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Translated Triangles Perspective to a Reference Triangle

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Abstract. Suppose A, B, C, D, E, F are points and L is a line other than the line at infinity. This work examines cases in which a translation $D'E'F'$ of DEF in the direction of L is perspective to ABC, in the sense that the lines AD', BE', CF' concur.

1. Introduction

In the transfigured plane of a triangle ABC, let L^∞ be the line at infinity and L a line other than L^∞. (To say "transfigured plane" means that the sidelengths a, b, c of triangle ABC are variables or indeterminates, and points are defined as functions of a, b, c, so that the "plane" of ABC is infinite dimensional.) Suppose that D, E, F are distinct points, none on L^∞, such that the set $\{A, B, C, D, E, F\}$ consists of at least five distinct points. We wish to translate triangle DEF in the direction of L and to discuss cases in which the translated triangle $D'E'F'$ is perspective to ABC, in the sense that the lines AD', BE', CF' concur. One of these cases is the limiting case that $D' = L \cap L^\infty$; call this point U, and note that $D' = E' = F' = U$.

Points and lines will be given (indeed, are *defined*) by homogeneous trilinear coordinates. The line L^∞ at infinity is given by $a\alpha + b\beta + c\gamma = 0$, and L, by an equation $l\alpha + m\beta + n\gamma = 0$, where $l : m : n$ is a point. Then the point $U = u : v : w$ is given by

$$u = bn - cm, \ v = cl - an, \ w = am - bl. \tag{1}$$

Write the vertices of DEF as

$$D = d_1 : e_1 : f_1, \qquad E = d_2 : e_2 : f_2, \qquad F = d_3 : e_3 : f_3,$$

and let

$$\delta = ad_1 + be_1 + cf_1, \qquad \epsilon = ad_2 + be_2 + cf_2, \qquad \varphi = ad_3 + be_3 + cf_3.$$

The hypothesis that none of D, E, F is on L^∞ implies that none of $\delta, \epsilon, \varphi$ is 0. The line L is given parametrically as the locus of point $D' = D_t = x_1 : y_1 : z_1$ by

$$x_1 = d_1 + \delta tu, \qquad y_1 = e_1 + \delta tv, \qquad z_1 = f_1 + \delta tw.$$

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The point E' traverses the line through E parallel to L, so that $E' = E_t = x_2 : y_2 : z_2$ is given by

$$x_2 = d_2 + \epsilon tu, \qquad y_2 = e_2 + \epsilon tv, \qquad z_2 = f_2 + \epsilon tw.$$

The point F' traverses the line through F parallel to L, so that $F' = F_t = x_3 : y_3 : z_3$ is given by

$$x_3 = d_3 + \varphi tu \quad y_3 = e_3 + \varphi tv \qquad z_3 = f_3 + \varphi tw.$$

In these parameterizations, t represents a homogeneous function of a, b, c. The degree of homogeneity of t is that of $(x_1 - d_1)/(\delta u)$.

2. Two basic theorems

Theorem 1. *Suppose ABC and DEF are triangles such that $\{A, B, C, D, E, F\}$ consists of at least five distinct points. Suppose L is a line and $U = L \cap L^\infty$. As D_t traverses the line DU, the triangle $D_tE_tF_t$ of translation of DEF in the direction of L is either perspective to ABC for all t or else perspective to ABC for at most two values of t.*

Proof. The lines AD_t, BE_t, CF_t are given by the equations

$$-z_1\beta + y_1\gamma = 0, \quad z_2\alpha - x_2\gamma = 0, \quad -y_3\alpha + x_3\beta = 0,$$

respectively. Thus, the concurrence determinant,

$$\begin{vmatrix} 0 & -z_1 & y_1 \\ z_2 & 0 & -x_2 \\ -y_3 & x_3 & 0 \end{vmatrix} \tag{2}$$

is a polynomial P, formally of degree 2 in t:

$$P(t) = p_0 + p_1 t + p_2 t^2, \tag{3}$$

where

$$p_0 = d_3e_1f_2 - d_2e_3f_1, \tag{4}$$
$$p_1 = u(\varphi e_1f_2 - \epsilon e_3f_1) + v(\delta d_3f_2 - \varphi d_2f_1) + w(\epsilon e_1d_3 - \delta e_3d_2), \tag{5}$$
$$p_2 = \delta vw(\epsilon d_3 - \varphi d_2) + \epsilon wu(\varphi e_1 - \delta e_3) + \varphi uv(\delta f_2 - \epsilon f_1). \tag{6}$$

Thus, either p_0, p_1, p_0 are all zero, in which case $D_tE_tF_t$ is perspective to ABC for all t, or else $P(t)$ is zero for at most two values of t. □

If triangle DEF is homothetic to ABC, then $D_tE_tF_t$ is homothetic to ABC and hence perspective to ABC, for every t. This is well known in geometry. The geometric theorem, however, does not imply the "same" theorem in the more general setting of triangle algebra, in which the objects are defined in terms of variables or indeterminants. Specifically, perspectivity and parallelism (hence homothety) are defined by zero determinants. When such determinants are "symbolically zero", they are zero not only for Euclidean triangles, for which a, b, c are positive real numbers satisfying $(a > b + c, b > c + a, c > a + b)$ or $(a \geq b + c, b \geq c + a,$

$c \geq a+b$), but also for a, b, c as indeterminates. Among geometric theorems that readily generalize to algebraic theorems are these:

If $L_1 \| L_2$ and $L_2 \| L_3$, then $L_1 \| L_3$.
If T_1 is homothetic to T_2 and T_2 is homothetic to T_3, then T_1 is homothetic to T_3.
If T_1 is homothetic to T_2, then T_1 is perspective to T_2.

Theorem 2. *Suppose ABC and DEF in Theorem 1 are homothetic. Then $D_tE_tF_t$ is perspective to ABC for all t.*

Proof. ABC and DEF are homothetic, and DEF and $D_tE_tF_t$ are homothetic. Therefore $D_tE_tF_t$ is homothetic to ABC, which implies that $D_tE_tF_t$ is perspective to ABC. □

It is of interest to express the coefficients p_0, p_1, p_2 more directly in terms of a, b, c and the coordinates of D, E, F. To that end, we shall use cofactors, as defined by the identity

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix},$$

where

$$\Delta = \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} = d_1D_1 + e_1E_1 + f_1F_1;$$

that is, $D_1 = e_2f_3 - f_2e_3$, etc. For example, in the case that ABC and DEF are homothetic, line EF is parallel to line BC, as defined by a zero determinant (e.g., [1], p. 29); likewise, the lines FD and CA are parallel, as are DE and AB. The zero determinants yield

$$bF_1 = cE_1, \qquad cD_2 = aF_2, \qquad aE_3 = bD_3. \tag{7}$$

These equations can be used to give a direct but somewhat tedious proof of Theorem 2; we digress to prove only that $p_0 = 0$. Let $\mathcal{L}$ and $\mathcal{R}$ denote the products of the left-hand sides and the right-hand sides in (7). Then $\mathcal{L} - \mathcal{R}$ factors as $abc\Psi\Delta$, where

$$\Psi = e_3d_2f_1 - e_1d_3f_2,$$

and $abc\Psi\Delta = 0$ by (7). It is understood that A, B, C are not collinear, so that D, E, F are not collinear. As the defining equation for collinearity of D, E, F is the determinant equation $\Delta = 0$, we have $\Delta \neq 0$. Therefore, $\Psi = 0$, so that $p_0 = 0$.

Next, substitute from (1) for u, v, w in (5) and (6), getting

$$p_1 = lp_{1l} + mp_{1m} + np_{1n} \qquad \text{and} \qquad p_2 = mnp_{2l} + nlp_{2m} + lm_{2n},$$

where

$$\begin{aligned} p_{1l} =& b^2e_1F_1 + c^2f_1E_1 - abd_2F_2 - cad_3E_3, \\ p_{1m} =& c^2f_2D_2 + a^2d_2F_2 - bce_3D_3 - abe_1F_1, \\ p_{1n} =& a^2d_3E_3 + b^2e_3D_3 - caf_1E_1 - bcf_2D_2, \end{aligned}$$

and

$$\begin{aligned} p_{2l} =& 2a^2bc(e_1d_2f_3 - e_2d_3f_1) - ab^2e_3F_3 + ac^2f_2E_2 \\ & - bc^2f_1D_1 + ba^2d_3F_3 - ca^2d_2E_2 + cb^2e_1D_1, \\ p_{2m} =& 2b^2ca(f_2e_3d_1 - f_3e_1d_2) - bc^2f_1D_1 + ba^2d_3F_3 \\ & - ca^2d_2E_2 + cb^2e_1D_3 - ab^2e_3F_3 + ac^2f_2E_2, \\ p_{2n} =& 2c^2ab(e_3d_1e_2 - d_1f_2e_3) - ca^2d_2E_2 + cb^2e_1D_1 \\ & - ab^2e_3F_3 + ac^2f_2E_1 - bc^2f_1D_1 + ba^2d_3F_3. \end{aligned}$$

The task of expressing the coefficients p_0, p_1, p_2 more directly in terms of a, b, c and the coordinates of D, E, F is now completed.

3. Intersecting conics

We begin with a lemma proved in [7]; see also [2].

Lemma 3. *Suppose a point $P = p : q : r$ is given parametrically by*

$$\begin{aligned} p =& p_1t^2 + q_1t + r_1, \\ q =& p_2t^2 + q_2t + r_2, \\ r =& p_3t^2 + q_3t + r_3, \end{aligned}$$

where the matrix

$$M = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$$

is nonsingular with adjoint (cofactor) matrix

$$M^{\#} = \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{pmatrix}.$$

Then P lies on this conic:

$$(Q_1\alpha + Q_2\beta + Q_3\gamma)^2 = (P_1\alpha + P_2\beta + P_3\gamma)(R_1\alpha + R_2\beta + R_3\gamma). \qquad (8)$$

In Theorem 4, we shall show that the point of concurrence of the lines AD_t, BE_t, CF_t is also the point of concurrence of three conics. Let

$$A_t = BE_t \cap CF_t, \quad B_t = CF_t \cap AD_t, \quad C_t = AD_t \cap BE_t.$$

Theorem 4. *If, in Theorem 1, the line L is not parallel to a sideline of triangle ABC, then the locus of each of the points A_t, B_t, C_t is a conic.*

Proof. The point $A_t = a_t : b_t : c_t$ is given by

$$a_t = x_2 x_3, \qquad b_t = x_2 y_3, \qquad c_t = z_2 x_3,$$

so that

$$a_t = (d_2 + \epsilon t u)(d_3 + \varphi t u) = \epsilon\varphi u^2 t^2 + (\varphi d_2 u + \epsilon d_3 u)t + d_2 d_3, \tag{9}$$

$$b_t = (d_2 + \epsilon t u)(e_3 + \varphi t v) = \epsilon\varphi u v t^2 + (\varphi d_2 v + \epsilon e_3 u)t + d_2 e_3, \tag{10}$$

$$c_t = (f_2 + \epsilon t w)(d_3 + \varphi t u) = \epsilon\varphi u w t^2 + (\varphi f_2 u + \epsilon d_3 w)t + f_2 d_3. \tag{11}$$

By Lemma 3, the locus $\{A_t\}$ is a conic unless $\epsilon\varphi uvw = 0$, in which case u, v, or w must be zero. Consider the case $u = 0$; then $u : v : w = 0 : c : -b$, but this is the point in which line BC meets L^∞, contrary to the hypothesis. Likewise, the loci $\{B_t\}$ and $\{C_t\}$ are conics. Note that the conic $\{A_t\}$ passes through B and C, as indicated by Figure 1. [1] □

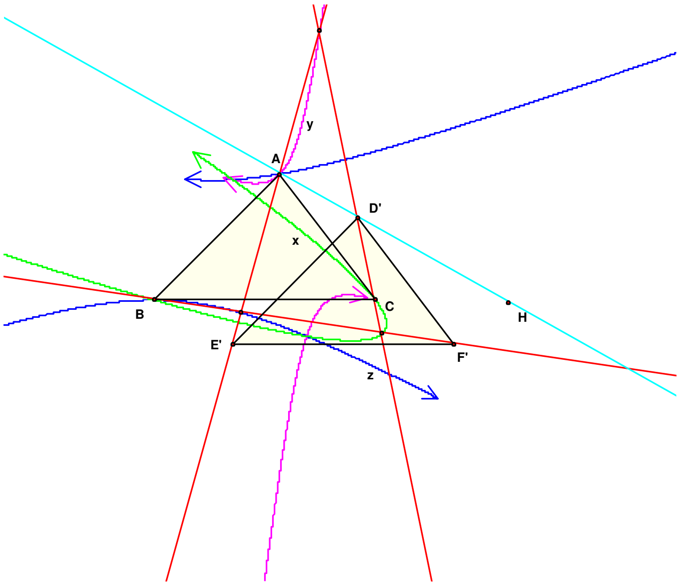

Figure 1. Intersecting conics

Lemma 3 shows how to write out equations of conics starting with a matrix M. As the lemma applies only to nonsingular M, we can, by factoring $|M|$, determine

[1]Figure 1 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. The choice of triangle DEF is given by the equations $D = C$, $E = A$, $F = B$. (The labels D, E, F are not shown.) Point H on line AD' is an independent point, and triangle $D'E'F'$ is a translation of DEF in the direction of line AH. Except for special cases, as D' traverses line AH, points x, y, z traverse conics as in Theorem 4, and the conics meet twice (with $x = y = z$), at the perspectors described in Theorem 1.

criteria for nonsingularity. In connection with $\{A_t\}$ and (9)-(11),

$$\begin{aligned}|M| &= \begin{vmatrix} \epsilon\varphi u^2 & \varphi d_2 u + \epsilon d_3 u & d_2 d_3 \\ \epsilon\varphi uv & \varphi d_2 v + \epsilon e_3 u & d_2 e_3 \\ \epsilon\varphi wu & \varphi f_2 u + \epsilon d_3 w & f_2 d_3 \end{vmatrix} \\ &= \epsilon\varphi u \left(uf_2 - wd_2\right)\left(vd_3 - ue_3\right)\left(be_3 d_2 - be_2 d_3 + cd_2 f_3 - cd_3 f_2\right).\end{aligned}$$

By hypothesis, $\epsilon\varphi u \neq 0$. Also, $(uf_2 - wd_2)(vd_3 - ue_3) \neq 0$, as it is assumed that $E \neq U$ and $F \neq U$. Finally, the factor $be_3 d_2 - be_2 d_3 + cd_2 f_3 - cd_3 f_2$ is 0 if and only if line EF is parallel to line BC. In conclusion, if EF is not parallel to BC, and FD is not parallel to CA, and DE is not parallel to AB, then the three loci are conics and Theorem 3 applies.

4. Terminology and notation

The main theorem in this paper is Theorem 1. For various choices of DEF and L, the perspectivities indicated by Theorem 1 are of particular interest. Such choices are considered in Sections 3-6; they are, briefly, that DEF is a cevian triangle of a point, or an anticevian triangle, or a rotation of triangle ABC about its circumcenter. In order to describe the configurations, it will be helpful to adopt certain terms and notations.

Unless otherwise noted, the points $U = u : v : w$ and $P = p : q : r$ are arbitrary. If at least one of the products up, vq, wr is not zero, the product $U \cdot P$ is defined by the equation

$$U \cdot P = up : vq : wr.$$

The multiplicative inverse of P, defined if $pqr \neq 0$, is the isogonal conjugate of P, given by

$$P^{-1} = p^{-1} : q^{-1} : r^{-1}.$$

The quotient U/P is defined by

$$U/P = U \cdot P^{-1}.$$

The *isotomic conjugat*e of P is defined if $pqr \neq 0$ by the trilinears

$$a^{-2}p^{-1} : b^{-2}q^{-1} : c^{-2}r^{-1}.$$

Geometric definitions of isogonal and isotomic conjugates are given at *MathWorld* [8]. We shall also employ these terms and notations:

crossdifference of U and $P = CD(U,P) = rv - qw : pw - ru : qu - pv$,

crosssum of U and $P = CS(U,P) = rv + qw : pw + ru : qu + pv$,

crosspoint of U and $P = CP(U,P) = pu(rv + qw) : qv(pw + ru) : rw(qu + pv)$.

Geometric interpretations of these "cross operations" are given at [4].

Notation of the form X_i as in [3] will be used for certain special points, such as

$$\begin{aligned} \text{incenter} =& X_1 = 1:1:1 = \text{ the multiplicative identity}, \\ \text{centroid} =& X_2 = 1/a : 1/b : 1/c, \\ \text{circumcenter} =& X_3 = \cos A : \cos B : \cos C, \\ \text{symmedian point} =& X_6 = a : b : c. \end{aligned}$$

Instead of the trigonometric trilinears for X_3, we shall sometimes use trilinears for X_3 expressed directly in terms of a, b, c. As $\cos A = (b^2 + c^2 - a^2)/(2bc)$, we shall use abbreviations:

$$a_1 = (b^2+c^2-a^2)/(2bc), \quad b_1 = (c^2+a^2-b^2)/(2ca), \quad c_1 = (a^2+b^2-c^2)/(2ab);$$

thus, $X_3 = a_1 : b_1 : c_1$.

The line X_2X_3 is the Euler line, and the line X_3X_6 is the Brocard axis. When working with lines algebraically, it is sometimes helpful to do so with reference to a parameter and the point in which the line meets L^∞. In the case of the Euler line, this point is

$$X_{30} = a_1 - 2b_1c_1 : b_1 - 2c_1a_1 : c_1 - 2a_1b_1,$$

and a parametric representation is given by $x : y : z = x(s) : y(s) : z(s)$, where

$$\begin{aligned} x(s) =& a_1 + s(a_1 - 2b_1c_1), \\ y(s) =& b_1 + s(b_1 - 2c_1a_1), \\ z(s) =& c_1 + s(c_1 - 2a_1b_1). \end{aligned}$$

The point X_{30} will be called the direction of the Euler line. More generally, for any line, its point of intersection with L^∞ will be called the *direction* of the line. The parameter s is not necessarily a numerical variable; rather, it is a function of a, b, c. In this paper, trilinears for any point are homogeneneous functions of a, b, c, all having the same degree of homogeneity; thus in a parametric expression of the form $p + su$, the degree of homogeneity of s is that of p minus that of u.

Two families of cubics will occur in the sequel. The cubic $\mathcal{Z}(U, P)$ is given by

$$(vqy - wrz)px^2 + (wrz - upx)qy^2 + (upx - vqy)rz^2 = 0,$$

and the cubic $\mathcal{ZC}(U, P)$, by

$$L(wy - vz)x^2 + M(uz - wx)y^2 + N(vx - uy)z^2 = 0.$$

For details on these and other families of cubics, see [5].

The remainder of this article is mostly about special translations. It will be helpful to introduce some related terminology. Suppose DEF is a triangle in the transfigured plane of ABC, and U is a direction (i.e., a point on L^∞). A triangle $D'E'F'$, other than DEF itself, such that $D'E'F'$ is a U-translation of DEF and $D'E'F'$ is perspective to ABC (in the sense that the lines AD', BE', CF' concur) will be called a *U-ppt of DEF*. The designation "ppt" means "proper perspective translation".

In view of Theorem 1, except for special cases, each DEF has, for every U, at most two U-ppt's. Thus, if DEF is perspective to ABC, as when DEF is a cevian triangle or an anticevian tiangle, there is "usually" just one ppt. That one ppt is of primary interest in the next three sections; especially in Case 5.4 and Case 6.4.

5. Translated cevian triangles

In this section, DEF is the cevian triangle of a point $X = x : y : z$; thus DEF is given as a matrix by

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix},$$

and the perspectivity determinant (2) is given by $-t(\Delta_0 + t\Delta_1)$, where

$$\Delta_0 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ u & v & w \\ x & y & z \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a^2ux^2 & b^2vy^2 & c^2wz^2 \\ u & v & w \\ x & y & z \end{vmatrix}.$$

In particular, the equations $\Delta_0 = 0$ and $\Delta_1 = 0$ represent cubics in x, y, z, specifically, $\mathcal{Z}(X_2, X_6 \cdot U^{-1})$ and $\mathcal{Z}(X_{75} \cdot U^{-1}, X_{31})$, respectively. We shall consider four cases:

Case 5.1: $\Delta_0 = 0$ and $\Delta_1 = 0$. In this case, $D_tE_tF_t$ is perspective to ABC for every t. Clearly this holds for $X = X_2$, for all U. Now for any given U, let X be the isotomic conjugate of U. Rows 1 and 3 of the determinant Δ_1 are equal, so that $\Delta_1 = 0$. Also,

$$\begin{aligned} \Delta_0 =& bcvw(b^2v^2 - c^2w^2) + cawu(c^2w^2 - a^2u^2) + abuv(a^2u^2 - b^2v^2) \\ =& -(bv - cw)(cw - au)(au - bv)(au + bv + cw) \\ =& 0. \end{aligned}$$

The cevian triangle DEF of X is not homothetic to ABC, yet $D_tE_tF_t$ is perspective to ABC for every t. Another such example is obtained by simply taking X to be U. Further results in Case 1 are given in Theorem 5.

Case 5.2: $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For given U, the point $X = CP(X_2, U)$ satisfies $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For quite a different example, let

$$U = X_{511} = \cos(A + \omega) : \cos(B + \omega) : \cos(C + \omega),$$

where ω denotes the Brocard angle. Then the cubic $\Delta_0 = 0$ passes through the following points, X_3 (the circumcenter), X_6 (the symmedian point), X_{297}, X_{325}, X_{694}, X_{2009}, and X_{2010}, none of which lies on the cubic $\Delta_1 = 0$. Other points on the cubic $\Delta_0 = 0$ are given at [5], where the cubic $\Delta_1 = 0$ is classified as $\mathcal{ZC}(511, L(30, 511))$.

Case 5.3: $\Delta_0 \neq 0$ and $\Delta_1 = 0$. In this case, for any U, there is no U-ppt. For example, take $U = X_{523}$. Then the cubic $\Delta_1 = 0$ passes through the two points in which the Euler line meets the circumcircle, these being X_{1113} and X_{1114}, and these points do not also lie on the cubic $\Delta_0 = 0$.

Case 5.4: $\Delta_0 \neq 0$ and $\Delta_1 \neq 0$. In this case, $D_tE_tF_t$ is perspective to ABC for $t = -\Delta_0/\Delta_1$. The perspector is the point $x_2x_3 : x_2y_3 : z_2x_3$, which, after cancellation of common factors, is the point $X' = x' : y' : z'$ given by

$$x' = \frac{by - cz}{(bv - cw)yz + ax(vz - wy)}, \tag{12}$$

$$y' = \frac{cz - ax}{(cw - au)zx + by(wx - uz)}, \tag{13}$$

$$z' = \frac{ax - by}{(au - bv)xy + cz(uy - vx)}. \tag{14}$$

Note that if X and U are triangle centers for which X' is a point, then X' is a triangle center. For the special case $U = X_{511}$, pairs X and X' are shown here:

X	4	7	54	68	69	99	183	190	385	401	668	670	671	903
X'	3	256	52	52	6	690	262	900	325	297	691	888	690	900

Returning to Case 5.1, in the subcase that X is the isotomic conjugate of U, it is natural to ask about the perspectors, and to find the following theorem.

Theorem 5. *Suppose U is any point on L^∞ but not on a sideline BC, CA, AB. Let X be the isotomic conjugate of U. The locus of the perspector P_t of triangles $D_tE_tF_t$ and ABC is a conic that passes through A, B, C, and the point*

$$X^2 = b^4c^4v^2w^2 : c^4a^4w^2u^2 : a^4b^4u^2v^2. \tag{15}$$

Proof. The perspector is the point $P_t = x_2x_3 : x_2y_3 : z_2x_3$. Substituting and simplifying give

$$\begin{aligned} P_t = & b^3c^3vw(bv - acuwt)(cw - abuvt) \\ & : c^3a^3wu(cw - bavut)(au - bcvwt) \\ & : a^3b^3uv(au - cbwvt)(bv - cawut). \end{aligned}$$

By Theorem 3, the locus of P_t is a conic. Clearly, P_t passes through A, B, C for $t = au/(bcvw)$, $bv/(cawu)$, $cw/(abuv)$, respectively, and P_0 is the point given by (15). See Figure 2. [2] $\square$

An equation for the circumconic described in Theorem 5 is found from (8):

$$b^2c^2(b^2v^2 - c^2w^2)\beta\gamma + c^2a^2(c^2w^2 - a^2u^2)\gamma\alpha + a^2b^2(a^2u^2 - b^2v^2)\alpha\beta = 0.$$

Theorem 6. *Suppose X is the isotomic conjugate of a point U_1 on L^∞ but not on a sideline BC, CA, AB. Then the perspector X' in Case 5.4 is invariant of the point U. In fact, $X' = CD(X_6, U_1^{-1})$, and X' is on L^∞.*

[2] Figure 2 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. An arbitrary point U on L^∞ is given by $U = Au \cap L^\infty$, where u is an independent point; i.e., the user can vary u freely. The cevian triangle of U is def, the cevian triangle of the isotomic conjugate of U is DEF. Point D' is movable on line DU. Triangle $D'E'F'$ is thus a movable translation of DEF in the direction of U, and $D'E'F'$ stays perspective to ABC. The perspector P traverses a circumconic.

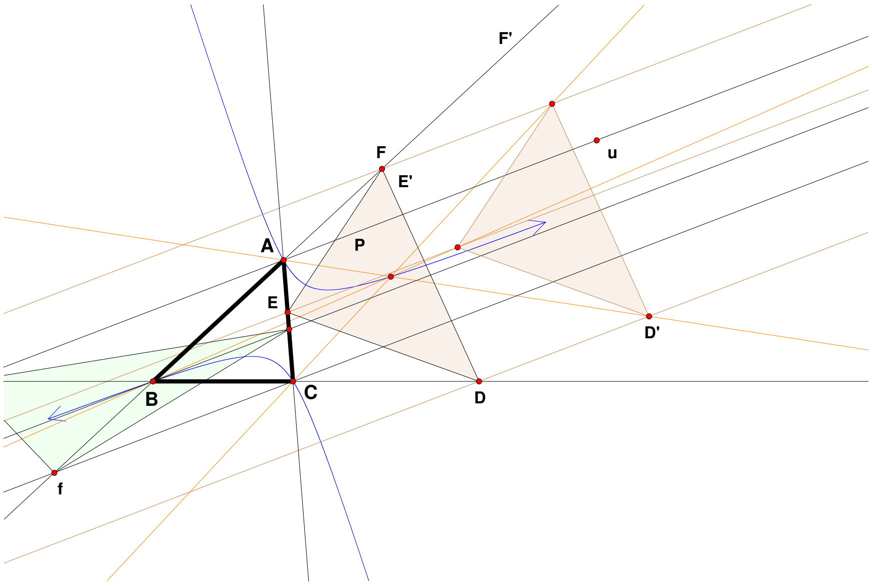

Figure 2. Cevian triangle and circumconic as in Theorem 5.

Proof. Write $X = x : y : z = b^2c^2v_1w_1 : c^2a^2w_1u_1 : a^2b^2u_1v_1$ where the point $U_1 = u_1 : v_1 : w_1$ is on L^∞ and $u_1v_1w_1 \neq 0$. Represent U_1 parametrically by

$$u_1 = (b-c)(1+bcs), \quad v_1 = (c-a)(1+cas), \quad w_1 = (a-b)(1+abs), \quad (16)$$

so that

$$(bv-cw)yz+ax(vz-wy) = \Lambda\left(bv - av - aw + cw + as(b^2v - abv + c^2w - acw)\right),$$

where

$$\Lambda = -a^3b^3c^3(b-c)(c-a)(a-b)(1+bcs)(1+cas)(1+abs).$$

Thus

$$(bv-cw)yz + ax(vz-wy) = \Lambda(-a(u+v+w) + as(a^2u + b^2v + c^2w)),$$

and by (12),

$$\begin{aligned} x' =& \frac{by - cz}{(bv-cw)yz + ax(vz-wy)} \\ =& \frac{(by-cz)/a}{\Lambda(-(u+v+w) + s(a^2u+b^2v+c^2w))}. \end{aligned}$$

Coordinates y' and z' are found in the same manner, and multiplying through by the common denominator gives

$$\begin{aligned} x' : y' : z' =& (by-cz)/a : (cz-ax)/b : (ax-by)/c \\ =& u_1(bv_1 - cw_1) : v_1(cw_1 - au_1) : w_1(au_1 - bv_1) \\ =& CD(X_6, U_1^{-1}). \end{aligned}$$

Clearly, $ax' + by' + cz' = 0$, which is to say that the perspector X' is on L^∞. $\square$

Corollary 7. *As X traverses the Steiner circumellipse, the perspector X' traverses the line at infinity.*

Proof. The Steiner circumellipse is given by

$$bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0.$$

The corollary follows from the easy-to-verify fact that the isotomic conjugacy mapping carries the Steiner circumellipse to L^{∞}, to which Theorem 6 applies. □

Theorem 7 is exemplified by taking $X = X_{190}$; the isotomic conjugate of X is then X_{514}, for which the perspector is $X' = X_{900} = CD(X_6, X_{101})$. Other examples (X, X') are these: (X_{99}, X_{690}), (X_{668}, X_{891}), (X_{670}, X_{888}), (X_{671}, X_{690}), (X_{903}, X_{900}). These examples show that the mapping $X \to X'$ is not one-to-one.

6. Translated anticevian triangles

In this section, DEF is the anticevian triangle of a point $X = x : y : z$; given as a matrix by

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix}.$$

The perspectivity determinant (2) is given by $-2t(\Delta_0 + t\Delta_2)$, where

$$\Delta_0 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ u & v & w \\ x & y & z \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ (bv+cw)u & (cw+au)v & (au+bv)w \\ x & y & z \end{vmatrix}.$$

The cubic $\Delta_0 = 0$ is already discussed in Section 3. The equation $\Delta_2 = 0$ represents the cubic $\mathcal{Z}(X_2/CS(X_6, U^{-1}))$. We consider four cases as in Section 3.

Case 6.1: $\Delta_0 = 0$ and $\Delta_2 = 0$. In this case, $D_tE_tF_t$ is perspective to ABC for every t. Clearly this holds for $X = X_2$, for all U. Now for any given U, the point $X = U$ is on both cubics. It is easy to prove that the point $CP(X_2, U)$ also lies on both cubics.

Case 6.2: $\Delta_0 = 0$ and $\Delta_2 \neq 0$. For given U, the isotomic conjugate X of U satisfies $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For a diffferent example, let $U = X_{511}$; then the points listed for Case 5.2 in the cevian case are also points for which $\Delta_0 = 0$ and $\Delta_2 \neq 0$.

Case 6.3: $\Delta_0 \neq 0$ and $\Delta_2 = 0$. In this case, for any U, there is no ppt. For example, take $U = X_{523}$. Then the cubic $\Delta_2 = 0$ passes through the points in which the Brocard axis X_3X_6 meets the circumcircle these being X_{1312} and X_{1313}; these points do not also lie on the cubic $\Delta_0 = 0$.

Case 6.4: $\Delta_0 \neq 0$ and $\Delta_2 \neq 0$. In this case, $D_tE_tF_t$ is perspective to ABC for $t = -\Delta_0/\Delta_2$. The perspector is the point $x_2x_3 : x_2y_3 : z_2x_3$, which on cancellation of common factors, is the point $X' = x' : y' : z'$ given by

$$x' = \frac{by - cz}{bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy)}, \tag{17}$$

$$y' = \frac{cz - ax}{cuz^2 - awx^2 + (cw - au)zx + by(wx - uz)}, \tag{18}$$

$$z' = \frac{ax - by}{avx^2 - buy^2 + (au - bv)xy + cz(uy - vx)}. \tag{19}$$

Theorem 8. *Suppose $X = CP(X_2, U_1)$, where U_1 is a point on L^∞ but not on a sideline BC, CA, AB. Then the perspector X' in Case 6.4 is invariant of the point U. In fact, $X' = CD(X_6, U_1^{-1})$, and X' lies on the circumconic given by*

$$au_1^2(bv_1 - cw_1)\beta\gamma + bv_1^2(cw_1 - au_1)\gamma\alpha + cw_1^2(au_1 - bv_1)\alpha\beta = 0. \tag{20}$$

Proof. Let $X = x : y : z = bu_1v_1 + cu_1w_1 : cv_1w_1 + av_1u_1 : aw_1u_1 + bw_1v_1$. Following the steps of the proof of Theorem 6, we have

$$\begin{aligned} &bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy) \\ =&\widehat{\Lambda}(bv - av - aw + cw + as(b^2v - abv + c^2w - acw), \end{aligned}$$

where

$$\widehat{\Lambda} = 2abc(b - c)(c - a)(a - b)(1 + bcs)(1 + cas)(1 + abs).$$

Thus

$$bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy) = \widehat{\Lambda}(-a(u+v+w) + as(a^2u + b^2v + c^2w)),$$

and by (17),

$$\begin{aligned} x' =& \frac{by - cz}{bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy)} \\ =& \frac{(by - cz)/a}{\widehat{\Lambda}(-(u + v + w) + s(a^2u + b^2v + c^2w))}. \end{aligned}$$

Coordinates y' and z' are found in the same manner, and multiplying through by the common denominator gives

$$x' : y' : z' = CD(X_6, U_1^{-1}),$$

the same point as at the end of the proof of Theorem 6. It is easy to check that this point satisfies (20). □

Corollary 9. *As X traverses the Steiner inellipse, the perspector X' traverses the circumconic (20).*

Proof. The Steiner inellipse is given by

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - 2bc\beta\gamma - 2ca\gamma\alpha - 2ab\alpha\beta = 0. \tag{21}$$

First, we note that, using (16), it is easy to show that if U_1 is on L^∞, then the point $X = x : y : z = CP(X_2, U_1)$ satisfies (21). Now, the mapping $U_1 \rightarrow$

$CP(X_2, U_1) = X$ is invertible; specifically, for given $X = x : y : z$ on the Steiner inellipse, the point $U_1 = u_1 : v_1 : w_1$ given by

$$u_1 : v_1 : w_1 = \frac{bc}{by + cz - ax} : \frac{ca}{cz + ax - by} : \frac{ab}{ax + by - cz}$$

is on L^∞, and Theorem 8 applies. $\square$

Corollary 9 is exemplified by taking $X = X_{1086}$, which is $CP(X_2, X_{514})$; the perspector is then $X' = X_{900} = CD(X_6, X_{101})$. Other examples (X, X') are these: (X_{115}, X_{690}), (X_{1015}, X_{891}), (X_{1084}, X_{888}), (X_{2482}, X_{690}). Note that the mapping $X \to X'$ is not one-to-one.

7. Translation along the Euler line

In this section, the perspectivity problem for both families, cevian and anticevian triangles, is discussed for translations in a single direction, namely the direction of the Euler line. Two points on the Euler line are the circumcenter, $a_1 : b_1 : c_1 = \cos A : \cos B : \cos C$ and

$$U = X_{30} = u : v : w = a_1 - 2b_1c_1 : b_1 - 2c_1a_1 : c_1 - 2a_1b_1,$$

the latter being the point in which the Euler line meets L^∞.

Theorem 10. *If X is the isotomic conjugate of a point X' on the Euler line other than X_2, then the perspector, in the case of the cevian triangle of X as given by (12)-(14), is X'.*

Proof. An arbitrary point X' on the Euler line is given parametrically by

$$a_1 + su : b_1 + sv : c_1 + sw,$$

and the isotomic conjugate $X = x : y : z$ by

$$a^{-2}(a_1 + su)^{-1} : b^{-2}(b_1 + sv)^{-1} : c^{-2}(c_1 + sw)^{-1}.$$

Substituting for x, y, z in (12) gives a product of several factors, of which exactly two involve s. The same holds for the results of substituting in (13) and (14). After canceling all common factors that do not contain s, the remaining coordinates for X' have a common factor $3s + 1$. This equals 0 for $s = -1/3$, for which $a_1 + su : b_1 + sv : c_1 + sw = X_2$. As $X' \neq X_2$, we can and do cancel $3s + 1$. The remaining coordinates are equivalent to those given just above for X'. $\square$

Theorem 11. *Suppose P is on the Euler line and $P \neq X_2$. Let $X = CP(X_2, P)$. Then the perspector X', in the case of the anticevian triangle of X, as given by (17)-(19), is the point P.*

Proof. Write

$$p = a_1 + su, \qquad q = b_1 + sv, \qquad r = c_1 + sw,$$

where $(u, v, w) = (a_1 - 2b_1c_1, b_1 - 2c_1a_1, c_1 - 2a_1b_1)$, so that the point $X = CP(X_2, P)$ is given by

$$x = p(bq + cr), \qquad y = q(cr + ap), \qquad z = r(ap + bq).$$

Substituting into (17)-(19) and factoring give expressions with several common factors. Canceling those, including the factor $3s+1$ which corresponds to the disallowed X_2, leaves trilinears for P. □

Theorem 12. *If P is on the circumcircle and $X = CS(X_6, P)$, then the perspector X', in the case of the anticevian triangle of X as given by (17)-(19), is the point $CD(X_6, P)$.*

Proof. Represent an arbitrary point $P = p : q : r$ on the circumcircle parametrically by

$$(p,q,r) = \left(\frac{1}{(b-c)(bc+s)}, \frac{1}{(c-a)(ca+s)}, \frac{1}{(a-b)(ab+s)}\right).$$

Then the point $X = CS(X_6, P)$ is given by

$$x = br + cq, \qquad y = cp + ar, \qquad z = aq + bp.$$

Substituting into (17)-(19) and factoring gives expressions with several common factors. Canceling those leaves trilinears for $CD(X_6, P)$. □

We conclude this section with a pair of examples. First, let $X = X_{618}$, the complement of the Fermat point (or 1st isogonic center), X_{13}. The perspector in the case of the anticevian triangle of X is the point X_{13}. Finally, let $X = X_{619}$, the complement of the 2nd isogonic center, X_{14}. The perspector in this case is the point X_{14}.

8. Translated rotated reference triangle

Let DEF be the rotation of ABC about the circumcenter of ABC. Let $U = u : v : w$ be a point on L^∞. In this section, we wish to translate DEF in the direction of line DU, seeking translations $D'E'F'$ that are perspective to ABC. Except for rotations of 0 and π, triangle DEF is not perspective to ABC, so that by Theorem 1, there are at most two perspective translations.

Yff's parameterization of the circumcircle ([1, p.39]) is used to express the rotation DEF of ABC counterclockwise with angle 2θ as follows:

$$\begin{aligned} D &= \csc\theta : \csc(C-\theta) : -\csc(B+\theta), \\ E &= -\csc(C+\theta) : \csc\theta : \csc(A-\theta), \\ F &= \csc(B-\theta) : -\csc(A+\theta) : \csc\theta. \end{aligned}$$

Let

$$\begin{aligned} r &= ((a+b+c)(b+c-a)(c+a-b)(a+b-c))^{1/2}/(2abc), \\ \theta_1 &= \sin\theta, \\ \theta_2 &= \cos\theta. \end{aligned}$$

Then the vertices D, E, F are given by the rows of the matrices

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} \theta_1^{-1} & (rc\theta_2 - c_1\theta_1)^{-1} & (rb\theta_2 + b_1\theta_1)^{-1} \\ (rc\theta_2 + c_1\theta_1)^{-1} & \theta_1^{-1} & (ra\theta_2 - a_1\theta_1)^{-1} \\ (rb\theta_2 - b_1\theta_1)^{-1} & (ra\theta_2 + a_1\theta_1)^{-1} & \theta_1^{-1} \end{pmatrix},$$

where

$$\begin{aligned}(a_1, b_1, c_1) =&(\cos A, \cos B, \cos C)\\ =&((b^2+c^2-a^2)/(2bc), (c^2+a^2-b^2)/(2ca), (a^2+b^2-c^2)/(2ab)).\end{aligned}$$

The perspectivity determinant (2) is factored using a computer. Only one of the factors involves t, and it is a polynomial $P(t)$ as in (3), with coefficients

$$\begin{aligned}p_0 =&4abc\theta_1^2,\\ p_1 =&4\theta_1^2 abc\,(au+bv+cw) = 0,\\ p_2 =&(a+b-c)\,(a-b+c)\,(b-a+c)\,(a+b+c)\,(avw+buw+cuv)\,,\end{aligned}$$

hence roots

$$\pm(\theta_1/r)(-abcs)^{-1/2}, \tag{22}$$

where $s = avw + buw + cuv$.

Conjecture. The perspectors given by (22) are a pair of antipodes on the circumcircle.

See Figure 3. [3] It would perhaps be of interest to study, for fixed H, the loci of D', E', F' as θ varies from 0 to π.

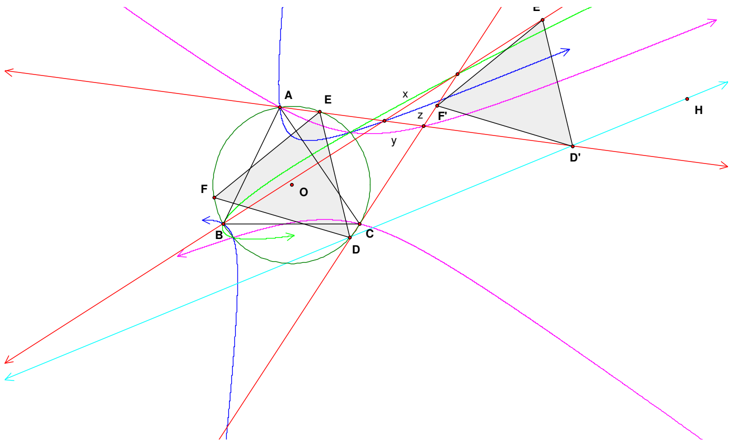

Figure 3. Translated rotation of ABC.

[3] Figure 3 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. Triangle DEF is a variable rotation of ABC about its circumcenter O. Independent point H determines line DH. Point D' is movable on line DH. Triangle $D'E'F'$is thus a movable translation of DEF in the direction of DH. Three conics as in Theorem 4 meet in two points, which according to the Conjecture are a pair of antipodes on the circumcircle.

References

[1] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[2] C. Kimberling, Conics associated with a cevian nest, *Forum Geom.*, 1 (2001) 141–150.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000-, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[4] C. Kimberling, *Glossary*, 2002-, http://faculty.evansville.edu/ck6/encyclopedia/glossary.html.
[5] C. Kimberling, *Points on Cubics*, 2003-, http://faculty.evansville.edu/ck6/encyclopedia/Intro&Zcubics.html .
[6] C. Kimberling, *Translated Triangles*, 2005, http://faculty.evansville.edu/ck6/encyclopedia/TransTri.html .
[7] E. A. Maxwell, *General Homogeneous Coordinates*, Cambridge University Press, Cambridge, 1957.
[8] E. Weisstein, *MathWorld*, http://mathworld.wolfram.com/ .

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