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Some Geometric Constructions

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Abstract. We solve some problems of geometric construction. Some of them cannot be solved with ruler and compass only and require the drawing of a rectangular hyperbola: (i) construction of the Simson lines passing through a given point, (ii) construction of the lines with a given orthopole, and (iii) a problem of congruent incircles whose analysis leads to some remarkable properties of the internal Soddy center.

1. Simson lines through a given point

1.1. *Problem.* A triangle ABC and a point P are given, $P \neq H$, the orthocenter, and does not lie on the sidelines of the triangle. We want to construct the points of the circumcircle Γ of ABC whose Simson lines pass through P.

1.2. *Analysis.* We make use of the following results of Trajan Lalesco [3].

Proposition 1 (Lalesco). *If the Simson lines of A', B', C' concur at P, then*
(a) *P is the midpoint of HH', where H' is the orthocenter of $A'B'C'$,*
(b) *for any point $M \in \Gamma$, the Simson lines $S(M)$ and $S'(M)$ of M with respect to ABC and $A'B'C'$ are parallel.*

Let h be the rectangular hyperbola through A, B, C, P. If the hyperbola h intersects Γ again at U, the center W of h is the midpoint of HU. Let h' be the rectangular hyperbola through A', B', C', U. The center W' of h' is the midpoint of $H'U$. Hence, by (a) above, $W' = T(W)$, where T is the translation by the vector $\overrightarrow{HP}$.

Let D, D' be the endpoints of the diameter of Γ perpendicular to the Simson line $S(U)$. The asymptotes of h are $S(D)$ and $S(D')$; as, by (b), $S(U)$ and $S'(U)$ are parallel, the asymptotes of h' are $S'(D)$ and $S'(D')$ and, by (b), they are parallel to the asymptotes of h.

It follows that T maps the asymptotes of h to the asymptotes of h'. Moreover, T maps $P \in h$ to $H' \in h'$. As a rectangular hyperbola is determined by a point and the asymptotes, it follows that $h' = T(h)$.

Construction 1. *Given a point $P \neq H$ and not on any of the sidelines of triangle ABC, let the rectangular hyperbola h through A, B, C, P intersect the circumcircle Γ again at U. Let h' be the image of h under the translation T by the vector*

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$\overrightarrow{HP}$. *Then* h' *passes through* U. *The other intersections of* h' *with* Γ *are the points whose Simson lines pass through* P. See Figure 1.

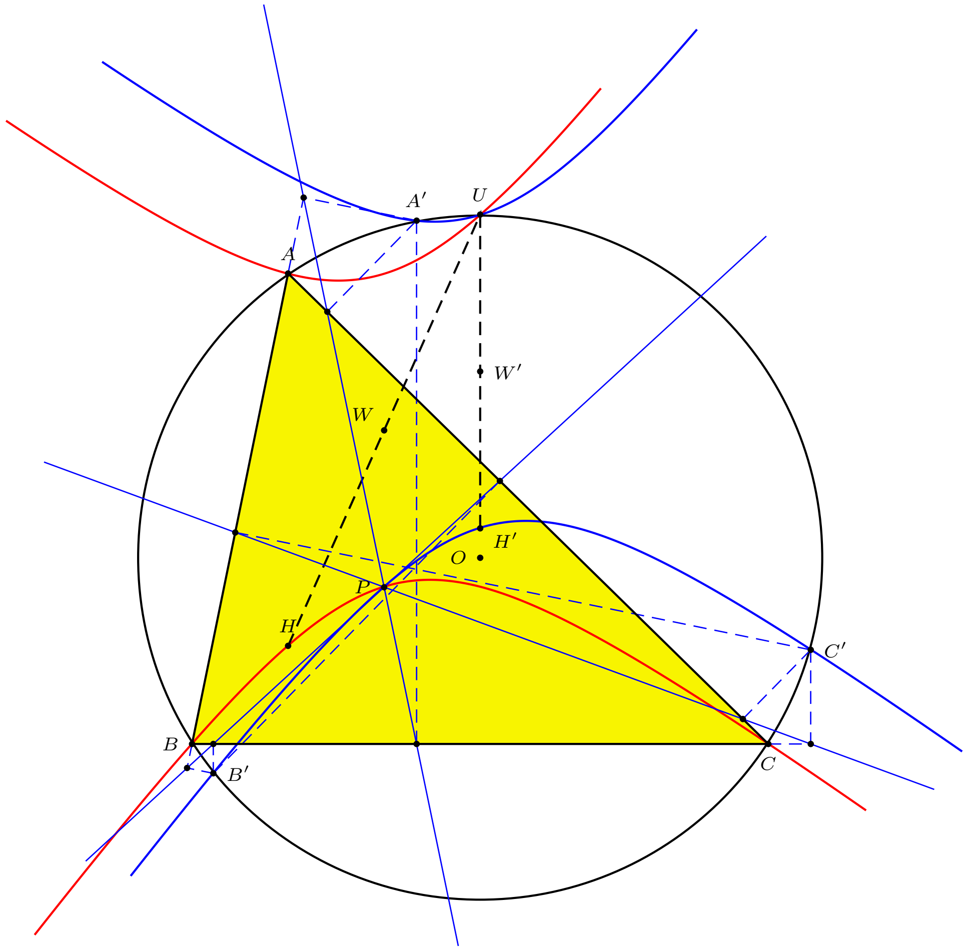

Figure 1.

1.3. *Orthopole.* The above construction leads to a construction of the lines whose orthopole is P. It is well known that, if M and N lie on the circumcircle, the orthopole of the line MN is the common point of the Simson lines of M and N (see [1]). Thus, if we have three real points A', B', C' whose Simson lines pass through P, the lines with orthopole P are the sidelines of $A'B'C'$.

Moreover, the orthopole of a line L lies on the directrix of the insribed parabola touching L (see [1, pp.241–242]). Thus, in any case and in order to avoid imaginary lines, we can proceed this way: for each point M whose Simson line passes through P, let Q be the isogonal conjugate of the infinite point of the direction orthogonal to HP. The line through Q parallel to the Simson line of M intersects the line HP at R. Then P is the orthopole of the perpendicular bisector of QR.

2. Two congruent incircles

2.1. *Problem.* Construct a point P inside ABC such that if B' and C' are the traces of P on AC and AB respectively, the quadrilateral $AB'PC'$ has an incircle congruent to the incircle of PBC.

2.2. *Analysis.* Let h_a be the hyperbola through A with foci B and C, and D_a the projection of the incenter I of triangle ABC upon the side BC.

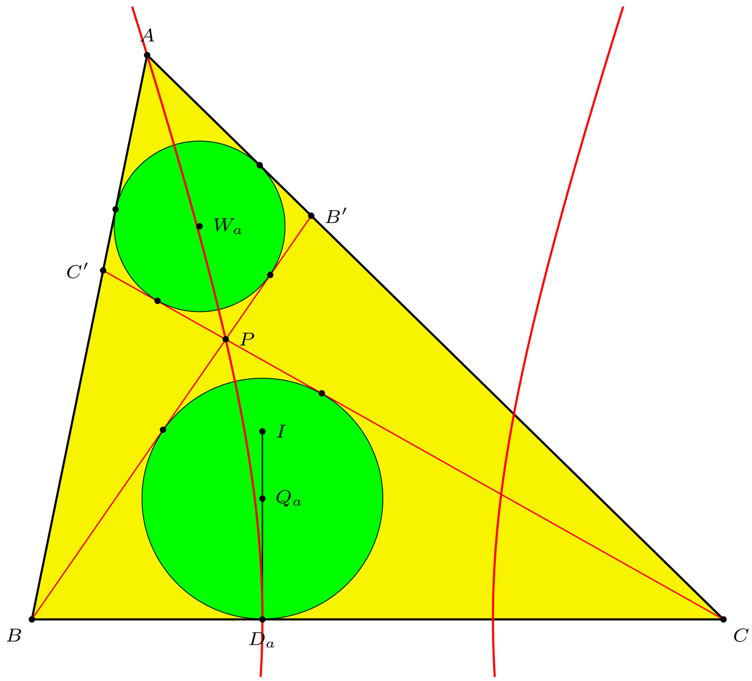

Figure 2.

Proposition 2. *Let P be a point inside ABC and Q_a the incenter of PBC. The following statements are equivalent.*
(a) $PB - PC = AB - AC$.
(b) *P lies on the open arc AD_a of h_a.*
(c) *The quadrilateral $AB'PC'$ has an incircle.*
(d) $IQ_a \perp BC$.
(e) *The incircles of PAB and PAC touch each other.*

Proof. (a) $\Longleftrightarrow$ (b). As $2BD_a = AB+BC-AC$ and $2CD_a = AC+BC-AB$, we have $BD_a - CD_a = AB - AC$ and D_a is the vertex of the branch of h_a through A.

(b)$\Longrightarrow$ (c). AI and PQ_a are the lines tangent to h_a respectively at A and P. If W_a is their common point, BW_a is a bisector of $\angle ABP$. Hence, W_a is equidistant from the four sides of the quadrilateral.

(c)$\Longrightarrow$ (b). If the incircle of $AB'PC'$ touches PB', PC', AC, AB respectively at U, U', V, V', we have $PB-PC = BU-CV = BV'-CU' = AB-AC$.

(a) $\Longleftrightarrow$ (d). If S_a is the projection of Q_a upon BC, we have $2BS_a = PB + BC - PC$. Hence, $PB - PC = AB - AC \Longleftrightarrow S_a = D_a \Longleftrightarrow IQ_a \perp BC$.

(a) $\Longleftrightarrow$ (e). If the incircles of PAC and PAB touch the line AP respectively at S_b and S_c, we have $2AS_b = AC + PA - PC$ and $2AS_c = AB + PA - PB$. Hence, $PB - PC = AB - AC \Longleftrightarrow S_b = S_c$. See Figure 3. $\square$

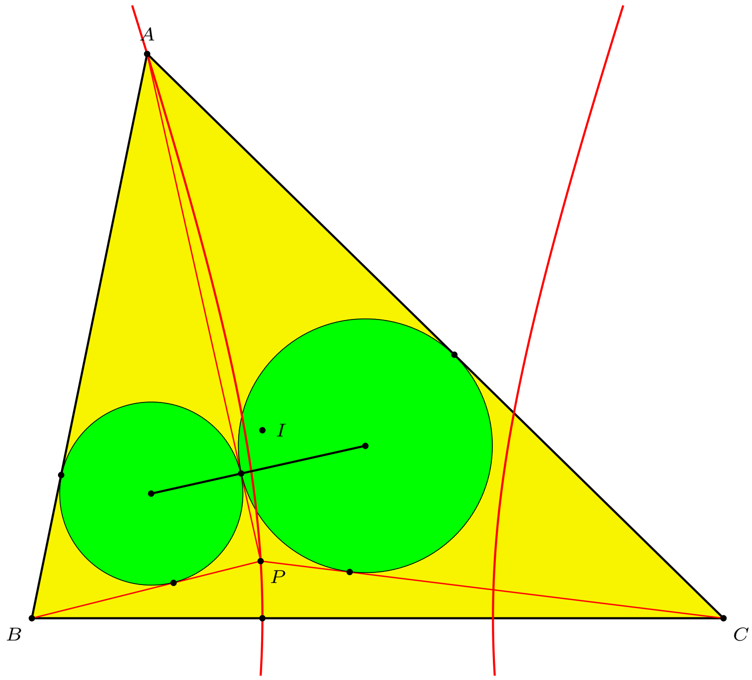

Figure 3.

Proposition 3. *When the conditions of Proposition 2 are satisfied, the following statements are equivalent.*

(a) *The incircles of PBC and $AB'PC'$ are congruent.*

(b) *P is the midpoint of W_aQ_a.*

(c) *W_aQ_a and AD_a are parallel.*

(d) *P lies on the line M_aI where M_a is the midpoint of BC.*

Proof. (a) $\Longleftrightarrow$ (b) is obvious.

Let's notice that, as I is the pole of AD_a with respect to h_a, M_aI is the conjugate diameter of the direction of AD_a with respect to h_a.

So (c) $\Longleftrightarrow$ (d) because W_aQ_a is the tangent to h_a at P.

As the line M_aI passes through the midpoint of AD_a, (b') $\Longleftrightarrow$ (c). $\square$

Now, let us recall the classical construction of an hyperbola knowing the foci and a vertex: For any point M on the circle with center M_a passing through D_a, if L is the line perpendicular at M to BM, and N the reflection of B in M, L touches h_a at $L \cap CN$.

Construction 2. *The perpendicular from B to AD_a and the circle with center M_a passing through D_a have two common points. For one of them M, the perpendicular at M to BM will intersect M_aI at P and the lines D_aI and AI respectively at Q_a and W_a.* See Figure 4.

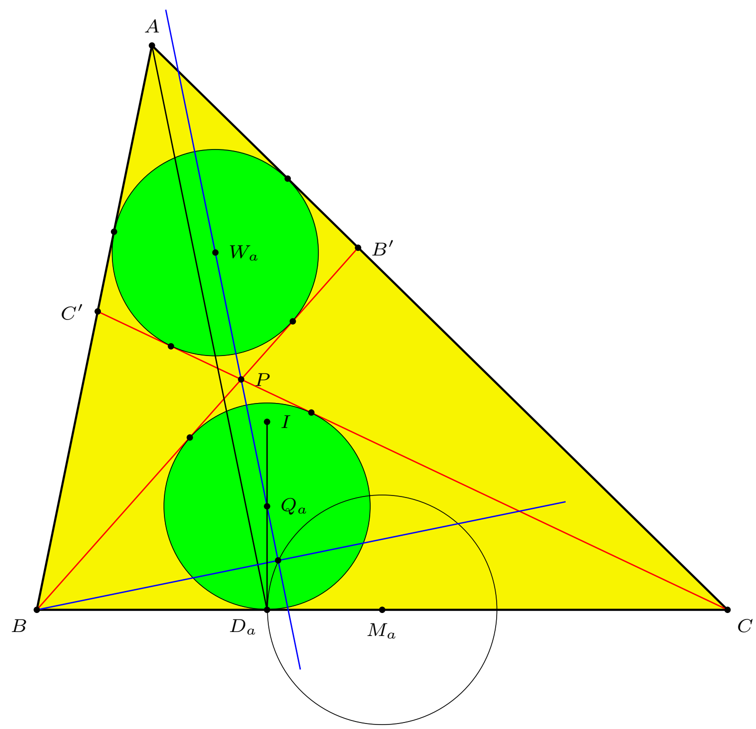

Figure 4.

Remark. We have already known that $PB - PC = c - b$. A further investigation leads to the following results.
(i) $PB + PC = \sqrt{as}$ where s is the semiperimeter of ABC.
(ii) The homogeneous barycentric coordinates of P, Q_a, W_a are as follows.

$$\begin{aligned} P: &\quad (a,\ b - s + \sqrt{as},\ c - s + \sqrt{as}) \\ Q_a: &\quad \left(a,\ b + 2(s-c)\sqrt{\frac{s}{a}},\ c + 2(s-b)\sqrt{\frac{s}{a}}\right) \\ W_a: &\quad (a + 2\sqrt{as},\ b,\ c) \end{aligned}$$

(iii) The common radius of the two incircles is $r_a\left(1 - \sqrt{\frac{a}{s}}\right)$, where r_a is the radius of the $A-$excircle.

3. The internal Soddy center

Let Δ, s, r, and R be respectively the area, the semiperimeter, the inradius, and the circumradius of triangle ABC.

The three circles $(A, s-a)$, $(B, s-b)$, $(C, s-c)$ touch each other. The internal Soddy circle is the circle tangent externally to each of these three circles. See Figure 5. Its center is $X(176)$ in [2] with barycentric coordinates

$$\left(a+\frac{\Delta}{s-a},\ b+\frac{\Delta}{s-b},\ c+\frac{\Delta}{s-c}\right)$$

and its radius is

$$\rho=\frac{\Delta}{2s+4R+r}.$$

See [2] for more details and references.

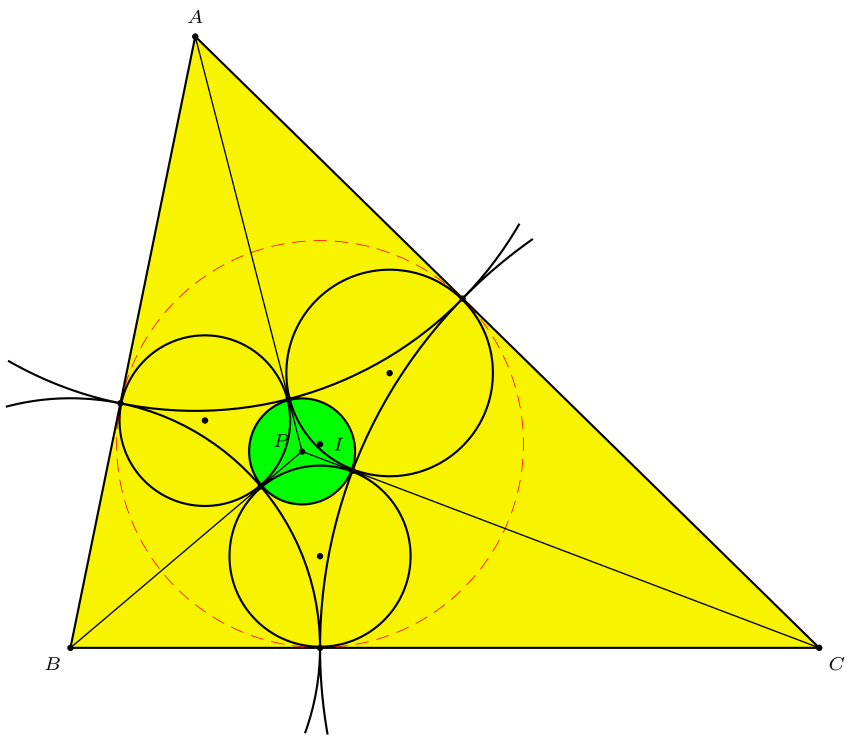

Figure 5.

Proposition 4. *The inner Soddy center $X(176)$ is the only point P inside ABC*
(a) *for which the incircles of PBC, PCA, PAB touch each other;*
(b) *with cevian triangle $A'B'C'$ for which each of the three quadrilaterals $AB'PC'$, $BC'PA'$, $CA'PB'$ have an incircle.* See Figure 6.

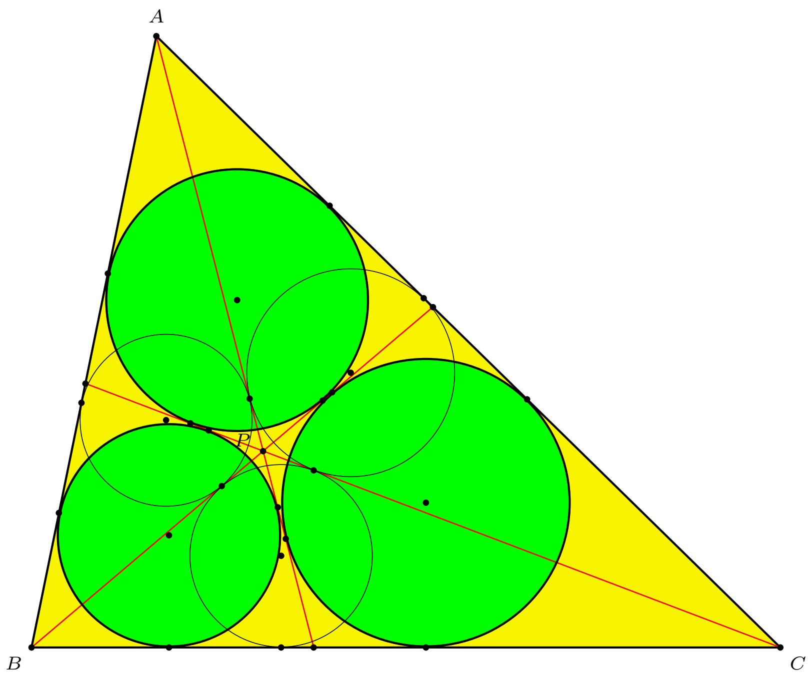

Figure 6.

Proof. Proposition 2 shows that the conditions in (a) and (b) are both equivalent to

$$PB - PC = c - b,\ PC - PA = a - c,\ PA - PB = b - a.$$

As $PA = \rho + s - a$, $PB = \rho + s - b$, $PC = \rho + s - c$, these conditions are satisfied for $P = X(176)$. Moreover, a point P inside ABC verifying these conditions must lie on the open arc AD_a of h_a and on the open arc BD_b of h_b and these arcs cannot have more than a common point. □

Remarks. (1) It follows from Proposition 2(d) that the contact points of the incircles of PBC, PCA, PAB with BC, CA, AB respectively are the same ones D_a, D_b, D_c than the contact points of incircle of ABC.[1]

(2) The incircles of PBC, PCA, PAB touch each other at the points where the internal Soddy circle touches the circles $(A, s-a)$, $(B, s-b)$, $(C, s-c)$.

(3) If Q_a is the incenter of PBC, and W_a the incenter of $AB'PC'$, we have $\dfrac{\overline{Q_aD_a}}{\overline{IQ_a}} = \dfrac{r_a}{a}$, and $\dfrac{\overline{W_aI}}{\overline{AW_a}} = \dfrac{r_a}{s}$, where r_a is the radius of the A-excircle.

(4) The four common tangents of the incircles of $BC'PA'$ and $CA'PB'$ are BC, Q_bQ_c, AP and D_aI.

(5) The lines AQ_a, BQ_b, CQ_c concur at

$$X(482) = \left(a + \frac{2\Delta}{s-a},\ b + \frac{2\Delta}{s-b},\ c + \frac{2\Delta}{s-c}\right).$$

[1]Thanks to François Rideau for this nice remark.

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[1] J. W. Clawson, The complete quadrilateral, *Annals of Mathematics*, ser. 2, 20 (1919)? 232–261.
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