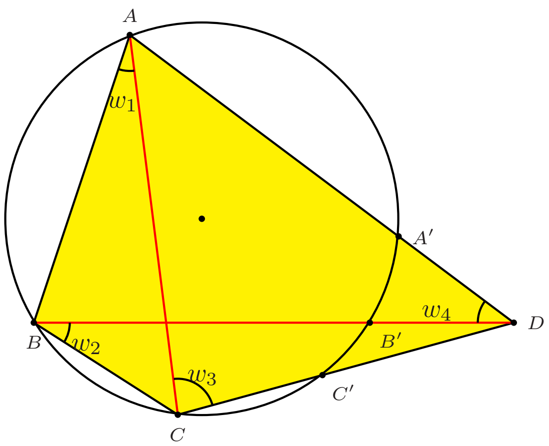

Figure 2

Proof. Since $ABCD$ is not a cyclic quadrilateral the lines DA, DB, DC meet the circumcircle of triangle ABC at the distinct points A', B', C'. The triangle $A'B'C'$ is the circumcevian triangle of D relative to ABC. Note that

$$\begin{aligned} 2w_1 =& \text{arc } BC, \\ 2w_2 =& \text{arc } CC' + \text{arc } C'B', \\ 2w_3 =& \text{arc } C'A, \\ 2w_4 =& \text{arc } AB + \text{arc} A'B'. \end{aligned}$$

From these, $w_1 + w_2 + w_3 + w_4 = \pm 180^\circ$ if and only if

$$(\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB) + \text{arc } C'B' + \text{arc } A'B' = \pm 360^\circ.$$

Since $\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB = \pm 360^\circ$, the above condition holds if and only if $\text{arc } C'B' = \text{arc } B'A'$. This means that the circumcevian triangle of D is isosceles, *i.e.*,

$$B'A' = B'C'. \tag{3}$$

It is well known that $A'B'C'$ is similar to with the pedal triangle $A''B''C''$ of D. See [2, §7.18] The condition (3) is equivalent to

$$B''A'' = B''C''.$$

This, in turn, is equivalent to the fact that D lies on the B-Apollonius circle of ABC because for a pedal triangle we know that

$$B''A'' = DC \cdot \sin C = B''C'' = DA \cdot \sin A$$

or

$$\frac{DC}{DA} = \frac{\sin A}{\sin C} = \frac{BC}{BA}.$$

From this we have $AB \cdot CD = BC \cdot DA$. $\square$

References

[1] E. J. Atzema, A theorem by Giusto Bellavitis on a class of quadrilaterals, *Forum Geom.*, 6 (2006) 181–185.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

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