

## On the Derivative of a Vertex Polynomial

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**Abstract.** A geometric characterization of the critical points of a complex polynomial  $f$  is given, in the special case where the zeros of  $f$  are the vertices of a polygon affine-equivalent to a regular polygon.

### 1. Steiner Polygons

The relationship between the locations of the zeros of a complex polynomial  $f$  and those of its derivative has been extensively studied. The best-known theorem in this area is the Gauss-Lucas Theorem, that the zeros of  $f'$  lie in the convex hull of the zeros of  $f$ . The following theorem [1, p.93], due to Linfield, is also of interest:

**Theorem 1.** *Let  $\lambda_j \in \mathbf{R} \setminus \{0\}$ ,  $j = 1, \dots, k$ , and let  $z_j$ ,  $j = 1, \dots, k$  be distinct complex numbers. Then the zeros of the rational function  $R(z) := \sum_{j=1}^k \frac{\lambda_j}{z-z_j}$  are the foci of the curve of class  $k-1$  which touches each of the  $k(k-1)/2$  line segments  $\overline{z_\mu, z_\nu}$  in a point dividing that line segment in the ratio  $\lambda_\mu : \lambda_\nu$ .*

Since  $f' = f \cdot \sum_{j=1}^k \frac{1}{z-z_j}$ , where the  $z_j$  are the zeros of  $f$ , Linfield's Theorem can be used to locate the zeros of  $f'$  which are not zeros of  $f$ .

In this paper, we will consider the case of a polynomial whose zeros form the vertices of a polygon which is affine-equivalent to a regular polygon; the zeros of the derivative can be geometrically characterized in a manner resembling Linfield's Theorem. First, let  $\zeta$  be a primitive  $n$ th root of unity, for some  $n \geq 3$ . Define  $G(\zeta)$  to be the  $n$ -gon whose vertices are  $\zeta^0, \zeta^1, \dots, \zeta^{n-1}$ .

**Proposition 2.** *Let  $n \geq 3$ , and let  $G$  be an  $n$ -gon with vertices  $v_0, \dots, v_{n-1}$ , no three of which are collinear. The following are equivalent.*

- (1) *There is an ellipse which is tangent to the edges of  $G$  at their midpoints.*
- (2)  *$G$  is affine-equivalent to  $G(\zeta)$  for some primitive  $n$ th root of unity  $\zeta$ .*
- (3) *There is a primitive  $n$ th root of unity  $\zeta$  and complex constants  $g, u, v$  such that  $|u| \neq |v|$  and, for  $k = 0, \dots, n-1$ ,  $v_k = g + u\zeta^k + v\zeta^{-k}$ .*

*Proof.* 1) $\implies$ 2): Applying an affine transformation if necessary, we may assume that the ellipse is a circle centered at 0 and that  $v_0 = 1$ . Let  $m_0$  be the midpoint of the edge  $v_0v_1$ .  $v_0v_1$  is then perpendicular to  $Om_0$ , and  $v_0, v_1$  are equidistant from  $m_0$ ; it follows that the right triangles  $Om_0v_0$  and  $Om_0v_1$  are congruent, and in

particular that  $v_1$  also lies on the unit circle. Now let  $m_1$  be the midpoint of  $v_1v_2$ ; since  $m_0$  and  $m_1$  are equidistant from 0 and the triangles  $0m_0v_1, 0m_1v_1$  are right, they are congruent, and  $m_0, m_1$  are equidistant from  $v_1$ . It follows that the edges  $v_0v_1$  and  $v_1v_2$  have the same length. Furthermore, the triangles  $0v_0v_1$  and  $0v_1v_2$  are congruent, whence  $v_2 = v_1^2$ . Similarly we obtain  $v_k = v_1^k$  for all  $k$ , and in particular that  $v_1^n = v_0 = 1$ .  $\zeta = v_1$  is a primitive  $n$ th root of unity since none of  $v_0, \dots, v_{n-1}$  coincide, and  $G = G(\zeta)$ .

2) $\implies$ 1):  $G(\zeta)$  has an ellipse – indeed, a circle – tangent to its edges at their midpoints; an affine transformation preserves this.

2) $\iff$ 3): Any real-linear transformation of  $\mathbf{C}$  can be put in the form  $z \mapsto uz + v\bar{z}$  for some choice of  $u, v$ , and conversely; the transformation is invertible iff  $|u| \neq |v|$ .  $\square$

We will refer to an  $n$ -gon satisfying these conditions as a *Steiner  $n$ -gon*; when needed, we will say it *has root  $\zeta$* . The ellipse is its *Steiner inellipse*. (This is a generalization of the case  $n = 3$ ; every triangle is a Steiner triangle.) The parameters  $g, u, v$  are its *Fourier coordinates*. Note that a Steiner  $n$ -gon is regular iff either  $u$  or  $v$  vanishes.

## 2. The Foci of the Steiner Inellipse

Now, let  $S_\zeta$  be the set of Steiner  $n$ -gons with root  $\zeta$  for which the constant  $g$ , above, is 0. We may use the Fourier coordinates  $u, v$  to identify it with an open subset of  $\mathbf{C}^2$ . Let  $\Phi$  be the map taking the  $n$ -gon with vertices  $v_0, v_1, \dots, v_{n-1}$  to the  $n$ -gon with vertices  $v_1, \dots, v_{n-1}, v_0$ . If  $f$  is a complex-valued function whose domain is a subset of  $S_\zeta$  which is closed under  $\Phi$ , write  $\varphi f$  for  $f \circ \Phi$ . Note that  $\varphi u = \zeta u$  and  $\varphi v = \zeta^{-1}v$ ; this will prove useful. Note also that special points associated with  $n$ -gons may be identified with complex-valued functions on appropriate subsets of  $S_\zeta$ .

We define several useful fields associated with  $S_\zeta$ . First, let  $F = \mathbf{C}(u, v, \bar{u}, \bar{v})$ , where  $u, v$  are as in 3) of the above proposition.  $\varphi$  is an automorphism of  $F$ . Let  $K = \mathbf{C}(x, y, \bar{x}, \bar{y})$  be an extension field of  $F$  satisfying  $x^2 = u, y^2 = v, \bar{x}^2 = \bar{u}, \bar{y}^2 = \bar{v}$ . Let  $\theta$  be a fixed square root of  $\zeta$ ; we extend  $\varphi$  to  $K$  by setting  $\varphi x = \theta x, \varphi y = \theta^{-1}y, \varphi \bar{x} = \theta^{-1}\bar{x}, \varphi \bar{y} = \theta \bar{y}$ . Let  $K_0$  be the fixed field of  $\varphi$  and  $K_1$  the fixed field of  $\varphi^n$ . Elements of  $F$  may be regarded as complex-valued functions defined on dense open subsets of  $S_\zeta$ . Functions corresponding to elements of  $K$  may only be defined locally; however, given  $G \in S_\zeta$  such that  $uv \neq 0$  and  $f \in K_1$  defined at  $G$ , one may choose a small neighborhood  $U_0$  of  $G$  which is disjoint from  $\Phi^k(U_0), k = 1, \dots, n - 1$  and on which neither  $u$  nor  $v$  vanish;  $f$  may then be defined on  $U = \bigcup_{k=0}^{n-1} \Phi^k(U_0)$ .

For the remainder of this section,  $G$  is a fixed Steiner  $n$ -gon with root  $\zeta$ . The vertices of  $G$  are  $v_0, \dots, v_{n-1}$ . We have the following.

**Proposition 3.** *The foci of the Steiner inellipse of  $G$  are located at  $f_\pm = g \pm (\theta + \theta^{-1})xy$ .*

*Proof.* Translating if necessary, we may assume that  $g = 0$ , i.e.,  $G \in S_\zeta$ . Note first that  $f_\pm \in K_0$ . (This is to be expected, since the Steiner inellipse and its foci do not depend on the choice of initial vertex.) For  $k = 0, \dots, n-1$ , let  $m_k = (v_k + v_{k+1})/2$ , the midpoint of the edge  $v_k v_{k+1}$ . Let  $d_\pm$  be the distance from  $f_\pm$  to  $m_0$ ; we will first show that  $d_+ + d_-$  is invariant under  $\varphi$ . (This will imply that the sum of the distances from  $f_\pm$  to  $m_k$  is the same for all  $k$ .) Now,  $m_0 = (v_0 + v_1)/2 = ((1+\zeta)u + (1+\zeta^{-1})v)/2 = (\theta + \theta^{-1})(\theta x^2 + \theta^{-1}y^2)/2$ . Thus,  $m_0 - f_+ = (\theta + \theta^{-1})(\theta x^2 - 2xy + \theta^{-1}y^2)/2 = (\zeta + 1)(x - \theta^{-1}y)^2/2$ . Hence  $d_+ = |m_0 - f_+| = |\zeta + 1|(x - \theta^{-1}y)(\bar{x} - \theta\bar{y})/2$ . Similarly,  $d_- = |\zeta + 1|(x + \theta^{-1}y)(\bar{x} + \theta\bar{y})/2$ , and so  $d_+ + d_- = |\zeta + 1|(x\bar{x} + y\bar{y})$ , which is invariant under  $\varphi$  as claimed. This shows that there is an ellipse with foci  $f_\pm$  passing through the midpoints of the edges of  $G$ . If  $n \geq 5$ , this is already enough to show that this ellipse is the Steiner inellipse; however, for  $n = 3, 4$  it remains to show that this ellipse is tangent to the sides, or, equivalently, that the side  $v_k v_{k+1}$  is the external bisector of the angle  $\angle f_+ m_k f_-$ . It suffices to show that  $A_k = (m_k - v_k)(m_k - v_{k+1})$  is a positive multiple of  $B_k = (m_k - f_+)(m_k - f_-)$ . Now  $A_0 = -(\zeta - 1)^2(u - \zeta^{-1}v)^2/4$ , and  $B_0 = (\zeta + 1)^2(x - \theta^{-1}y)^2(x + \theta^{-1}y)^2/4 = (\zeta + 1)^2(u - \zeta^{-1}v)^2/4$ ; thus,  $A_0/B_0 = -(\zeta - 1)^2/(\zeta + 1)^2 = -(\theta - \theta^{-1})^2/(\theta + \theta^{-1})^2$ , which is evidently positive. This quantity is invariant under  $\varphi$ ; hence  $A_k/B_k$  is also positive for all  $k$ .  $\square$

**Corollary 4.** *The Steiner inellipse of  $G$  is a circle iff  $G$  is similar to  $G(\zeta)$ .*

*Proof.*  $f_+ = f_-$  iff  $xy = 0$ , i.e., iff one of  $u$  and  $v$  is zero. (Note that  $\theta + \theta^{-1} \neq 0$ .)  $\square$

Define the *vertex polynomial*  $f_G(z)$  of  $G$  to be  $\prod_{k=0}^{n-1}(z - v_k)$ . We have the following.

**Proposition 5.** *The foci of the Steiner inellipse of  $G$  are critical points of  $f_G$ .*

*Proof.* Again, we may assume  $G \in S_\zeta$ . Since  $f'_G/f_G = \sum_{k=0}^{n-1}(z - v_k)^{-1}$ , it suffices to show that this sum vanishes at  $f_\pm$ . Now  $f_+$  is invariant under  $\varphi$ , and  $v_k = \varphi^k v_0$ ; hence  $\sum_{k=0}^{n-1}(f_+ - v_k)^{-1} = \sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1}$ .  $(f_+ - v_0)^{-1} = -\theta/((\theta y - x)(y - \theta x))$ . Now let  $g = \theta^2/((\theta^2 - 1)x(\theta y - x))$ . Note that  $g \in K_1$ ; that is,  $\varphi^n g = g$ . A straightforward calculation shows that  $(f_+ - v_0)^{-1} = g - \varphi g$ ; therefore,  $\sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1} = \sum_{k=0}^{n-1}(\varphi^k g - \varphi^{k+1} g) = g - \varphi^n g = 0$ , as desired. The proof that  $f_-$  is a critical point of  $f_G$  is similar.  $\square$

### 3. Holomorphs

Again, we let  $G$  be a Steiner  $n$ -gon with root  $\zeta$  and vertices  $v_0, \dots, v_{n-1}$ . For any integer  $m$ , we set  $v_m = v_l$  where  $l = 0, \dots, n-1$  is congruent to  $m \pmod n$ . The following lemma is trivial.

**Lemma 6.** *Let  $k = 1, \dots, \lfloor n/2 \rfloor$ . Then:*

- (1) *If  $k$  is relatively prime to  $n$ , let  $G^k$  be the  $n$ -gon with vertices  $v_0^k, \dots, v_{n-1}^k$  given by  $v_j^k = v_{jk}$ . Then  $G^k$  is a Steiner  $n$ -gon with root  $\zeta^k$ , and its Fourier coordinates are  $g, u, v$ .*

- (2) If  $d = \gcd(k, n)$  is greater than 1 and less than  $n/2$ , set  $m = n/d$ . Then, for  $l = 0, \dots, d - 1$ , let  $G^{k,l}$  be the  $m$ -gon with vertices  $v_0^{k,l}, \dots, v_{m-1}^{k,l}$  given by  $v_j^{k,l} = v_{kj+l}$ . Then, for each  $l$ ,  $G^{k,l}$  is a Steiner  $m$ -gon with root  $\zeta^k$ , and the Fourier coordinates of  $G^{k,l}$  are  $g, \zeta^l u, \zeta^{-l} v$ . The  $G^{k,l}$  all have the same Steiner inellipse.
- (3) If  $k = n/2$ , the line segments  $v_j v_{j+k}$  all have midpoint  $g$ .

In the three given cases, we will say  $k$ -holomorph of  $G$  to refer to  $G^k$ , the union of the  $m$ -gons  $G^{k,l}$ , or the union of the line segments  $v_j v_{j+k}$ . We extend the definition of *Steiner inellipse* to the  $k$ -holomorphs in Cases 2 and 3, meaning the common Steiner inellipse of the  $G^{k,l}$  or the point  $g$ , respectively. The propositions of Section II clearly extend to Case 2; since the foci are critical points of the vertex polynomials of each of the  $G^{k,l}$ , they are also critical points of their product. In Case 3, taking  $g$  as a degenerate ellipse – indeed, circle – with focus at  $g$ , the propositions likewise extend; in this case,  $\theta = \pm i$ , so  $\theta + \theta^{-1} = 0$ , and the sole critical point of  $(z - v_j)(z - v_{j+k})$  is  $(v_j + v_{j+k})/2 = g$ .

In Cases 1 and 2, it should be noted that the Steiner inellipse is a circle iff the Steiner inellipse of  $G$  itself is a circle – i.e.,  $G$  is similar to  $G(\zeta)$ . It should also be noted that the vertex polynomials of the holomorphs of  $G$  are equal to  $f_G$  itself; hence they have the same critical points. Suppose that  $G$  is not similar to  $G(\zeta)$ . If  $n$  is odd,  $G$  has  $(n - 1)/2$  holomorphs, each with a noncircular Steiner inellipse and hence two distinct Steiner foci; these account for the  $n - 1$  critical points of  $f_G$ . If  $n$  is even,  $G$  has  $(n - 2)/2$  holomorphs in Cases 1 and 2, each with two distinct Steiner foci, and in addition the Case 3 holomorph, providing one more Steiner focus; again, these account for  $n - 1$  critical points of  $f_G$ . On the other hand, if  $G$  is similar to  $G(\zeta)$ , then  $f_G = (z - g)^n - r^n$  for some real  $r$ ; the Steiner foci of the holomorphs of  $G$  collapse together, and  $f_G$  has an  $(n - 1)$ -fold critical point at  $g$ . We have proven the following.

**Theorem 7.** *If  $G$  is a Steiner  $n$ -gon, the critical points of  $f_G$  are the foci of the Steiner inellipses of the holomorphs of  $G$ , counted with multiplicities if  $G$  is regular. They are collinear, lying at the points  $g + (2 \cos k\pi/n)xy$ , as  $k$  ranges from 0 to  $n - 1$ .*

(For the last statement, note that  $\cos(n - k)\pi/n = -\cos k\pi/n$ .)

## Reference

- [1] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Clarendon Press, Oxford, 2002.

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