Square Wreaths Around Hexagons

Floor van Lamoen

Abstract. We investigate the figures that arise when squares are attached to a
triple of non-adjacent sides of a hexagon, and this procedure is repeated with
alternating choice of the non-adjacent sides. As a special case we investigate the
figure that starts with a triangle.

1. Square wreaths around hexagons

Consider a hexagon $\mathcal{H}_1 = H_{1,1}H_{2,1}H_{3,1}H_{4,1}H_{5,1}H_{6,1}$ with counterclockwise
orientation. We attach squares externally on the sides $H_{1,1}H_{2,1}$, $H_{3,1}H_{4,1}$ and
$H_{5,1}H_{6,1}$, to form a new hexagon $\mathcal{H}_2 = H_{1,2}H_{2,2}H_{3,2}H_{4,2}H_{5,2}H_{6,2}$. Following
Nottrot, [8], we say we have made the first square wreath around $\mathcal{H}_1$. Then we
attach externally squares to the sides $H_{6,2}H_{1,2}$, $H_{2,2}H_{3,2}$ and $H_{4,2}H_{5,2}$, to get a
third hexagon $\mathcal{H}_3$, creating the second square wreath. We may repeat this operation
to find a sequence of hexagons $\mathcal{H}_n = H_{1,n}H_{2,n}H_{3,n}H_{4,n}H_{5,n}H_{6,n}$ and square
wreaths. See Figure 1.

![Figure 1](image)

We introduce complex number coordinates, and abuse notations by identifying
a point with its affix. Thus, we shall also regard $H_{m,n}$ as a complex number, the
first subscript $m$ taken modulo 6.
Assuming a standard orientation of the given hexagon \( \mathcal{H}_0 \) in the complex plane, we easily determine the vertices of the hexagons in the above iterations.

If \( n \) is even, then \( H_{1,n}, H_{2,n}, H_{3,n}, H_{4,n}, H_{5,n}, H_{6,n} \) are the opposite sides of the squares erected on \( H_{1,n-1}H_{2,n-1}, H_{3,n-1}H_{4,n-1} \) and \( H_{5,n-1}H_{6,n-1} \) respectively. This means, for \( k = 1, 2, 3, \)

\[
H_{2k-1,n} = H_{2k-1,n-1} - i(H_{2k,n-1} - H_{2k-1,n-1})
= (1 + i)H_{2k-1,n-1} - i \cdot H_{2k,n-1},
\]

\[
H_{2k,n} = H_{2k,n-1} + i(H_{2k-1,n-1} - H_{2k,n-1})
= i \cdot H_{2k,n-1} + (1 - i)H_{2k,n-1}.
\]

If \( n \) is odd, then \( H_{2,n}H_{3,n}, H_{4,n}H_{5,n}, H_{6,n}H_{1,n} \) are the opposite sides of the squares erected on \( H_{2,n-1}H_{3,n-1}, H_{4,n-1}H_{5,n-1} \) and \( H_{6,n-1}H_{1,n-1} \) respectively. This means, for \( k = 1, 2, 3, \) reading first subscripts modulo 6, we have

\[
H_{2k,n} = (1 + (-1)^n i)H_{2k,n-1} + (-1)^n i \cdot H_{2k+1(-1)^n,n-1},
\]

\[
H_{2k+1,n} = (1 + (-1)^n i)H_{2k+1,n-1} + (-1)^{n+1} i \cdot H_{2k+2(-1)^n,n-1},
\]

or even more succinctly,

\[
H_{m,n} = (1 + (-1)^n i)H_{m,n-1} + (-1)^{m+n} i \cdot H_{m+(-1)^n,n}.
\]

**Proposition 1.** Triangles \( H_{1,n}H_{3,n}H_{5,n} \) and \( H_{1,n-2}H_{3,n-2}H_{5,n-2} \) have the same centroid, so do triangles \( H_{2,n}H_{4,n}H_{6,n} \) and \( H_{2,n-2}H_{4,n-2}H_{6,n-2} \).

**Proof.** Applying the relations \( 1, 2, 3, 4 \) twice, we have

\[
H_{1,n} = -(1 - i)H_{6,n-2} + 2H_{1,n-2} + (1 - i)H_{2,n-2} - H_{3,n-2},
\]

\[
H_{3,n} = -(1 - i)H_{2,n-2} + 2H_{3,n-2} + (1 - i)H_{4,n-2} - H_{5,n-2},
\]

\[
H_{5,n} = -(1 - i)H_{4,n-2} + 2H_{5,n-2} + (1 - i)H_{6,n-2} - H_{1,n-2}.
\]

The triangle \( H_{1,n}H_{3,n}H_{5,n} \) has centroid

\[
\frac{1}{3}(H_{1,n} + H_{3,n} + H_{5,n}) = \frac{1}{3}(H_{1,n-2} + H_{3,n-2} + H_{5,n-2}),
\]

which is the centroid of triangle \( H_{1,n-2}H_{3,n-2}H_{5,n-2} \). The proof for the other pair is similar. □
Theorem 2. For each $m = 1, 2, 3, 4, 5, 6$, the sequence of vertices $H_{m,n}$ satisfies the recurrence relation

$$H_{m,n} = 6H_{m,n-2} - 6H_{m,n-4} + H_{m,n-6}. \quad (5)$$

Proof. By using the recurrence relations (1, 2, 3, 4), we have

\[
\begin{align*}
H_{1,2} &= (1 + i)H_{1,1} - iH_{2,1}, \\
H_{2,3} &= 2H_{1,1} - (1 + i)H_{2,1} - H_{5,1} + (1 + i)H_{6,1}, \\
H_{1,4} &= 3(1 + i)H_{1,1} - 4iH_{2,1} - (1 + i)H_{3,1} + iH_{4,1} - (1 + i)H_{5,1} + 2iH_{6,1}, \\
H_{1,5} &= 8H_{1,1} - 5(1 + i)H_{2,1} - H_{4,1} - 6H_{5,1} + 5(1 + i)H_{6,1}, \\
H_{1,6} &= 13(1 + i)H_{1,1} - 18iH_{2,1} - 6(1 + i)H_{3,1} + 6iH_{4,1} - 6(1 + i)H_{5,1} + 11iH_{6,1}, \\
H_{1,7} &= 37H_{1,1} - 24(1 + i)H_{2,1} - 6H_{3,1} - 30H_{5,1} + 24(1 + i)H_{6,1}.
\end{align*}
\]

Elimination of $H_{m,1}, m = 2, 3, 4, 5, 6$, from these equations gives

$$H_{1,7} = 6H_{1,5} - 6H_{1,3} + H_{1,1}.$$ 

The same relations hold if we simultaneously increase each first subscript by 2, or each second subscript by 1. Thus, we have the recurrence relation (5) for $m = 1, 3, 5$. Similarly,

\[
\begin{align*}
H_{2,2} &= iH_{1,1} + (1 - i)H_{2,1}, \\
H_{2,3} &= - (1 - i)H_{1,1} + 2H_{2,1} + (1 - i)H_{3,1} - H_{4,1}, \\
H_{2,4} &= 4iH_{1,1} + 3(1 - i)H_{2,1} - 2iH_{3,1} - (1 - i)H_{4,1} - iH_{5,1} - (1 - i)H_{6,1}, \\
H_{2,5} &= - 5(1 - i)H_{1,1} + 8H_{2,1} + 5(1 - i)H_{3,1} - 6H_{4,1} - H_{6,1}, \\
H_{2,6} &= 18iH_{1,1} + 13(1 - i)H_{2,1} - 11iH_{3,1} - 6(1 - i)H_{4,1} - 6iH_{5,1} - 6(1 - i)H_{6,1}, \\
H_{2,7} &= - 24(1 - i)H_{1,1} + 37H_{2,1} + 24(1 - i)H_{3,1} - 30H_{4,1} - 6H_{6,1}.
\end{align*}
\]

Elimination of $H_{m,1}, m = 1, 3, 4, 5, 6$, from these equations gives

$$H_{2,7} - 6H_{2,5} + 6H_{2,3} - H_{2,1} = 0.$$ 

A similar reasoning shows that (5) holds for $m = 2, 4, 6$. \(\square\)

2. Midpoint triangles

Let $M_1, M_2, M_3$ be the midpoints of $H_{4,1}H_{5,1}, H_{6,1}H_{1,1}$ and $H_{2,1}H_{3,1}$, and $M'_1, M'_2, M'_3$ the midpoints of $H_{1,3}H_{2,3}, H_{3,3}H_{4,3}$ and $H_{5,3}H_{6,3}$ respectively. We have

\[
M'_1 = \frac{1}{2} (H_{1,3} + H_{2,3}) \\
= \frac{1}{2} ((1 + i)H_{1,1} + (1 - i)H_{2,1} + (1 - i)H_{3,1} - H_{4,1} - H_{5,1} + (1 + i)H_{6,1}) \\
= - M_1 + (1 + i)M_2 + (1 - i)M_3.
\]
Similarly,
\[ M'_2 = (1 - i)M_1 - M_2 + (1 + i)M_3, \]
\[ M'_3 = (1 + i)M_1 + (1 - i)M_2 - M_3. \]

**Proposition 3.** For a permutation \((j,k,\ell)\) of the integers \(1,2,3\), the segments \(M_jM'_k\) and \(M_kM'_j\) are perpendicular to each other and equal in length, while \(M_\ell M'_j\) is parallel to an angle bisector of \(M_jM'_k\) and \(M_kM'_j\), and is \(\sqrt{2}\) times as long as each of these segments.

**Proof.** From the above expressions for \(M'_j\), \(j = 1,2,3\), we have
\[ M'_2 - M_3 = (1 - i)M_1 - M_2 + i \cdot M_3, \quad (6) \]
\[ M'_3 - M_2 = (1 + i)M_1 - i \cdot M_2 - M_3, \]
\[ = i ((1 - i)M_1 - M_2 + iM_3), \]
\[ = i(M'_2 - M_3); \quad (7) \]
\[ M_1 - M'_1 = 2M_1 - (1 + i)M_2 - (1 - i)M_3 \]
\[ = (M'_2 - M_3) + (M'_3 - M_2). \quad (8) \]

From (6) and (7), \(M_2M'_3\) and \(M_3M'_2\) are perpendicular and have equal lengths. From (8), we conclude that \(M'_1M_1\) is parallel to an angle bisector of \(M_2M'_3\) and \(M_3M'_2\), and is \(\sqrt{2}\) times as long as each of these segments. The same results for \((k,\ell) = (3,1), (1,2)\) follow similarly. \(\square\)

The midpoints of the segments \(M_jM'_j\), \(j = 1,2,3\), are the points
\[ N_1 = \frac{1}{2} ((1 + i)M_2 + (1 - i)M_3), \]
\[ N_2 = \frac{1}{2} ((1 + i)M_3 + (1 - i)M_1), \]
\[ N_3 = \frac{1}{2} ((1 + i)M_1 + (1 - i)M_2). \]

Note that
\[ N_1 = \frac{M_2 + M_3}{2} + i \cdot \frac{M_2 - M_3}{2}, \]
\[ = \frac{M'_2 + M'_3}{2} - i \cdot \frac{M'_2 - M'_3}{2}. \]

Thus, \(N_1\) is the center of the square constructed externally on the side \(M_2M_3\) of triangle \(M_1M_2M_3\), and also the center of the square constructed internally on \(M'_1M'_2M'_3\). Similarly, for \(N_2\) and \(N_3\). From this we deduce the following corollaries. See Figure 2.
Corollary 4. The triangles $M_1M_2M_3$ and $M'_1M'_2M'_3$ are perspective. The perspector is the outer Vecten point of $M_1M_2M_3$ and the inner Vecten point of $M'_1M'_2M'_3$.  

Corollary 5. The segments $M_jN_\ell$ and $M_kN_\ell$ are equal in length and are perpendicular. The same is true for $M'_jN_\ell$ and $M'_kN_\ell$.

Let $M''_1M''_2M''_3$ be the desmic mate of $M_1M_2M_3$ and $M'_1M'_2M'_3$, i.e., $M''_1 = M_2M_3' \cap M_3M_2'$ etc. By Proposition 3, $\angle M_2M'_1M_3$ is a right angle, and $M''_1$ lies on the circle with diameter $M_2M_3$. Since the bisector angle $M_2M'_1M_3$ is parallel to the line $N_1M_1$, $M''_1N_1$ is perpendicular to this latter line. See Figure 3.

Proposition 6. If $(j,k,\ell)$ is a permutation of $1,2,3$, the lines $M_jM'_k$ and $M_kM'_j$ and the line through $N_\ell$ perpendicular to $M_\ell M'_\ell$ are concurrent (at $M''_\ell$).

Corollary 7. The circles $(M_jM_kN_\ell)$, $(M'_jM'_kN_\ell)$, $(M_\ell M''_\ell N_\ell)$ and $(M'_\ell M''_\ell N_\ell)$ are coaxial, so the midpoints of $M_jM_k$, $M'_jM'_k$, $M_\ell M''_\ell$ and $M'_\ell M''_\ell$ are collinear, the line being parallel to $M_\ell M'_\ell$.

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1The outer (respectively inner) Vecten point is the point $X_{485}$ (respectively $X_{486}$) of [4].
Since the lines $M_j M'_j, j = 1, 2, 3$, concur at the outer Vecten point of triangle $M_1 M_2 M_3$, the intersection of the lines is the inferior (complement) of the outer Vecten point.\(^2\) As such, it is the center of the circle through $N_1 N_2 N_3$ (see [4]).

**Corollary 8.** The three lines joining the midpoints of $M_1 M'_1, M_2 M'_2, M_3 M'_3$ are concurrent at the center of the circle through $N_1, N_2, N_3$, which also passes through $M''_1, M'_2$ and $M''_3$.

![Diagram](image)

**Figure 3**

### 3. Starting with a triangle

An interesting special case occurs when the initial hexagon $\mathcal{H}_6$ degenerates into a triangle with

$$H_{1,1} = H_{6,1} = A, \quad H_{2,1} = H_{3,1} = B, \quad H_{4,1} = H_{5,1} = C.$$  

This case has been studied before by Haight and Nottrot, who examined especially the side lengths and areas of the squares in each wreath. Under this assumption,
between two consecutive hexagons $\mathcal{H}_0$ and $\mathcal{H}_{n+1}$ are three squares and three alternating trapezoids of equal areas. The trapezoids between $\mathcal{H}_0$ and $\mathcal{H}_2$ degenerate into triangles. The sides of the squares are parallel and perpendicular to the sides or to the medians of triangle $ABC$ according as $n$ is odd or even. We shall assume the sidelengths of triangle $ABC$ to be $a$, $b$, $c$, and the median lengths $m_a$, $m_b$, $m_c$ respectively.

The squares of the first wreath are attached to the triangle sides outwardly.\(^3\) Haight [2] has computed the ratios of the sidelengths of the squares.

If $n = 2k - 1$, the squares have sidelengths $a$, $b$, $c$ multiplied by $a_1(k)$, where

$$a_1(k) = 5a_1(k - 1) - a_1(k - 2),$$

$$a_1(1) = 1, \quad a_1(2) = 4.$$  

This is sequence A004253 in Sloane’s Online Encyclopedia of Integer sequences [9]. This also means that

$$H_{m,2k} - H_{m,2k-1} = a_1(k)(H_m,2 - H_{m,1}). \tag{9}$$

If $n = 2k$, the squares have sidelengths $2m_a$, $2m_b$, $2m_c$ multiplied by $a_2(k)$, where

$$a_2(k) = 5a_2(k - 1) - a_2(k - 2),$$

$$a_2(1) = 1, \quad a_2(2) = 5.$$  

This is sequence A004254 in [9]. This also means that

$$H_{m,2k+1} - H_{m,2k} = a_2(k)(H_{m,3} - H_{m,2}). \tag{10}$$

**Proposition 9.** Each trapezoid in the wreath bordered by $\mathcal{H}_0$ and $\mathcal{H}_{n+1}$ has area $a_2(n) \cdot \triangle ABC$.

**Lemma 10.** (1) $\sum_{j=1}^{k} a_1(j) = a_2(k)$.  
(2) The sums $a_3(k) = \sum_{j=1}^{k} a_2(j)$ satisfy the recurrence relation

$$a_3(k) = 6a_3(k - 1) - 6a_3(k - 2) + a_3(k - 3),$$

$$a_3(1) = 1, \quad a_3(2) = 6, \quad a_3(3) = 30.$$  

The sequence $a_3(k)$ is essentially sequence A089817 in [9].\(^4\)

It follows from (9) and (10) that

$$H_{m,2k} = H_{m,1} + \sum_{j=1}^{k} (H_{m,2j} - H_{m,2j-1}) + \sum_{j=1}^{k-1} (H_{m,2j+1} - H_{m,2j})$$

$$= H_{m,1} + \left( \sum_{j=1}^{k} \left( \begin{array}{c} a_1(j) \end{array} \right) \right) (H_{m,2} - H_{m,1}) + \left( \sum_{j=1}^{k-1} a_2(j) \right) (H_{m,3} - H_{m,2})$$

$$= H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k - 1)(H_{m,3} - H_{m,2}).$$

\(^3\) Similar results as those in §3, 4 can be found if these initial squares are constructed inwardly.

\(^4\) Note that sequences $a_1$ and $a_2$ follow this third order recurrence relation as well.
Also,

\[ H_{m,2k+1} = H_{m,1} + \sum_{j=1}^{k} (H_{m,2j} - H_{m,2j-1}) + \sum_{j=1}^{k} (H_{m,2j+1} - H_{m,2j}) \]

\[ = H_{m,1} + \left( \sum_{j=1}^{k} a_1(j) \right) (H_{m,2} - H_{m,1}) + \left( \sum_{j=1}^{k} a_2(j) \right) (H_{m,3} - H_{m,2}) \]

\[ = H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k)(H_{m,3} - H_{m,2}) \]

Here is a table of the absolute barycentric coordinates (with respect to triangle \(ABC\)) of the initial values in the above recurrence relations.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(H_{m,1})</th>
<th>(H_{m,2} - H_{m,1})</th>
<th>(H_{m,3} - H_{m,2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((1, 0, 0))</td>
<td>((1, -1, 0))</td>
<td>((-1, 0, 0))</td>
</tr>
<tr>
<td>2</td>
<td>((0, 1, 0))</td>
<td>((-a_3, -a_4, -c_2))</td>
<td>((-1, 0, 0))</td>
</tr>
<tr>
<td>3</td>
<td>((0, 1, 0))</td>
<td>((-a_3, -a_4, c_2))</td>
<td>((-1, 0, 0))</td>
</tr>
<tr>
<td>4</td>
<td>((0, 1, 0))</td>
<td>((-a_3, -a_4, -c_2))</td>
<td>((-1, 0, 0))</td>
</tr>
<tr>
<td>5</td>
<td>((0, 0, 1))</td>
<td>((-a_3, -b^2, -a_4))</td>
<td>((-1, 0, 0))</td>
</tr>
<tr>
<td>6</td>
<td>((1, 0, 0))</td>
<td>((-a_3, -b^2, a_4))</td>
<td>((-1, 0, 0))</td>
</tr>
</tbody>
</table>

From these we have the homogeneous barycentric coordinates

\[ H_{m,2k} = H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k - 1)(H_{m,3} - H_{m,2}), \]

\[ H_{m,2k+1} = H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k)(H_{m,3} - H_{m,2}). \]

These can be combined into a single relation

\[ H_{m,n} = H_{m,1} + a_2(n')(H_{m,2} - H_{m,1}) + a_3(n'')(H_{m,3} - H_{m,2}), \]

in which \(n' = \left\lfloor \frac{n}{2} \right\rfloor\) and \(n'' = \left\lfloor \frac{n-1}{2} \right\rfloor\).

Here are the coordinates of the points \(H_{m,n}\).

<table>
<thead>
<tr>
<th>(m)</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((2a_3(n') + 1)S + a_2(n')S_B)</td>
<td>(-a_3(n')S + a_2(n')S_A)</td>
<td>(-a_3(n')S - a_2(n')c_2)</td>
</tr>
<tr>
<td>2</td>
<td>(-a_3(n')S + a_2(n')S_B)</td>
<td>((2a_3(n'') + 1)S + a_2(n')S_A)</td>
<td>(-a_3(n''')S - a_2(n'')c_2^2)</td>
</tr>
<tr>
<td>3</td>
<td>(-a_3(n''')S - a_2(n'')a^2)</td>
<td>((2a_3(n''') + 1)S + a_2(n')S_C)</td>
<td>(-a_3(n''')S + a_2(n')S_B)</td>
</tr>
<tr>
<td>4</td>
<td>(-a_3(n''')S - a_2(n'')a^2)</td>
<td>(-a_3(n''')S + a_2(n')S_C)</td>
<td>((2a_3(n''') + 1)S + a_2(n')S_B)</td>
</tr>
<tr>
<td>5</td>
<td>(-a_3(n''')S + a_2(n')S_C)</td>
<td>(-a_3(n''')S - a_2(n')b^2)</td>
<td>((2a_3(n''') + 1)S + a_2(n')S_A)</td>
</tr>
<tr>
<td>6</td>
<td>((2a_3(n'') + 1)S + a_2(n')S_C)</td>
<td>(-a_3(n''')S + a_2(n')b^2)</td>
<td>(-a_3(n''')S + a_2(n')S_A)</td>
</tr>
</tbody>
</table>

Note that the coordinate sum of each of the points in the above is equal to \(S\).

Consider the midpoints of the following segments

<table>
<thead>
<tr>
<th>segment</th>
<th>(H_{1,n}H_{2,n})</th>
<th>(H_{2,n}H_{3,n})</th>
<th>(H_{3,n}H_{4,n})</th>
<th>(H_{4,n}H_{5,n})</th>
<th>(H_{5,n}H_{6,n})</th>
<th>(H_{6,n}H_{1,n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>midpoint</td>
<td>(C_{1,n})</td>
<td>(B_{2,n})</td>
<td>(A_{1,n})</td>
<td>(C_{2,n})</td>
<td>(B_{1,n})</td>
<td>(A_{2,n})</td>
</tr>
</tbody>
</table>
For \( j = 1, 2 \), denote by \( T_{j,n} \) the triangle with vertices \( A_{j,n}B_{j,n}C_{j,n} \).

\[
\begin{array}{llll}
A_{1,n} & -2a_3(n')^2S - 2a_2(n')a^2 & (a_3(n') + 1)S + 2a_2(n')SC & (a_3(n'') + 1)S + 2a_2(n')SB \\
B_{1,n} & (a_3(n') + 1)S + 2a_2(n')SC & -2a_3(n')S - 2a_2(n')b^2 & (a_3(n') + 1)S + 2a_2(n')SA \\
C_{1,n} & (a_3(n') + 1)S + 2a_2(n')SB & -2a_3(n')S - 2a_2(n')c^2 &
\end{array}
\]

\[
\begin{array}{llll}
A_{2,n} & -2a_3(n'' + 1)S - a_2(n')a^2 & 2a_3(n')S + a_2(n')SC & 2a_3(n')S + a_2(n')SB \\
B_{2,n} & 2a_3(n')S + a_2(n')SC & -2(2a_3(n') + 1)S - a_2(n')b^2 & 2a_3(n')S + a_2(n')SA \\
C_{2,n} & 2a_3(n')S + a_2(n')SB & 2a_3(n')S + a_2(n')SA & -2(2a_3(n') + 1)S - a_2(n')c^2 \\
\end{array}
\]

**Proposition 11.** The triangles \( T_{1,n} \) and \( T_{2,n} \) are perspective.

This is a special case of the following general result.

**Theorem 12.** Every triangle of the form

\[
\begin{align*}
-2fS - ga^2 & \quad : \quad (f + 1)S + gSC & \quad : \quad (f + 1)S + gSB \\
(f + 1)S + gSC & \quad : \quad -2fS - gb^2 & \quad : \quad (f + 1)S + gSA \\
(f + 1)S + gSB & \quad : \quad (f + 1)S + gSA & \quad : \quad -2fS - gc^2
\end{align*}
\]

where \( f \) and \( g \) represent real numbers, is perspective with the reference triangle. Any two such triangles are perspective.

**Proof.** Clearly the triangle given above is perspective with \( ABC \) at the point

\[
\left( \frac{1}{(f + 1)S + gSA}, \frac{1}{(f + 1)S + gSB}, \frac{1}{(f + 1)S + gSC} \right),
\]

which is the Kiepert perspector \( K(\phi) \) for \( \phi = \cot^{-1} \frac{f+1}{g} \).

Consider a second triangle of the same form, with \( f \) and \( g \) replaced by \( p \) and \( q \) respectively. We simply give a description of this perspector \( P \). This perspector is the centroid if and only if \((g - q) + 3(gp -fq) = 0 \). Otherwise, the line joining this perspector to the centroid \( G \) intersects the Brocard axis at the point

\[
Q = (a^2((g - q)SA - (f - p)S) : b^2((g - q)SB - (f - p)S) : c^2((g - q)SC - (f - p)S)),
\]

which is the isogonal conjugate of the Kiepert perspector \( K\left(-\cot^{-1} \frac{f-p}{g-q}\right) \). The perspector \( P \) in question divides \( GQ \) in the ratio

\[
GP : GQ = \frac{(g - q) + 3(gp - fq))((g - q)S - (f - p)S_\omega)}{(3(f - p)^2 + (g - q)^2)S + 2(f - p)(g - q)S_\omega}.
\]

Note that \( T_{j,n} \) for \( j \in \{1, 2\} \) and the Kiepert triangles \( K_{\phi}^5 \) are of this form. Also the medial triangle of a triangle of this form is again of the same form. The perspectors of \( T_{j,n} \) and \( ABC \) lie on the Kiepert hyperbola. It is the Kiepert perspector \( K(\phi_{j,n}) \) where

\[
\cot \phi_{1,n} = \frac{a_3(n'') + 1}{2a_2(n')},
\]

\[5\text{See for instance [6].}\]
Lemma 13. (a) If \( n \geq 2 \) is even, then

\[
\begin{align*}
A_{1,n}A_{1,n+1} & = -\frac{1}{2}H_{6,n}H_{6,n+1} = -\frac{1}{2}H_{1,n}H_{1,n+1}, \\
B_{1,n}B_{1,n+1} & = -\frac{1}{2}H_{2,n}H_{2,n+1} = -\frac{1}{2}H_{3,n}H_{3,n+1}, \\
C_{1,n}C_{1,n+1} & = -\frac{1}{2}H_{4,n}H_{4,n+1} = -\frac{1}{2}H_{5,n}H_{5,n+1}.
\end{align*}
\]

(b) If \( n \geq 3 \) is odd, then

\[
\begin{align*}
A_{2,n}A_{2,n+1} & = -\frac{1}{2}H_{3,n}H_{3,n+1} = -\frac{1}{2}H_{4,n}H_{4,n+1}, \\
B_{2,n}B_{2,n+1} & = -\frac{1}{2}H_{5,n}H_{5,n+1} = -\frac{1}{2}H_{6,n}H_{6,n+1}, \\
C_{2,n}C_{2,n+1} & = -\frac{1}{2}H_{1,n}H_{1,n+1} = -\frac{1}{2}H_{2,n}H_{2,n+1}.
\end{align*}
\]

Proof: Consider the case of \( A_{1,n}A_{1,n+1} \) for even \( n \). Translate the trapezoid \( H_{5,n+1}H_{6,n+1}H_{6,n}H_{5,n} \) by the vector \( H_{5,n+1}H_{4,n+1} \) and the trapezoid \( H_{2,n+1}H_{2,n}H_{1,n+1} \) by the vector \( H_{2,n+1}H_{5,n+1} \). Together with the trapezoid \( H_{3,n}H_{4,n}H_{4,n+1}H_{3,n+1} \), these images form two triangles \( XH_{3,n+1}H_{4,n+1} \) and

and

\[
\cot \phi_{2,n} = \frac{2a_3(n'')}{a_2(n')}. \]

In particular the perspectors tend to limits when \( n \) tends to infinity. The perspectors and limits are given by

\[
\begin{array}{|c|c|c|}
\hline
\text{triangle} & \text{perspector } K_\phi \text{ with} & \text{limit for } k \to \infty \\
\hline
T_{1,2k} & \phi_{1,2k} = \cot^{-1} \frac{a_2(k-1)+1}{2a_2(k)} & \phi_{1,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{12} \\
\hline
T_{1,2k+1} & \phi_{1,2k+1} = \cot^{-1} \frac{a_2(k)+1}{2a_2(k)} & \phi_{1,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{12} \\
\hline
T_{2,2k} & \phi_{2,2k} = \cot^{-1} \frac{a_2(k-1)+1}{2a_2(k)} & \phi_{2,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{3} \\
\hline
T_{2,2k+1} & \phi_{2,2k+1} = \cot^{-1} \frac{a_2(k)+1}{2a_2(k)} & \phi_{2,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{3} \\
\hline
\end{array}
\]

Remarks. (1) \( T_{2,2} \) is the medial triangle of \( T_{1,3} \).

(2) The perspector of \( T_{2,2} \) and \( T_{2,3} \) is \( X_{591} \).

(3) The perspector of \( T_{2,2} \) and \( T_{2,4} \) is the common circumcenter of \( T_{1,3} \) and \( T_{2,4} \).

Nottrot [8], on the other hand, has found that the sum of the areas of the squares between \( \mathcal{H}_n \) and \( \mathcal{H}_{n+1} \) as \( a_4(n)(a^2 + b^2 + c^2) \), where wreaths.

\[
a_4(n) = 4a_4(n-1) + 4a_4(n-2) - a_4(n-3),
\]

\[
a_4(1) = 1, a_4(2) = 3, a_4(3) = 16.
\]

This is sequence A005386 in [9]. Note that \( a_4(n) = a_4(n) + a_4(n-1) \) for \( n \geq 2 \).
The triangles \( H_3, n H_4, n \) are homothetic at their common centroid \( G \). See Figure 4. It is clear that the points \( X, Y, A_{1,n}, A_{1,n+1} \) all lie on a line through the centroid \( G \). Furthermore, \( A_{1,n} A_{1,n+1} = \frac{1}{2} XY = -\frac{1}{2} H_{1,n} H_{1,n+1} \). The other cases follow similarly. \( \square \)

**Proposition 14.** (1) If \( n \geq 3 \) is odd, the following pairs of triangles are congruent.

(i) \( H_2, n B_{2,n+1} C_{1,n-1} \) and \( H_5, n B_{1,n-1} C_{2,n+1} \).

(ii) \( H_{3,n} A_{1,n-1} B_{2,n+1} \) and \( H_{6,n} A_{2,n+1} B_{1,n-1} \).

(iii) \( H_{4,n} C_{2,n+1} A_{1,n-1} \) and \( H_{1,n} C_{1,n-1} A_{2,n+1} \).

(2) If \( n \geq 2 \) is even, the following pairs of triangles are congruent.

(iv) \( H_{2,n} B_{2,n-1} C_{1,n+1} \) and \( H_{5,n} B_{1,n+1} C_{2,n-1} \).

(v) \( H_{3,n} A_{1,n+1} B_{2,n-1} \) and \( H_{6,n} A_{2,n-1} B_{1,n+1} \).

(vi) \( H_{4,n} C_{2,n-1} A_{1,n+1} \) and \( H_{1,n} C_{1,n+1} A_{2,n-1} \).

**Proof.** We consider the first of these cases. Let \( n \geq 3 \) be an odd number. Consider the triangles \( H_{2,n} B_{2,n+1} C_{1,n-1} \) and \( H_{5,n} B_{1,n-1} C_{2,n+1} \). We show that \( H_{2,n} B_{2,n+1} \) and \( H_{5,n} B_{1,n-1} \) are perpendicular to each other and equal in length, and the same for \( H_{2,n} C_{1,n-1} \) and \( H_{5,n} C_{2,n+1} \).

Consider the triangles \( H_{2,n} B_{2,n+1} B_{2,n+1} \) and \( B_{1,n-1} B_{1,n} H_{5,n} \). By Lemma 13, \( B_{2,n} B_{2,n+1} \) is parallel to \( H_{5,n} H_{5,n+1} \) and is half its length. It follows that \( B_{2,n} B_{2,n+1} \) is perpendicular to \( A_{1,n} \) and \( B_{1,n-1} B_{1,n} H_{5,n} \). Similarly, \( B_{2,n} H_{2,n} \) is perpendicular to \( A_{1,n} \) and \( B_{1,n-1} B_{1,n} H_{5,n} \). Therefore, the triangles \( H_{2,n} B_{2,n} B_{2,n+1} \) and \( B_{1,n-1} B_{1,n} H_{5,n} \) are congruent, and the segments \( H_{2,n} B_{2,n+1} \)

---

6As noted in the beginning of §3, the sides of the squares are parallel to and perpendicular to the sides of \( ABC \) or the medians of \( ABC \) according as \( n \) is odd or even. In the even case it is thus clear that the homothetic center is the centroid. In the odd case it can be seen as the lines perpendicular to the sides of \( ABC \) are parallel to the medians of the triangles in the first wreath (the flank triangles). The centroid and the orthocenter befriend each other. See [5].
and $B_{1,n-1}H_{5,n}$ are perpendicular and equal in length. See Figure 5. The same reasoning shows that the segments $H_{2,n}C_{1,n-1}$ and $H_{5,n}C_{2,n+1}$ are perpendicular and equal in length. Therefore, the triangles $H_{2,n}B_{2,n+1}C_{1,n-1}$ and $H_{5,n}B_{1,n-1}C_{2,n+1}$ are congruent.

The other cases can be proved similarly.

Remark. The fact that the segments $B_{2,n+1}C_{1,n-1}$ and $B_{1,n-1}C_{2,n+1}$ are perpendicular and equal in length has been proved in Proposition 3 for square wreaths arising from an arbitrary hexagon.

5. A pair of Kiepert hyperbolas

The triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ are congruent to $ABC$. The counterparts of a point $P$ in these triangles are the points with the barycentric coordinates relative to these three triangles as $P$ relative to $ABC$. 
Theorem 15. The locus of point $P$ in $ABC$ whose counterparts in the triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ form a triangle $A'B'C'$ perspective to $ABC$ is the union of the line at infinity and the rectangular hyperbola

$$S \sum_{\text{cyclic}} (S_B - S_C)yz + (x + y + z) \left( \sum_{\text{cyclic}} (S_B - S_C)(S_A + S)x \right) = 0. \quad (11)$$

The locus of the perspector is the union of the line at infinity and the Kiepert hyperbola of triangle $ABC$.

Proof. The counterparts of $P = (x : y : z)$ form a triangle perspective with $ABC$ if and only if the parallels through $A$, $B$, $C$ to $A_{1,3}P$, $B_{1,3}P$ and $C_{1,3}P$ are concurrent. These parallels have equations

$$(S + S_B)(x + y + z) - Sz)Y - ((S + S_C)(x + y + z) - S_y)Z = 0,$$

$$-(S + S_A)(x + y + z) - S_x)X + ((S + S_C)(x + y + z) - S_x)Z = 0,$$

$$(S + S_A)(x + y + z) - S_y)X - ((S + S_B)(x + y + z) - S_x)Y = 0.$$

They are concurrent if and only if

$$(x + y + z) \left( \sum_{\text{cyclic}} (b^2 - c^2)((S_A + S)x^2 - S_Ayz) \right) = 0.$$

The locus therefore consists of the line at infinity and a conic. Rearranging the equation of the conic in the form (11), we see that it is homothetic to the Kiepert hyperbola.

For a point $P$ on the locus (11), let $Q = (x : y : z)$ be the corresponding perspector. This means that the parallels through $A_{1,3}$, $B_{1,3}$, $C_{1,3}$ to $AQ$, $BQ$, $CQ$ are concurrent. These parallels have equations

$$(S + S_B)y - (S + S_C)z)x + ((S + S_B)y - S_Cz)Y - ((S + S_C)z - S_By)Z = 0,$$

$$-(S + S_A)x - (S + S_C)z)x + ((S + S_C)z - (S + S_A)x)Y + ((S + S_C)z - S_Ax)Z = 0,$$

$$(S + S_A)x - (S + S_B)y - (S + S_C)x)Y + ((S + S_A)x - (S + S_B)y)Z = 0.$$

They are concurrent if and only if

$$(x + y + z)((S_B - S_C)yz + (S_C - S_A)zx + (S_A - S_B)xy) = 0.$$

This means the perspector lies on the union of the line at infinity and the Kiepert hyperbola.

Here are some examples of points on the locus (11) with the corresponding perspectors on the Kiepert hyperbola (see [1, 7]).
<table>
<thead>
<tr>
<th>Point</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$</td>
<td>on Kiepert hyperbola</td>
</tr>
<tr>
<td>$P$</td>
<td>on locus</td>
</tr>
<tr>
<td>$K$</td>
<td>$\left(\arctan \frac{2}{3}\right)$</td>
</tr>
<tr>
<td>centroid</td>
<td></td>
</tr>
<tr>
<td>orthocenter</td>
<td>de Longchamps point</td>
</tr>
<tr>
<td>centroid</td>
<td>$G' = (-2S_A + a^2 + S : \cdots : \cdots)$</td>
</tr>
<tr>
<td>outer Vecten point</td>
<td>outer Vecten point</td>
</tr>
<tr>
<td>$A$</td>
<td>$A' = B_{1,3}H_{5,2} \cap C_{1,3}H_{2,2}$</td>
</tr>
<tr>
<td>$B$</td>
<td>$B' = C_{1,3}H_{1,2} \cap A_{1,3}H_{4,2}$</td>
</tr>
<tr>
<td>$C$</td>
<td>$C' = A_{1,3}H_{3,2} \cap B_{1,3}H_{6,2}$</td>
</tr>
<tr>
<td>$A_0$</td>
<td>$A_{1,3} = (-S + S_B + S_C : S + S_C : S + S_B)$</td>
</tr>
<tr>
<td>$B_0$</td>
<td>$B_{1,3} = (S + S_C : -(S + S_C + S_A) : S + S_A)$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$C_{1,3} = (S + S_B : S + S_A : -(S + S_A + S_B))$</td>
</tr>
</tbody>
</table>

![Figure 6](image)

Figure 6.

Here, $A_0$ is the intersection of the Kiepert hyperbola with the parallel through $A$ to $BC$; similarly for $B_0$ and $C_0$. Since the triangle $A_{1,3}B_{1,3}C_{1,3}$ has centroid $G$, the rectangular hyperbola (11) is the Kiepert hyperbola of triangle $A_{1,3}B_{1,3}C_{1,3}$. See Figure 6. We show that it is also the Kiepert hyperbola of triangle $AB'C'$. 
Square wreaths around hexagons

This follows from the fact that \( A', B', C' \) have coordinates

\[
\begin{align*}
A' & : -S - 2S_A : S + S_A \\
B' & : S + S_B : -S - 2S_B : S + S_B \\
C' & : S + S_C : S + S_C : -S - 2S_C 
\end{align*}
\]

From these, the centroid of triangle \( A'B'C' \) is the point \( G' \) in the above table. The rectangular hyperbola (11) is therefore the Kiepert hyperbola of triangle \( AB'C' \).

References


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