Forum Geometricorum
Volume 7 (2007) 1–9.

FORUM GEOM
ISSN 1534-1178

Euler's Triangle Determination Problem

Joseph Stern

Abstract. We give a simple proof of Euler's remarkable theorem that for a non-degenerate triangle, the set of points eligible to be the incenter is precisely the orthocentroidal disc, punctured at the nine-point center. The problem is handled algebraically with complex coordinates. In particular, we show how the vertices of the triangle may be determined from the roots of a complex cubic whose coefficients are functions of the classical centers.

1. Introduction

Consider the determination of a triangle from its centers.[1] What relations must be satisfied by points O, H, I so that a unique triangle will have these points as circumcenter, orthocenter, and incenter? In Euler's groundbreaking article [3], *Solutio facilis problematum quorundam geometricorum difficillimorum*, this intriguing question is answered synthetically, but without any comment on the geometric meaning of the solution.

Euler proved the existence of the required triangle by treating the lengths of the sides as zeros of a real cubic, the coefficients being functions of OI, OH, HI. He gave the following algebraic restriction on the distances to ensure that the cubic has three real zeros:

$$OI^2 < OH^2 - 2 \cdot HI^2 < 2 \cdot OI^2.$$

Though Euler did not remark on the geometric implications, his restriction was later proven equivalent to the simpler inequality

$$GI^2 + IH^2 < GH^2,$$

where G is the point that divides OH in the ratio $1:2$ (G is the centroid). This result was presented in a beautiful 1984 paper [4] by A. P. Guinand. Its geometric meaning is immediate: I must lie inside the circle on diameter GH. It also turns out that I cannot coincide with the midpoint of OH, which we denote by N (the nine-point center). The remarkable fact is that *all and only* points inside the circle and different from N are eligible to be the incenter. This region is often called the *orthocentroidal disc*, and we follow this convention.[2] Guinand considered the

Publication Date: January 8, 2007. Communicating Editor: Paul Yiu.
Dedicated to the tercentenary of Leonhard Euler.
[1]The phrase "determination of a triangle" is borrowed from [7].
[2]Conway discusses several properties of the orthocentroidal disc in [1].

cosines of the angles as zeros of a real cubic. He showed that this cubic has three real zeros with positive inverse cosines summing to π. Thus the angles are known, and the scale may be determined subsequently from OH. The problem received fresh consideration in 2002, when B. Scimemi [7] showed how to solve it using properties of the Kiepert focus, and again in 2005, when G. C. Smith [8] used statics to derive the solution.

The approach presented here uses complex coordinates. We show that the vertices of the required triangle may be computed from the roots of a certain complex cubic whose coefficients depend only upon the classical centers. This leads to a relatively simple proof.

2. Necessity of Guinand's Locus

Given a nonequilateral triangle, we show first that the incenter must lie within the orthocentroidal disc and must differ from the nine-point center. The equilateral triangle is uninteresting, since all the centers coincide.

Let $\triangle ABC$ be nonequilateral. As usual, we write O, H, I, G, N, R, r for the circumcenter, orthocenter, incenter, centroid, nine-point center, circumradius and inradius. Two formulas will feature very prominently in our discussion:

$$OI^2 = R(R-2r) \quad \text{and} \qquad NI = \tfrac{1}{2}(R-2r).$$

The first is due to Euler and the second to Feuerbach.[3] They jointly imply

$$OI > 2 \cdot NI,$$

provided the triangle is nonequilateral. Now given a segment PQ and a number $\lambda > 1$, the Apollonius Circle Theorem states that

(1) the equation $PX = \lambda \cdot QX$ describes a circle whose center lies on PQ, with P inside and Q outside;
(2) the inequality $PX > \lambda \cdot QX$ describes the interior of this circle (see [6]).

Thus the inequality $OI > 2 \cdot NI$ places I inside the circle $OX = 2 \cdot NX$, the center of which lies on the Euler line ON. Since G and H lie on the Euler line and satisfy the equation of the circle, GH is a diameter, and this circle turns out to be the orthocentroidal circle. Finally, the formulas of Euler and Feuerbach show that if $I = N$, then $O = I$. This means that the incircle and the circumcircle are *concentric*, forcing $\triangle ABC$ to be equilateral. Thus N is ineligible to be the incenter.

3. Complex Coordinates

Our aim now is to express the classical centers of $\triangle ABC$ as functions of A, B, C, regarded as complex numbers.[4] We are free to put $O = 0$, so that

$$|A| = |B| = |C| = R.$$

[3] Proofs of both theorems appear in [2].

[4] See [5] for a more extensive discussion of this approach.

The centroid is given by $3G = A+B+C$. The theory of the Euler line shows that $3G = 2O + H$, and since $O = 0$, we have

$$H = A + B + C.$$

Finally, it is clear that $2N = O + H = H$.

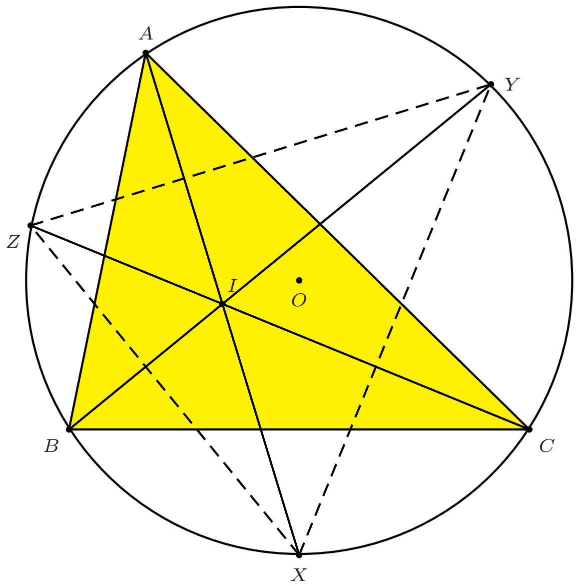

Figure 1.

To deal with the incenter, let X, Y, Z be the points at which the extended angle bisectors meet the circumcircle (Figure 1). It is not difficult to see that $AX \perp YZ$, $BY \perp ZX$ and $CZ \perp XY$. For instance, one angle between AX and YZ is the average of the minor arc from A to Z and the minor arc from X to Y. The first arc measures $\widehat{C}$, and the second, $\widehat{A} + \widehat{B}$. Thus the angle between AX and YZ is $\pi/2$. Evidently the angle bisectors of $\triangle ABC$ coincide with the altitudes of $\triangle XYZ$, and I is the orthocenter of $\triangle XYZ$. Since this triangle has circumcenter 0, its orthocenter is

$$I = X + Y + Z.$$

We now introduce complex square roots α, β, γ so that

$$\alpha^2 = A, \qquad \beta^2 = B, \qquad \gamma^2 = C.$$

There are two choices for each of α, β, γ. Observe that

$$|\beta\gamma| = R \qquad \text{and} \qquad \arg(\beta\gamma) = \tfrac{1}{2}(\arg B + \arg C),$$

so that $\pm\beta\gamma$ are the mid-arc points between B and C. It follows that $X = \pm\beta\gamma$, depending on our choice of signs. For reasons to be clarified later, we would like to arrange it so that

$$X = -\beta\gamma, \qquad Y = -\gamma\alpha, \qquad Z = -\alpha\beta.$$

These hold if α, β, γ are chosen so as to make $\triangle\alpha\beta\gamma$ *acute*, as we now show.

Let Γ denote the circle $|z| = \sqrt{R}$, on which α, β, γ must lie. Temporarily let α_1, α_2 be the two square roots of A, and β_1 a square root of B. Finally, let γ_1 be the square root of C on the side of $\alpha_1\alpha_2$ containing β_1 (Figure 2). Now $\triangle\alpha_i\beta_j\gamma_k$ is acute if and only if any two vertices are separated by the diameter of Γ through the remaining vertex. Otherwise one of its angles would be inscribed in a minor arc, rendering it obtuse. It follows that of all eight triangles $\triangle\alpha_i\beta_j\gamma_k$, only $\triangle\alpha_1\beta_2\gamma_1$ and $\triangle\alpha_2\beta_1\gamma_2$ are acute.

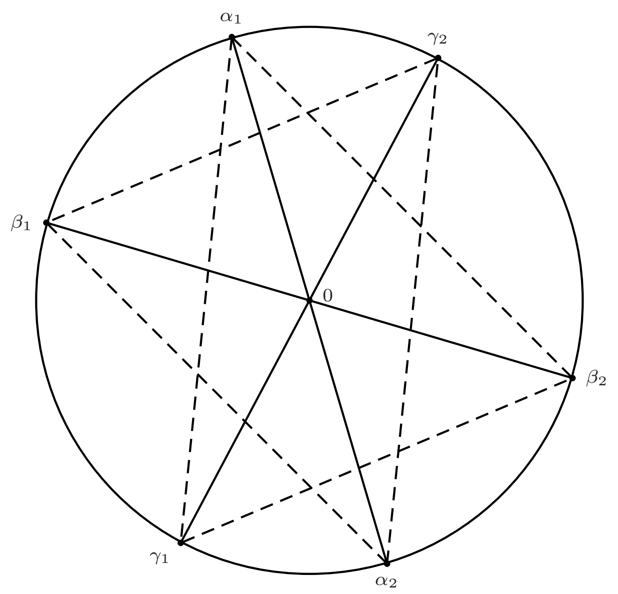

Figure 2.

Now let (α, β, γ) be either $(\alpha_1, \beta_2, \gamma_1)$ or $(\alpha_2, \beta_1, \gamma_2)$, so that $\triangle\alpha\beta\gamma$ is acute. Consider the stretch-rotation $z \mapsto \beta z$. This carries the diameter of Γ with endpoints $\pm\alpha$ to the diameter of $|z| = R$ with endpoints $\pm\alpha\beta$, one of which is Z. Now β and γ are separated by the diameter with endpoints $\pm\alpha$, and therefore B and $\beta\gamma$ are separated by the diameter with endpoints $\pm Z$. Thus to prove $X = -\beta\gamma$, we must only show that X and B are on the *same* side of the diameter with endpoints $\pm Z$. This will follow if the arc from Z to X passing through B is *minor* (Figure 3); but of course its measure is

$$\angle ZOB + \angle BOX = 2\angle ZCB + 2\angle BAX = \widehat{C} + \widehat{A} < \pi.$$

Hence $X = -\beta\gamma$. Similar arguments show that $Y = -\gamma\alpha$ and $Z = -\alpha\beta$.

To summarize, the incenter of $\triangle ABC$ may be expressed as

$$I = -(\beta\gamma + \gamma\alpha + \alpha\beta),$$

where α, β, γ are complex square roots of A, B, C for which $\triangle\alpha\beta\gamma$ is acute. Note that this expression is indifferent to the choice between $(\alpha_1, \beta_2, \gamma_1)$ and $(\alpha_2, \beta_1, \gamma_2)$, since each of these triples is the negative of the other.

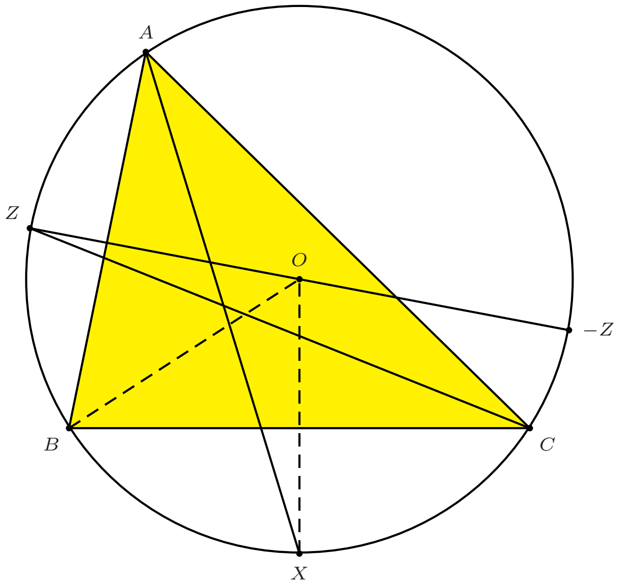

Figure 3.

4. Sufficiency of Guinand's Locus

Place O and H in the complex plane so that O lies at the origin. Define N and G as the points which divide OH internally in the ratios $1 : 1$ and $1 : 2$, respectively. Suppose that I is a point different from N selected from within the circle on diameter GH. Since $H - 2I = 2(N - I)$ is nonzero, we are free to scale coordinates so that $H - 2I = 1$. Let $u = |I|$. Guinand's inequality $OI > 2 \cdot NI$, which we write in complex coordinates as

$$|I| > 2|N - I|$$

now acquires the very simple form $u > 1$.

Consider the cubic equation

$$z^3 - z^2 - Iz + u^2 I = 0.$$

By the Fundamental Theorem of Algebra, this has three complex zeros α, β, γ. These turn out to be square roots of the required vertices. From the standard relations between zeros and coefficients, one has the important equations:

$$\alpha + \beta + \gamma = 1, \qquad \beta\gamma + \gamma\alpha + \alpha\beta = -I, \qquad \alpha\beta\gamma = -u^2 I.$$

Let us first show that the zeros lie on a circle centered at the origin. In fact,

$$|\alpha| = |\beta| = |\gamma| = u.$$

If z is a zero of the cubic, then $z^2(z-1) = I(z - u^2)$. Taking moduli, we get

$$|z|^2|z-1| = u|z - u^2|.$$

Squaring both sides and applying the rule $|w|^2 = w\bar{w}$, we find that

$$|z|^4(z-1)(\bar{z}-1) = u^2(z-u^2)(\bar{z}-u^2),$$

$$(|z|^6 - u^6) - (|z|^4 - u^4)(z + \bar{z}) + |z|^2(|z|^2 - u^2) = 0.$$

Assume for contradiction that a certain zero z has modulus $\neq u$. Then we may divide the last equation by the nonzero number $|z|^2 - u^2$, getting

$$|z|^4 + u^2|z|^2 + u^4 - (|z|^2 + u^2)(z + \bar{z}) + |z|^2 = 0,$$

or after a slight rearrangement,

$$(|z|^2 + u^2)(|z|^2 - (z + \bar{z})) + u^4 + |z|^2 = 0.$$

An elementary inequality of complex algebra says that

$$-1 \leq |z|^2 - (z + \bar{z}).$$

From this inequality and the above equation, we find that

$$(|z|^2 + u^2)(-1) + u^4 + |z|^2 \leq 0,$$

or after simplifying,

$$u^4 - u^2 \leq 0.$$

As this result is inconsistent with the hypothesis $u > 1$, we have proven that all the zeros of the cubic equation have modulus u.

Now *define* A, B, C by

$$A = \alpha^2, \qquad B = \beta^2, \qquad C = \gamma^2.$$

Clearly $|A| = |B| = |C| = u^2$. Since three points of a circle cannot be collinear, $\triangle ABC$ will be nondegenerate so long as A, B, C are distinct. Thus suppose for contradiction that $A = B$. It follows that $\alpha = \pm\beta$. If $\alpha = -\beta$, then $\gamma = \alpha + \beta + \gamma = 1$, yielding the falsehood $u = |\gamma| = 1$. The only remaining alternative is $\alpha = \beta$. In this case, $2\alpha + \gamma = 1$ and $\alpha(2\gamma + \alpha) = -I$, so that

$$|\alpha||2\gamma + \alpha| = |I|, \qquad \text{or} \qquad |2\gamma + \alpha| = 1.$$

Since $2\alpha + \gamma = 1$, one has $|2 - 3\alpha| = |2\gamma + \alpha| = 1$. Squaring this last result gives

$$4 - 6(\alpha + \bar{\alpha}) + 9|\alpha|^2 = 1, \qquad \text{or} \qquad 2(\alpha + \bar{\alpha}) = 1 + 3u^2.$$

Since $|\alpha + \bar{\alpha}| = 2|\mathrm{Re}(\alpha)| \leq 2|\alpha|$, we have $1 + 3u^2 \leq 4u$. Therefore the value of u is bounded between the zeros of the quadratic

$$3u^2 - 4u + 1 = (3u - 1)(u - 1),$$

yielding the falsehood $\frac{1}{3} \leq u \leq 1$. By this kind of reasoning, one shows that any two of A, B, C are distinct, and hence that $\triangle ABC$ is nondegenerate.

As in §3, since $\triangle ABC$ has circumcenter 0, its orthocenter is

$$\begin{aligned} A + B + C &= \alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta) \\ &= 1 + 2I \\ &= H. \end{aligned}$$

Here we see the rationale for having chosen $I = -(\beta\gamma + \gamma\alpha + \alpha\beta)$.

Lastly we must show that the incenter of $\triangle ABC$ lies at I. It has already appeared that $I = -(\beta\gamma + \gamma\alpha + \alpha\beta)$. As in §3, exactly two of the eight possible triangles formed from square roots of A, B, C are acute, and these are mutual images under the map $z \mapsto -z$. Moreover, the incenter of $\triangle ABC$ is necessarily the value of the expression $-(z_2z_3 + z_3z_1 + z_1z_2)$ whenever $\triangle z_1z_2z_3$ is one of these two acute triangles. Thus to identify the incenter with I, we must only show that $\triangle\alpha\beta\gamma$ is acute.

Angle $\widehat{\alpha}$ is acute if and only if

$$|\beta - \gamma|^2 < |\alpha - \beta|^2 + |\alpha - \gamma|^2.$$

On applying the rule $|w|^2 = w\bar{w}$, this becomes

$$2u^2 - (\beta\bar{\gamma} + \bar{\beta}\gamma) < 4u^2 - (\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\gamma} + \bar{\alpha}\gamma),$$

$$\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\gamma} + \bar{\alpha}\gamma + \beta\bar{\gamma} + \bar{\beta}\gamma < 2(u^2 + \beta\bar{\gamma} + \bar{\beta}\gamma).$$

Here the left-hand side may be simplified considerably as

$$(\alpha + \beta + \gamma)(\bar{\alpha} + \bar{\beta} + \bar{\gamma}) - |\alpha|^2 - |\beta|^2 - |\gamma|^2 = 1 - 3u^2.$$

In a similar way, the right-hand side simplifies as

$$\begin{aligned} & 2u^2 + 2(\beta + \gamma)(\bar{\beta} + \bar{\gamma}) - 2|\beta|^2 - 2|\gamma|^2 \\ = \;& 2(1 - \alpha)(1 - \bar{\alpha}) - 2u^2 \\ = \;& 2(1 + |\alpha|^2 - \alpha - \bar{\alpha} - u^2) \\ = \;& 2 - 2(\alpha + \bar{\alpha}). \end{aligned}$$

To complete the proof that $\widehat{\alpha}$ is acute, it remains only to show that

$$2(\alpha + \bar{\alpha}) < 1 + 3u^2.$$

However, $2(\alpha + \bar{\alpha}) \leq 4|\alpha| = 4u$, and we have already seen that

$$4u < 1 + 3u^2,$$

since the opposite inequality yields the falsehood $\frac{1}{3} \leq u \leq 1$. Similar arguments establish that $\widehat{\beta}$ and $\widehat{\gamma}$ are acute.

To summarize, we have produced a nondegenerate triangle $\triangle ABC$ which has classical centers at the given points O, H, I. We now return to original notation and write $R = u^2$ for the circumradius of $\triangle ABC$.

5. Uniqueness

Suppose some other triangle $\triangle DEF$ has O, H, I as its classical centers. The formulas of Euler and Feuerbach presented in §2 have a simple but important consequence: If a triangle has O, N, I as circumcenter, nine-point center, and incenter, then its *circumdiameter* is OI^2/NI. This means that $\triangle ABC$ and $\triangle DEF$ share not only the same circumcenter, but also the same circumradius. It follows that $|D| = |E| = |F| = R$.

Since $\triangle DEF$ has circumcenter 0, its orthocenter H is equal to $D + E + F$. Choose square roots δ, ϵ, ζ of D, E, F so that the incenter I will satisfy

$$I = -(\epsilon\zeta + \zeta\delta + \delta\epsilon).$$

Then

$$\begin{aligned}(\delta+\epsilon+\zeta)^2 &= \delta^2+\epsilon^2+\zeta^2+2(\epsilon\zeta+\zeta\delta+\delta\epsilon)\\ &= D+E+F-2I\\ &= H-2I\\ &= 1.\end{aligned}$$

Since the map $z \mapsto -z$ leaves I invariant, but reverses the sign of $\delta+\epsilon+\zeta$, we may change the signs of δ, ϵ, ζ if necessary to make it so that

$$\delta+\epsilon+\zeta = 1.$$

Observe next that $|\delta\epsilon\zeta| = u^3 = |u^2 I|$. Thus we may write

$$\delta\epsilon\zeta = -\theta u^2 I, \qquad \text{where} \qquad |\theta| = 1.$$

The elementary symmetric functions of δ, ϵ, ζ are now

$$\delta+\epsilon+\zeta = 1, \qquad \epsilon\zeta+\zeta\delta+\delta\epsilon = -I, \qquad \delta\epsilon\zeta = -\theta u^2 I.$$

It follows that δ, ϵ, ζ are the roots of the cubic equation

$$z^3 - z^2 - Iz + \theta u^2 I = 0.$$

As in §4, we rearrange and take moduli of both sides to obtain

$$|z|^2|z-1| = u|z-\theta u|.$$

Squaring both sides of this result, we get

$$|z|^4(|z|^2 - z - \bar{z} + 1) = u^2(|z|^2 - u^2 z\bar{\theta} - u^2\bar{z}\theta + u^4).$$

Since all zeros of the cubic have modulus u, we may replace every occurrence of $|z|^2$ by u^2. This dramatically simplifies the equation, reducing it to

$$z + \bar{z} = z\bar{\theta} + \bar{z}\theta.$$

Substituting δ, ϵ, ζ here successively for z and adding the results, one finds that

$$2 = \bar{\theta} + \theta,$$

since

$$\delta+\epsilon+\zeta = \bar{\delta}+\bar{\epsilon}+\bar{\zeta} = 1.$$

It follows easily that $\theta = 1$. Evidently δ, ϵ, ζ are determined from the same cubic as α, β, γ. Therefore (D, E, F) is a permutation of (A, B, C), and the solution of the determination problem is unique.

References

[1] J. H. Conway, Hyacinthos messge, 7644, August 19, 2003.

[2] H. S. M. Coxeter and S. Greitzer, *Geometry Revisited*, Math. Assoc. of America, 1967.

[3] L. Euler, Solutio facilis problematum quorundam geometricorum difficillimorum, *Novi Comm. Acad. Scie. Petropolitanae* 11 (1765); reprinted in *Opera Omnia*, serie prima, vol. 26 (A. Speiser, ed.), n. 325, 139–157.

[4] A. Guinand, Euler lines, tritangent centers, and their triangles, *Amer. Math. Monthly*, 91 (1984) 290-300.

[5] L. Hahn, *Complex Numbers and Geometry*, Math. Assoc. of America, 1994.

[6] D. Pedoe, *Geometry: A Comprehensive Course*, Cambridge Univ. Press, 1970.

[7] B. Scimemi, Paper-folding and Euler's theorem revisited, *Forum Geom.*, 2 (2002) 93–104.
[8] G. C. Smith, Statics and the moduli space of triangles, *Forum Geom.*, 5 (2005) 181–190.

Joseph Stern: Department of Mathematics, Stuyvesant High School, 345 Chambers Street, New York, New York 10282, USA
E-mail address: jstern@stuy.edu