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Intersecting Circles and their Inner Tangent Circle

Max M. Tran

Abstract. We derive the general equation for the radius of the inner tangent circle that is associated with three pairwise intersecting circles. We then look at three special cases of the equation.

It seems to the author that there should be one equation that gives the radius of the inner tangent circle inscribed in a triangular region bounded by either straight lines or circular arcs. As a step toward this goal of a single equation, consider three circles $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$ with radii α, β, γ respectively. $\mathcal{C}_A$ intersects $\mathcal{C}_B$ at an angle θ. $\mathcal{C}_B$ intersects $\mathcal{C}_C$ at an angle ρ. And $\mathcal{C}_C$ intersects $\mathcal{C}_A$ at an angle ϕ, with $0 \leqslant \theta, \rho, \phi \leqslant \pi$. We seek the radius of the circle $\mathcal{C}$, tangent externally to each of the given circles. See Figure 1. If the three intersecting cirlces were just touching instead, the inner tangent circle would be the inner Soddy circle. See [1]. The points of tangency of the inner tangent circle form the vertices of an inscribed triangle. We set up a coordinate system with the origin at the center of $\mathcal{C}$. See Figure 1.

Let the points of tangency A, B, C be represented by complex numbers of moduli R, the radius of $\mathcal{C}$. With these labels, the triangle ABC and the inscribed triangle is one and the same. Letting the lengths of the sides BC, CA, AB be a, b, c respectively, then

$$\parallel A - B \parallel = c \quad \text{and} \quad \langle A, B \rangle = R^2 - \frac{c^2}{2}. \tag{1}$$

Corresponding relations hold for the pairs B, C and C, A. With the above coordinate system, the centers of the circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ are respectively $\frac{R+\alpha}{R}A$, $\frac{R+\beta}{R}B$, $\frac{R+\gamma}{R}C$.

The circles $\mathcal{C}_A$ and $\mathcal{C}_B$ intersect at angle θ if and only if

$$\left\| \frac{R+\alpha}{R}A - \frac{R+\beta}{R}B \right\| = \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta.$$

By an application of (1) and the use of a half angle formula, the above can be shown to be equivalent to

$$c^2 = \frac{4R^2\alpha\beta\cos^2\frac{\theta}{2}}{(R+\alpha)(R+\beta)}.$$

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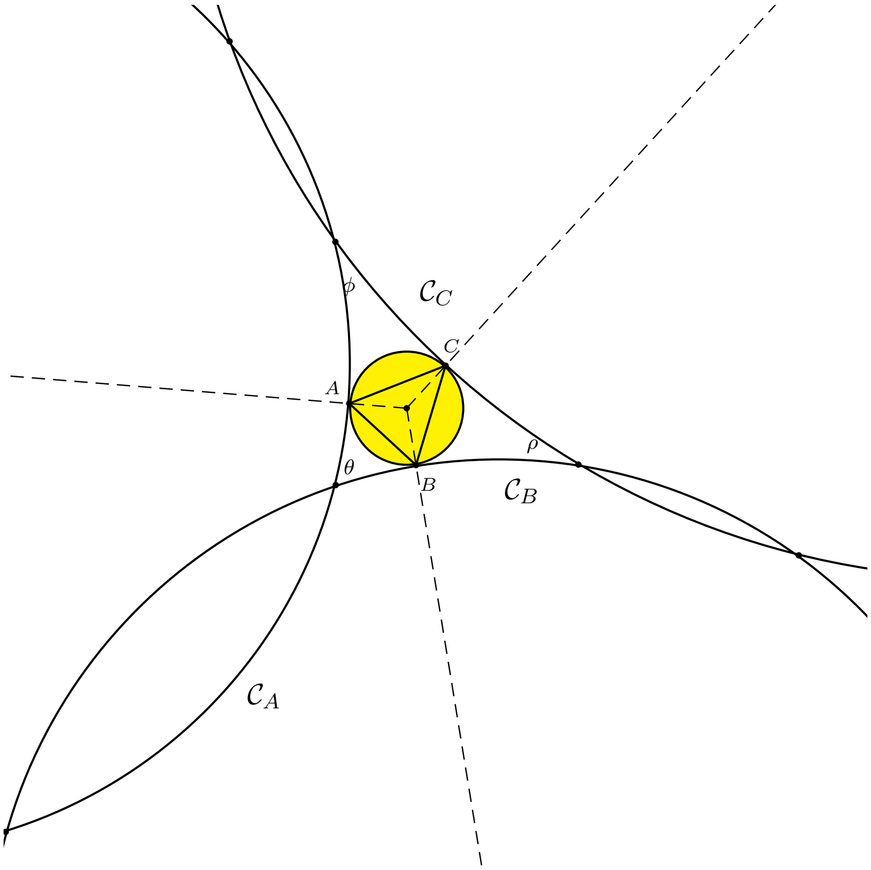

Figure 1

Thus the three circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ intersect each other at the given angles if and only if

$$a^2 = \frac{4R^2\beta\gamma\cos^2\frac{\rho}{2}}{(R+\beta)(R+\gamma)},$$
$$b^2 = \frac{4R^2\alpha\gamma\cos^2\frac{\phi}{2}}{(R+\alpha)(R+\gamma)}, \tag{2}$$
$$c^2 = \frac{4R^2\alpha\beta\cos^2\frac{\theta}{2}}{(R+\alpha)(R+\beta)}.$$

These equations are then used to solve for R in terms of α, β, γ, θ, ϕ and ρ. In the first step of this process, we multiply the equations in (2) and take square root to obtain

$$abc = \frac{8\alpha\beta\gamma R^3\cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\rho}{2}}{(R+\alpha)(R+\beta)(R+\gamma)}. \tag{3}$$

Using (3) and (2) we obtain,

$$\begin{aligned}\frac{\alpha}{R+\alpha} &= \frac{bc\cos\frac{\rho}{2}}{2Ra\cos\frac{\theta}{2}\cos\frac{\phi}{2}},\\ \frac{\beta}{R+\beta} &= \frac{ac\cos\frac{\phi}{2}}{2Rb\cos\frac{\theta}{2}\cos\frac{\rho}{2}},\\ \frac{\gamma}{R+\gamma} &= \frac{ab\cos\frac{\theta}{2}}{2Rc\cos\frac{\rho}{2}\cos\frac{\phi}{2}}.\end{aligned} \tag{4}$$

The area, $\triangle$, of the inscribed triangle ABC is given by

$$\triangle = \frac{abc}{4R}. \tag{5}$$

Consequently, equations (4) and (5) lead to

$$\begin{aligned}a^2 &= \frac{(R+\alpha)\triangle\cos\frac{\rho}{2}}{\alpha\cos\frac{\theta}{2}\cos\frac{\phi}{2}},\\ b^2 &= \frac{(R+\beta)\triangle\cos\frac{\phi}{2}}{\beta\cos\frac{\theta}{2}\cos\frac{\rho}{2}},\\ c^2 &= \frac{(R+\gamma)\triangle\cos\frac{\theta}{2}}{\gamma\cos\frac{\rho}{2}\cos\frac{\phi}{2}}.\end{aligned} \tag{6}$$

Now, Heron's formula for the triangle ABC can be written in the form

$$16\triangle^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4.$$

Using the above equation together with equations (6) will enable us to get an equation for R in terms of the parameters of the intersecting circles. This process involves substituting the value of a^2, b^2, c^2 into Heron's formula, dividing by $\triangle^2$, and performing a lengthy algebraic manipulation to yield the equation:

$$\begin{aligned}0 = \frac{1}{R^2}&\Big[4\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2}\cos^2\frac{\phi}{2} + \cos^4\frac{\phi}{2} + \cos^4\frac{\rho}{2} + \cos^4\frac{\theta}{2}\\ &\quad -2\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2} - 2\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2} - 2\cos^2\frac{\phi}{2}\cos^2\frac{\rho}{2}\Big]\\ -\frac{1}{R}&\Big[\frac{2\cos^2\frac{\rho}{2}}{\alpha}(\cos^2\frac{\theta}{2} + \cos^2\frac{\phi}{2} - \cos^2\frac{\rho}{2})\\ &\quad + \frac{2\cos^2\frac{\phi}{2}}{\beta}(\cos^2\frac{\theta}{2} + \cos^2\frac{\rho}{2} - \cos^2\frac{\phi}{2})\\ &\quad + \frac{2\cos^2\frac{\theta}{2}}{\gamma}(\cos^2\frac{\phi}{2} + \cos^2\frac{\rho}{2} - \cos^2\frac{\theta}{2})\Big]\\ &+ \frac{\cos^4\frac{\rho}{2}}{\alpha^2} + \frac{\cos^4\frac{\phi}{2}}{\beta^2} + \frac{\cos^4\frac{\theta}{2}}{\gamma^2}\\ &\quad - \frac{2\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}}{\beta\gamma} - \frac{2\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2}}{\alpha\gamma} - \frac{2\cos^2\frac{\phi}{2}\cos^2\frac{\rho}{2}}{\alpha\beta}.\end{aligned} \tag{7}$$

Although the equation can be formal solved in general, it is rather unwieldy. Let us consider some special cases.

When the three circles $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$ are mutually tangent, θ, ρ and ϕ equals zero, thus giving the equation:

$$0 = \frac{1}{R^2} - \frac{2}{R}\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] + \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\alpha\gamma}.$$

Solving for $\frac{1}{R}$ gives the standard Descartes formula for the Inner Soddy circle. See [2].

When $\mathcal{C}_C$ is a line tangent to $\mathcal{C}_A$ and $\mathcal{C}_B$, we have $\beta = \infty$ and $\theta = \rho = 0$, and equation (7) becomes

$$0 = \frac{1}{R^2}\left[\cos^4\frac{\phi}{2}\right] - \frac{2\cos^2\frac{\phi}{2}}{R}\left[\frac{1}{\alpha} + \frac{1}{\gamma}\right] + \left[\frac{1}{\alpha} - \frac{1}{\gamma}\right]^2.$$

Solving for 1/R, and using the fact that $\frac{1}{R} > \frac{1}{\alpha}$ and $\frac{1}{R} > \frac{1}{\gamma}$, gives the equation

$$\frac{1}{R} = \frac{1}{\cos^2\frac{\phi}{2}}\left[\frac{1}{\alpha} + \frac{1}{\gamma} + 2\sqrt{\frac{1}{\alpha\gamma}}\right].$$

When the circles $\mathcal{C}_A$ and $\mathcal{C}_C$ are lines that intersect at an angle $\phi > 0$ and are both tangent to the circle $\mathcal{C}_B$, we get a cone and equation (7) becomes

$$0 = \frac{1}{R^2}\left[\cos^4\frac{\phi}{2}\right] - \frac{2\cos^2\frac{\phi}{2}}{\gamma R}\left[2 - \cos^2\frac{\phi}{2}\right] + \left[\frac{\cos^2\frac{\phi}{2}}{\gamma}\right]^2.$$

After solving for 1/R, using some trigonometric identities and the fact that $\frac{1}{R} > \frac{1}{\gamma}$, we get the equation

$$\frac{1}{R} = \frac{2}{\gamma}\left[\frac{1 + \sin\frac{\phi}{2}}{1 - \sin\frac{\phi}{2}}\right]^2,$$

the same as obtained from working with the cone directly.

Unfortunately, equation (7) no longer gives any useful result when all three circles, $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$, becomes lines. The inner tangent circle in this case is just the inscribed circle in the triangle.

References

[1] H. S. M. Coxeter, *Introduction to Geometry, Second Edition,* John Wiley and Sons, New york 1969.

[2] W. Reyes, The Lucas Circles and the Descartes Formula, *Forum Geom.*, 3 (2003) 95–100.

Max M. Tran: Mathematics and Computer Science Department, Kingsborough Community College, 2001 Oriental Boulevard, Brooklyn, NY 11235-2398, USA

E-mail address: `mtran@kingsborough.edu`