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On the Derivative of a Vertex Polynomial

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Abstract. A geometric characterization of the critical points of a complex polynomial f is given, in the special case where the zeros of f are the vertices of a polygon affine-equivalent to a regular polygon.

1. Steiner Polygons

The relationship between the locations of the zeros of a complex polynomial f and those of its derivative has been extensively studied. The best-known theorem in this area is the Gauss-Lucas Theorem, that the zeros of f' lie in the convex hull of the zeros of f. The following theorem [1, p.93], due to Linfield, is also of interest:

Theorem 1. *Let $\lambda_j \in \mathbf{R}\backslash\{0\}$, $j = 1, \dots, k$, and let z_j, $j = 1, \dots, k$ be distinct complex numbers. Then the zeros of the rational function $R(z) := \sum_{j=1}^{k} \frac{\lambda_j}{z-z_j}$ are the foci of the curve of class $k-1$ which touches each of the $k(k-1)/2$ line segments $\overline{z_\mu, z_\nu}$ in a point dividing that line segment in the ratio $\lambda_\mu : \lambda_\nu$.*

Since $f' = f \cdot \sum_{j=1}^{k} \frac{1}{z-z_j}$, where the z_j are the zeros of f, Linfield's Theorem can be used to locate the zeros of f' which are not zeros of f.

In this paper, we will consider the case of a polynomial whose zeros form the vertices of a polygon which is affine-equivalent to a regular polygon; the zeros of the derivative can be geometrically characterized in a manner resembling Linfield's Theorem. First, let ζ be a primitive nth root of unity, for some $n \geq 3$. Define $G(\zeta)$ to be the n-gon whose vertices are $\zeta^0, \zeta^1, \dots, \zeta^{n-1}$.

Proposition 2. *Let $n \geq 3$, and let G be an n-gon with vertices $v_0, \dots, v_{n-1}$, no three of which are collinear. The following are equivalent.*

(1) *There is an ellipse which is tangent to the edges of G at their midpoints.*
(2) *G is affine-equivalent to $G(\zeta)$ for some primitive nth root of unity ζ.*
(3) *There is a primitive nth root of unity ζ and complex constants g, u, v such that $|u| \neq |v|$ and, for $k = 0, \dots, n-1$, $v_k = g + u\zeta^k + v\zeta^{-k}$.*

Proof. 1)$\Longrightarrow$2): Applying an affine transformation if necessary, we may assume that the ellipse is a circle centered at 0 and that $v_0 = 1$. Let m_0 be the midpoint of the edge v_0v_1. v_0v_1 is then perpendicular to $0m_0$, and v_0, v_1 are equidistant from m_0; it follows that the right triangles $0m_0v_0$ and $0m_0v_1$ are congruent, and in

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particular that v_1 also lies on the unit circle. Now let m_1 be the midpoint of v_1v_2; since m_0 and m_1 are equidistant from 0 and the triangles $0m_0v_1$, $0m_1v_1$ are right, they are congruent, and m_0, m_1 are equidistant from v_1. It follows that the edges v_0v_1 and v_1v_2 have the same length. Furthermore, the triangles $0v_0v_1$ and $0v_1v_2$ are congruent, whence $v_2 = v_1^2$. Similarly we obtain $v_k = v_1^k$ for all k, and in particular that $v_1^n = v_0 = 1$. $\zeta = v_1$ is a primitive nth root of unity since none of $v_0, \dots, v_{n-1}$ coincide, and $G = G(\zeta)$.

2)$\Longrightarrow$1): $G(\zeta)$ has an ellipse – indeed, a circle – tangent to its edges at their midpoints; an affine transformation preserves this.

2)$\Longleftrightarrow$3): Any real-linear transformation of $\mathbf{C}$ can be put in the form $z \mapsto uz + v\overline{z}$ for some choice of u, v, and conversely; the transformation is invertible iff $|u| \neq |v|$. □

We will refer to an n-gon satisfying these conditions as a *Steiner n-gon*; when needed, we will say it *has root* ζ. The ellipse is its *Steiner inellipse*. (This is a generalization of the case $n = 3$; every triangle is a Steiner triangle.) The parameters g, u, v are its *Fourier coordinates*. Note that a Steiner n-gon is regular iff either u or v vanishes.

2. The Foci of the Steiner Inellipse

Now, let S_ζ be the set of Steiner n-gons with root ζ for which the constant g, above, is 0. We may use the Fourier coordinates u, v to identify it with an open subset of $\mathbf{C}^2$. Let Φ be the map taking the n-gon with vertices $v_0, v_1, \dots, v_{n-1}$ to the n-gon with vertices $v_1, \dots, v_{n-1}, v_0$. If f is a complex-valued function whose domain is a subset of S_ζ which is closed under Φ, write φf for $f \circ \Phi$. Note that $\varphi u = \zeta u$ and $\varphi v = \zeta^{-1} v$; this will prove useful. Note also that special points associated with n-gons may be identified with complex-valued functions on appropriate subsets of S_ζ.

We define several useful fields associated with S_ζ. First, let $F = \mathbf{C}(u, v, \overline{u}, \overline{v})$, where u, v are as in 3) of the above proposition. φ is an automorphism of F. Let $K = \mathbf{C}(x, y, \overline{x}, \overline{y})$ be an extension field of F satisfying $x^2 = u$, $y^2 = v$, $\overline{x}^2 = \overline{u}, \overline{y}^2 = \overline{v}$. Let θ be a fixed square root of ζ; we extend φ to K by setting $\varphi x = \theta x, \varphi y = \theta^{-1} y, \varphi \overline{x} = \theta^{-1} \overline{x}, \varphi \overline{y} = \theta \overline{y}$. Let K_0 be the fixed field of φ and K_1 the fixed field of φ^n. Elements of F may be regarded as complex-valued functions defined on dense open subsets of S_ζ. Functions corresponding to elements of K may only be defined locally; however, given $G \in S_\zeta$ such that $uv \neq 0$ and $f \in K_1$ defined at G, one may choose a small neighborhood U_0 of G which is disjoint from $\Phi^k(U_0)$, $k = 1, \dots, n-1$ and on which neither u nor v vanish; f may then be defined on $U = \bigcup_{k=0}^{n-1} \Phi^k(U_0)$.

For the remainder of this section, G is a fixed Steiner n-gon with root ζ. The vertices of G are $v_0, \dots, v_{n-1}$. We have the following.

Proposition 3. *The foci of the Steiner inellipse of G are located at $f_\pm = g \pm (\theta + \theta^{-1})xy$.*

Proof. Translating if necessary, we may assume that $g = 0$, i.e., $G \in S_\zeta$. Note first that $f_\pm \in K_0$. (This is to be expected, since the Steiner inellipse and its foci do not depend on the choice of initial vertex.) For $k = 0, \ldots, n-1$, let $m_k = (v_k + v_{k+1})/2$, the midpoint of the edge $v_k v_{k+1}$. Let $d_\pm$ be the distance from $f_\pm$ to m_0; we will first show that $d_+ + d_-$ is invariant under φ. (This will imply that the sum of the distances from $f_\pm$ to m_k is the same for all k.) Now, $m_0 = (v_0+v_1)/2 = ((1+\zeta)u+(1+\zeta^{-1})v)/2 = (\theta+\theta^{-1})(\theta x^2+\theta^{-1}y^2)/2$. Thus, $m_0-f_+ = (\theta+\theta^{-1})(\theta x^2-2xy+\theta^{-1}y^2)/2 = (\zeta+1)(x-\theta^{-1}y)^2/2$. Hence $d_+ = |m_0-f_+| = |\zeta+1|(x-\theta^{-1}y)(\overline{x}-\theta\overline{y})/2$. Similarly, $d_- = |\zeta+1|(x+\theta^{-1}y)(\overline{x}+\theta\overline{y})/2$, and so $d_+ + d_- = |\zeta+1|(x\overline{x}+y\overline{y})$, which is invariant under φ as claimed. This shows that there is an ellipse with foci $f_\pm$ passing through the midpoints of the edges of G. If $n \geq 5$, this is already enough to show that this ellipse is the Steiner inellipse; however, for $n = 3, 4$ it remains to show that this ellipse is tangent to the sides, or, equivalently, that the side $v_k v_{k+1}$ is the external bisector of the angle $\angle f_+ m_k f_-$. It suffices to show that $A_k = (m_k - v_k)(m_k - v_{k+1})$ is a positive multiple of $B_k = (m_k - f_+)(m_k - f_-)$. Now $A_0 = -(\zeta-1)^2(u-\zeta^{-1}v)^2/4$, and $B_0 = (\zeta+1)^2(x-\theta^{-1}y)^2(x+\theta^{-1}y)^2/4 = (\zeta+1)^2(u-\zeta^{-1}v)^2/4$; thus, $A_0/B_0 = -(\zeta-1)^2/(\zeta+1)^2 = -(\theta-\theta^{-1})^2/(\theta+\theta^{-1})^2$, which is evidently positive. This quantity is invariant under φ; hence A_k/B_k is also positive for all k. □

Corollary 4. *The Steiner inellipse of G is a circle iff G is similar to $G(\zeta)$.*

Proof. $f_+ = f_-$ iff $xy = 0$, i.e., iff one of u and v is zero. (Note that $\theta + \theta^{-1} \neq 0$.) □

Define the *vertex polynomial* $f_G(z)$ of G to be $\prod_{k=0}^{n-1}(z - v_k)$. We have the following.

Proposition 5. *The foci of the Steiner inellipse of G are critical points of f_G.*

Proof. Again, we may assume $G \in S_\zeta$. Since $f'_G/f_G = \sum_{k=0}^{n-1}(z-v_k)^{-1}$, it suffices to show that this sum vanishes at $f_\pm$. Now f_+ is invariant under φ, and $v_k = \varphi^k v_0$; hence $\sum_{k=0}^{n-1}(f_+ - v_k)^{-1} = \sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1}$. $(f_+ - v_0)^{-1} = -\theta/((\theta y - x)(y - \theta x))$ Now let $g = \theta^2/((\theta^2-1)x(\theta y - x))$. Note that $g \in K_1$; that is, $\varphi^n g = g$. A straightforward calculation shows that $(f_+ - v_0)^{-1} = g - \varphi g$; therefore, $\sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1} = \sum_{k=0}^{n-1}(\varphi^k g - \varphi^{k+1} g) = g - \varphi^n g = 0$, as desired. The proof that f_- is a critical point of f_G is similar. □

3. Holomorphs

Again, we let G be a Steiner n-gon with root ζ and vertices $v_0, \ldots, v_{n-1}$. For any integer m, we set $v_m = v_l$ where $l = 0, \ldots, n-1$ is congruent to m mod n. The following lemma is trivial.

Lemma 6. *Let $k = 1, \ldots, \lfloor n/2 \rfloor$. Then:*

(1) *If k is relatively prime to n, let G^k be the n-gon with vertices $v_0^k, \ldots, v_{n-1}^k$ given by $v_j^k = v_{jk}$. Then G^k is a Steiner n-gon with root ζ^k, and its Fourier coordinates are g, u, v.*

(2) *If* $d = \gcd(k, n)$ *is greater than* 1 *and less than* $n/2$*, set* $m = n/d$*. Then, for* $l = 0, \dots, d-1$*, let* $G^{k,l}$ *be the* m*-gon with vertices* $v_0^{k,l}, \dots, v_{m-1}^{k,l}$ *given by* $v_j^{k,l} = v_{kj+l}$*. Then, for each* l*,* $G^{k,l}$ *is a Steiner* m*-gon with root* ζ^k*, and the Fourier coordinates of* $G^{k,l}$ *are* $g, \zeta^l u, \zeta^{-l} v$*. The* $G^{k,l}$ *all have the same Steiner inellipse.*

(3) *If* $k = n/2$*, the line segments* $v_j v_{j+k}$ *all have midpoint* g*.*

In the three given cases, we will say k*-holomorph of* G to refer to G^k, the union of the m-gons $G^{k,l}$, or the union of the line segments $v_j v_{j+k}$. We extend the definition of *Steiner inellipse* to the k-holomorphs in Cases 2 and 3, meaning the common Steiner inellipse of the $G^{k,l}$ or the point g, respectively. The propositions of Section II clearly extend to Case 2; since the foci are critical points of the vertex polynomials of each of the $G^{k,l}$, they are also critical points of their product. In Case 3, taking g as a degenerate ellipse – indeed, circle – with focus at g, the propositions likewise extend; in this case, $\theta = \pm i$, so $\theta + \theta^{-1} = 0$, and the sole critical point of $(z - v_j)(z - v_{j+k})$ is $(v_j + v_{j+k})/2 = g$.

In Cases 1 and 2, it should be noted that the Steiner inellipse is a circle iff the Steiner inellipse of G itself is a circle – i.e., G is similar to $G(\zeta)$. It should also be noted that the vertex polynomials of the holomorphs of G are equal to f_G itself; hence they have the same critical points. Suppose that G is not similar to $G(\zeta)$. If n is odd, G has $(n-1)/2$ holomorphs, each with a noncircular Steiner inellipse and hence two distinct Steiner foci; these account for the $n-1$ critical points of f_G. If n is even, G has $(n-2)/2$ holomorphs in Cases 1 and 2, each with two distinct Steiner foci, and in addition the Case 3 holomorph, providing one more Steiner focus; again, these account for $n-1$ critical points of f_G. On the other hand, if G is similar to $G(\zeta)$, then $f_G = (z-g)^n - r^n$ for some real r; the Steiner foci of the holomorphs of G collapse together, and f_G has an $(n-1)$-fold critical point at g. We have proven the following.

Theorem 7. *If* G *is a Steiner* n*-gon, the critical points of* f_G *are the foci of the Steiner inellipses of the holomorphs of* G*, counted with multiplicities if* G *is regular. They are collinear, lying at the points* $g + (2\cos k\pi/n)xy$*, as* k *ranges from* 0 *to* $n-1$*.*

(For the last statement, note that $\cos(n-k)\pi/n = -\cos k\pi/n$.)

Reference

[1] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Clarendon Press, Oxford, 2002.

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