Some Geometric Constructions

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Abstract. We solve some problems of geometric construction. Some of them cannot be solved with ruler and compass only and require the drawing of a rectangular hyperbola: (i) construction of the Simson lines passing through a given point, (ii) construction of the lines with a given orthopole, and (iii) a problem of congruent incircles whose analysis leads to some remarkable properties of the internal Soddy center.

1. Simson lines through a given point

1.1. Problem. A triangle $\triangle ABC$ and a point $P \neq H$, the orthocenter, and does not lie on the sidelines of the triangle. We want to construct the points of the circumcircle $\Gamma$ of $\triangle ABC$ whose Simson lines pass through $P$.

1.2. Analysis. We make use of the following results of Trajan Lalesco [3].

Proposition 1 (Lalesco). If the Simson lines of $A', B', C'$ concur at $P$, then
(a) $P$ is the midpoint of $HH'$, where $H'$ is the orthocenter of $A'B'C'$,
(b) for any point $M \in \Gamma$, the Simson lines $S(M)$ and $S'(M)$ of $M$ with respect to $\triangle ABC$ and $A'B'C'$ are parallel.

Let $h$ be the rectangular hyperbola through $A$, $B$, $C$, $P$. If the hyperbola $h$ intersects $\Gamma$ again at $U$, the center $W$ of $h$ is the midpoint of $HU$. Let $h'$ be the rectangular hyperbola through $A'$, $B'$, $C'$, $U$. The center $W'$ of $h'$ is the midpoint of $H'U$. Hence, by (a) above, $W' = T(W)$, where $T$ is the translation by the vector $\overrightarrow{HP}$.

Let $D$, $D'$ be the endpoints of the diameter of $\Gamma$ perpendicular to the Simson line $S(U)$. The asymptotes of $h$ are $S(D)$ and $S(D')$; as, by (b), $S(U)$ and $S'(U)$ are parallel, the asymptotes of $h'$ are $S'(D)$ and $S'(D')$ and, by (b), they are parallel to the asymptotes of $h$.

It follows that $T$ maps the asymptotes of $h$ to the asymptotes of $h'$. Moreover, $T$ maps $P \in h$ to $H' \in h'$. As a rectangular hyperbola is determined by a point and the asymptotes, it follows that $h' = T(h)$.

Construction 1. Given a point $P \neq H$ and not on any of the sidelines of triangle $\triangle ABC$, let the rectangular hyperbola $h$ through $A$, $B$, $C$, $P$ intersect the circumcircle $\Gamma$ again at $U$. Let $h'$ be the image of $h$ under the translation $T$ by the vector $\overrightarrow{HP}$.
\[ HP. \text{ Then } h' \text{ passes through } U. \text{ The other intersections of } h' \text{ with } \Gamma \text{ are the points whose Simson lines pass through } P. \text{ See Figure 1.} \]

Figure 1.

1.3. Orthopole. The above construction leads to a construction of the lines whose orthopole is \( P \). It is well known that, if \( M \) and \( N \) lie on the circumcircle, the orthopole of the line \( MN \) is the common point of the Simson lines of \( M \) and \( N \) (see [1]). Thus, if we have three real points \( A', B', C' \) whose Simson lines pass through \( P \), the lines with orthopole \( P \) are the sidelines of \( A'B'C' \).

Moreover, the orthopole of a line \( L \) lies on the directrix of the inscribed parabola touching \( L \) (see [1, pp.241–242]). Thus, in any case and in order to avoid imaginary lines, we can proceed this way: for each point \( M \) whose Simson line passes through \( P \), let \( Q \) be the isogonal conjugate of the infinite point of the direction orthogonal to \( HP \). The line through \( Q \) parallel to the Simson line of \( M \) intersects the line \( HP \) at \( R \). Then \( P \) is the orthopole of the perpendicular bisector of \( QR \).
2. Two congruent incircles

2.1. Problem. Construct a point \( P \) inside \( ABC \) such that if \( B' \) and \( C' \) are the traces of \( P \) on \( AC \) and \( AB \) respectively, the quadrilateral \( ABPC' \) has an incircle congruent to the incircle of \( PBC \).

2.2. Analysis. Let \( h_a \) be the hyperbola through \( A \) with foci \( B \) and \( C \), and \( D_a \) the projection of the incenter \( I \) of triangle \( ABC \) upon the side \( BC \).

![Figure 2](image_url)

**Proposition 2.** Let \( P \) be a point inside \( ABC \) and \( Q_a \) the incenter of \( PBC \). The following statements are equivalent.

(a) \( PB - PC = AB - AC \).
(b) \( P \) lies on the open arc \( AD_a \) of \( h_a \).
(c) The quadrilateral \( AB'PC' \) has an incircle.
(d) \( IQ_a \perp BC \).
(e) The incircles of \( PAB \) and \( PAC \) touch each other.

**Proof.** (a) \( \iff \) (b). As \( 2BD_a = AB + BC - AC \) and \( 2CD_a = AC + BC - AB \), we have \( BD_a - CD_a = AB - AC \) and \( D_a \) is the vertex of the branch of \( h_a \) through \( A \).

(b) \( \implies \) (c). \( AI \) and \( PQ_a \) are the lines tangent to \( h_a \) respectively at \( A \) and \( P \). If \( W_a \) is their common point, \( BW_a \) is a bisector of \( \angle ABP \). Hence, \( W_a \) is equidistant from the four sides of the quadrilateral.
(c)⇒(b). If the incircle of $AB'PC'$ touches $PB', PC', AC, AB$ respectively at $U, U', V, V'$, we have $PB - PC = BU - CV = BV' - CU' = AB - AC$.

(a)⇐⇒(d). If $S_a$ is the projection of $Q_a$ upon $BC$, we have $2BS_a = PB + BC - PC$. Hence, $PB - PC = AB - AC ⇐⇒ S_a = D_a ⇐⇒ IQ_a \perp BC$.

(a)⇐⇒(e). If the incircles of $PAC$ and $PAB$ touch the line $AP$ respectively at $S_b$ and $S_c$, we have $2AS_b = AC + PA - PC$ and $2AS_c = AB + PA - PB$. Hence, $PB - PC = AB - AC ⇐⇒ S_b = S_c$. See Figure 3.

\[\text{Figure 3.}\]

**Proposition 3.** When the conditions of Proposition 2 are satisfied, the following statements are equivalent.

(a) The incircles of $PBC$ and $AB'PC'$ are congruent.

(b) $P$ is the midpoint of $W_aQ_a$.

(c) $W_aQ_a$ and $AD_a$ are parallel.

(d) $P$ lies on the line $M_aI$ where $M_a$ is the midpoint of $BC$.

**Proof.** (a)⇐⇒(b) is obvious.

Let’s notice that, as $I$ is the pole of $AD_a$ with respect to $h_a$, $M_aI$ is the conjugate diameter of the direction of $AD_a$ with respect to $h_a$.

So (c)⇐⇒(d) because $W_aQ_a$ is the tangent to $h_a$ at $P$.

As the line $M_aI$ passes through the midpoint of $AD_a$, (b')⇐⇒(c). □

Now, let us recall the classical construction of an hyperbola knowing the foci and a vertex: For any point $M$ on the circle with center $M_a$ passing through $D_a$, 

\[\text{Figure 3.}\]
if \( L \) is the line perpendicular at \( M \) to \( BM \), and \( N \) the reflection of \( B \) in \( M \), \( L \) touches \( h_a \) at \( L \cap CN \).

**Construction 2.** The perpendicular from \( B \) to \( AD_a \) and the circle with center \( M_a \) passing through \( D_a \) have two common points. For one of them \( M \), the perpendicular at \( M \) to \( BM \) will intersect \( M_a I \) at \( P \) and the lines \( D_a I \) and \( AI \) respectively at \( Q_a \) and \( W_a \). See Figure 4.

![Figure 4.](image)

**Remark.** We have already known that \( PB - PC = c - b \). A further investigation leads to the following results.

(i) \( PB + PC = \sqrt{as} \) where \( s \) is the semiperimeter of \( ABC \).

(ii) The homogeneous barycentric coordinates of \( P, Q_a, W_a \) are as follows.

\[
\begin{align*}
P & : (a, b - s + \sqrt{as}, c - s + \sqrt{as}) \\
Q_a & : \left(a, b + 2(s - c) \sqrt{\frac{s}{a}}, c + 2(s - b) \sqrt{\frac{s}{a}} \right) \\
W_a & : (a + 2\sqrt{as}, b, c)
\end{align*}
\]

(iii) The common radius of the two incircles is \( r_a \left(1 - \sqrt{\frac{a}{s}}\right) \), where \( r_a \) is the radius of the \( A \)-excircle.
3. The internal Soddy center

Let $\Delta$, $s$, $r$, and $R$ be respectively the area, the semiperimeter, the inradius, and the circumradius of triangle $ABC$.

The three circles $(A, s-a)$, $(B, s-b)$, $(C, s-c)$ touch each other. The internal Soddy circle is the circle tangent externally to each of these three circles. See Figure 5. Its center is $X(176)$ in [2] with barycentric coordinates

$\left( a + \frac{\Delta}{s-a}, b + \frac{\Delta}{s-b}, c + \frac{\Delta}{s-c} \right)$

and its radius is

$$\rho = \frac{\Delta}{2s + 4R + r}.$$ 

See [2] for more details and references.

Proposition 4. The inner Soddy center $X(176)$ is the only point $P$ inside $ABC$ (a) for which the incircles of $PBC$, $PCA$, $PAB$ touch each other; (b) with cevian triangle $A'B'C'$ for which each of the three quadrilaterals $AB'PC'$, $BC'PA'$, $CA'PB'$ have an incircle. See Figure 6.
Proof. Proposition 2 shows that the conditions in (a) and (b) are both equivalent to
\[ PB - PC = c - b, \quad PC - PA = a - c, \quad PA - PB = b - a. \]

As \( PA = \rho + s - a, \) \( PB = \rho + s - b, \) \( PC = \rho + s - c, \) these conditions are satisfied for \( P = X(176). \) Moreover, a point \( P \) inside \( ABC \) verifying these conditions must lie on the open arc \( AD_a \) of \( h_a \) and on the open arc \( BD_b \) of \( h_b \) and these arcs cannot have more than a common point. \( \square \)

Remarks. (1) It follows from Proposition 2(d) that the contact points of the incircles of \( PBC, PCA, PAB \) with \( BC, CA, AB \) respectively are the same ones \( D_a, D_b, D_c \) than the contact points of incircle of \( ABC. \)

(2) The incircles of \( PBC, PCA, PAB \) touch each other at the points where the internal Soddy circle touches the circles \( (A, s - a), (B, s - b), (C, s - c). \)

(3) If \( Q_a \) is the incenter of \( PBC, \) and \( W_a \) the incenter of \( AB'PC', \) we have
\[ \frac{Q_aD_a}{IQ_a} = \frac{r_a}{a} \quad \text{and} \quad \frac{W_aI}{AW_a} = \frac{r_a}{s}, \]
where \( r_a \) is the radius of the \( A \)-excircle.

(4) The four common tangents of the incircles of \( BC'PA' \) and \( CA'PB' \) are \( BC, Q_bQ_c, AP \) and \( D_aI. \)

(5) The lines \( AQ_a, BQ_b, CQ_c \) concur at
\[ X(482) = \left( a + \frac{2\Delta}{s - a}, b + \frac{2\Delta}{s - b}, c + \frac{2\Delta}{s - c} \right). \]

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References


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