On a Porism Associated with the Euler and Droz-Farny Lines

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Abstract. The envelope of the Droz-Farny lines of a triangle is determined to be the inconic with foci at the circumcenter and orthocenter by using purely Euclidean means. The poristic triangles sharing this inconic and circumcircle have a common circumcenter, centroid and orthocenter.

1. Introduction

The triangle ABC has orthocenter H and circumcircle Σ. Suppose that a pair of perpendicular lines through H are drawn, then they meet the sides BC, CA, AB in pairs of points. The midpoints X, Y, Z of these pairs of points are known to be collinear on the Droz-Farny line [2]. The envelope of the Droz-Farny line is the inconic with foci at O and H, known recently as the Macbeath inconic, but once known as the Euler inconic [6]. We support the latter terminology because of its strong connection with the Euler line [3]. According to Goormaghtigh writing in [6] this envelope was first determined by Neuberg, and Goormaghtigh gives an extensive list of early articles related to the Droz-Farny line problem. We will not repeat the details since [6] is widely available through the archive service JSTOR.

We give a short determination of the Droz-Farny envelope using purely Euclidean means. Taken in conjunction with Ayme's recent proof [1] of the existence of the Droz-Farny line, this yields a completely Euclidean derivation of the envelope.

This envelope is the inconic of a porism consisting of triangles with a common Euler line and circumcircle. The sides of triangles in this porism arise as Droz-Farny lines of any one of the triangles in the porism. Conversely, if the orthocenter is interior to Σ, all Droz-Farny lines will arise as triangle sides.

2. The Droz-Farny envelope

Theorem. *Each Droz-Farny line of triangle ABC is the perpendicular bisector of a line segment joining the orthocenter H to a point on the circumcircle.*

Proof. Figure 1 may be useful. Let perpendicular lines l and l' through H meet BC, CA, AB at P and P', Q and Q', R and R' respectively and let X, Y, Z be the midpoints of PP', QQ', RR'.

The collinearity of X, Y, Z is the Droz-Farny theorem. Let K be the foot of the perpendicular from H to XYZ and produce HK to L with $HK = KL$. Now the circle HPP' has center X and $XH = XL$ so L lies on this circle. Let M, M' be the feet of the perpendiculars from L to l, l'. Note that $LMHM'$ is a rectangle
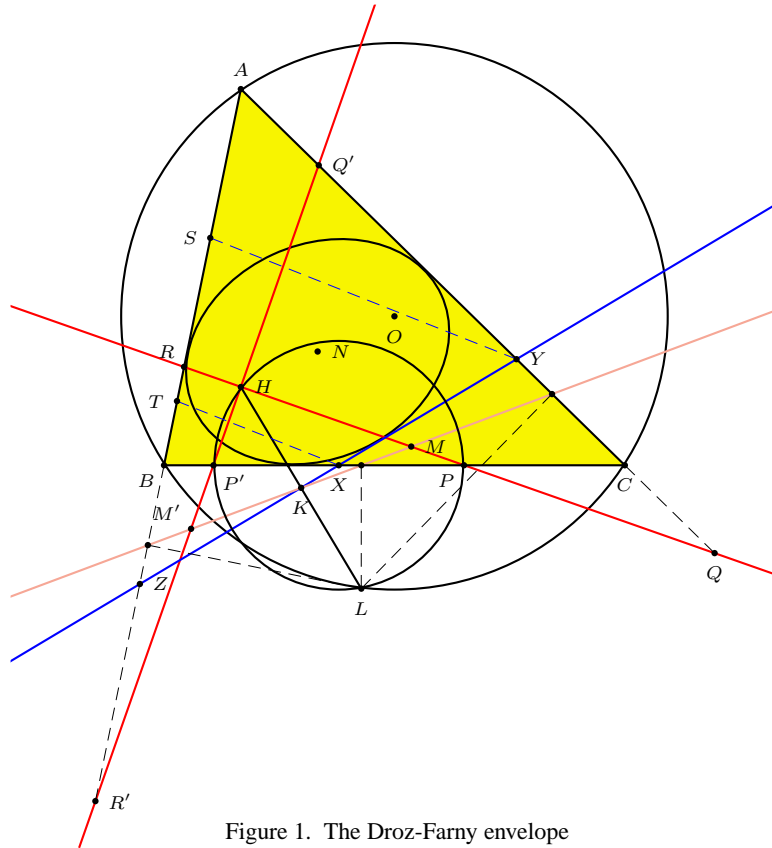
Figure 1. The Droz-Farny envelope

so K is on MM'. Then the foot of the perpendicular from L to the line PP' (*i.e.* BC) lies on MM' by the Wallace-Simson line property applied to the circumcircle of $PP'H$. Equally well, both perpendiculars dropped from L to AB and CA have feet on MM'. Hence L lies on circle ABC with MM' as its Wallace-Simson line. Therefore XYZ is a perpendicular bisector of a line segment joining H to a point on the circumcircle.   □

Note that K lies on the nine-point circle of ABC. An expert in the theory of conics will recognize that the nine-point circle is the auxiliary circle of the Euler inconic of ABC with foci at the circumcenter and orthocenter, and for such a reader this article is substantially complete. The points X, Y, Z are collinear and the line XYZ is tangent to the conic inscribed in triangle ABC and having O, H as foci. The direction of the Droz-Farny line is a continuous function of the direction of the mutual perpendiculars; the argument of the Droz-Farny line against a reference axis increases monotonically as the perpendiculars rotate (say) anticlockwise through θ, with the position of the Droz-Farny line repeating itself as θ increases by $\frac{\pi}{2}$. By the intermediate value theorem, the envelope of the Droz-Farny lines is the whole Euler inconic.

We present a detailed discussion of this situation in §3 for the lay reader.

Incidentally, the fact that XY is a variable tangent to a conic of which BC, CA are fixed tangents mean that the correspondence $X \sim Y$ is a projectivity between the two lines. There is a neat way of setting up this map: let the perpendicular bisectors of AH, BH meet AB at S and T respectively. Then SY and TX are parallel. With a change of notation denote the lines BC, CA, AB, XYZ by a, b, c, d respectively; let e, f be the perpendicular bisectors of AH, BH. All these lines are tangents to the conic in question. Consider the Brianchon hexagon of lines a, b, c, d, e, f. The intersections ae, fb are at infinity so their join is the line at infinity. We have $ec = S$, $bd = Y$, $cf = T$, $da = X$. By Brianchon's theorem SY is parallel to XT.
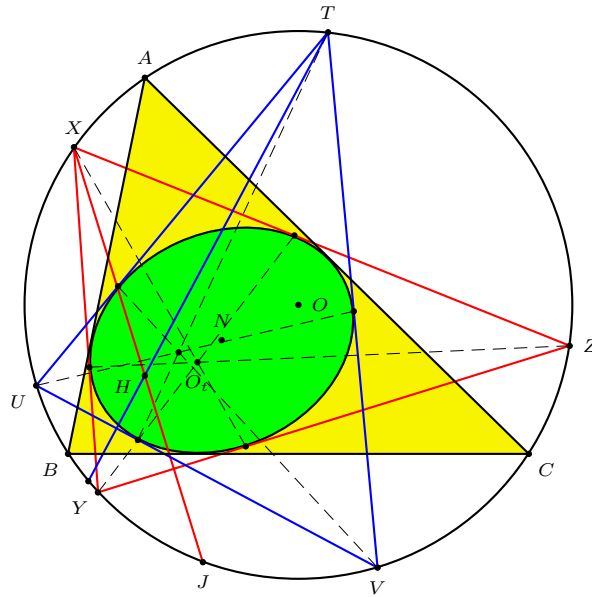
3. The porism

Figure 2. A porism associated with the Euler line

In a triangle with side lengths a, b and c, circumradius R and circumcenter O, the orthocenter H always lies in the interior of a circle center O and radius $3R$ since, as Euler showed, $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$.

We begin afresh. Suppose that we draw a circle Σ with center O and radius R in which is inscribed a non-right angled triangle ABC which has an orthocenter H, so $OH < 3R$ and H is not on Σ.

This H will serve as the orthocenter of infinitely many other triangles XYZ inscribed in the circle and a porism is obtained. We construct these triangles by choosing a point J on the circle. Next we draw the perpendicular bisector of HJ, and need this line to meet Σ again at Y and Z with XYZ anticlockwise. We can certainly arrange that the line and Σ meet by choosing X sufficiently close to A,

B or C. When this happens it follows from elementary considerations that triangle XYZ has orthocenter H, and is the only such triangle with circumcircle Σ and vertex X. In the event that H is inside the circumcircle (which happens precisely when triangle ABC is acute), then every point X on the circumcircle arises as a vertex of a triangle XYZ in the porism.

The construction may be repeated to create as many triangles ABC, TUV, PQR as we please, all inscribed in the circle and all having orthocenter H, as illustrated in Figure 2. Notice that the triangles in this porism have the same circumradius, circumcenter and orthocenter, so the sum of the squares of the side lengths of each triangle in the porism is the same.

We will show that all these triangles circumscribe a conic, with one axis of length R directed along the common Euler line, and with eccentricity $\frac{OH}{R}$. It follows that this inconic is an ellipse if H is chosen inside the circle, but a hyperbola if H is chosen outside.

Thus a porism arises which we call an *Euler line porism* since each triangle in the porism has the same circumcenter, centroid, nine-point center, orthocentroidal center, orthocenter *etc*. A triangle circumscribing a conic gives rise to a *Brianchon point* at the meet of the three Cevians which join each vertex to its opposite contact point.

We will show that the Brianchon point of a triangle in this porism is the isotomic conjugate O_t of the common circumcenter O.

In Figure 2 we pinpoint O_t for the triangle XYZ. The computer graphics system CABRI gives strong evidence for the conjecture that the locus of O_t, as one runs through the triangles of the porism, is a a subset of a conic.

It is possible to choose a point H at distance greater than $3R$ from O so there is no triangle inscribed in the circle which has orthocenter H and then there is no point J on the circle such that the perpendicular bisector of HJ cuts the circle.

The acute triangle case. See Figure 3. The construction is as follows. Draw AH, BH and CH to meet Σ at D, E and F. Draw DO, EO and FO to meet the sides at L, M, N. Let AO meet Σ at D^* and BC at L^*. Also let DO meet Σ at A^*. The points M^*, N^*, E^*, F^*, B^* and C^* are not shown but are similarly defined. Here A' is the midpoint of BC and the line through A' perpendicular to BC is shown.

3.1. *Proof of the porism.* Consider the ellipse defined as the locus of points P such that $HP + OP = R$, where R the circumradius of Σ. The triangle HLD is isosceles, so $HL + OL = LD + OL = R$; therefore L lies on the ellipse.

Now $\angle OLB = \angle CLD = \angle CLH$, because the line segment HD is bisected by the side BC. Therefore the ellipse is tangent to BC at L, and similarly at M and N. It follows that AL, BM, CN are concurrent at a point which will be identified shortly.

This ellipse depends only on O, H and R. It follows that if TUV is any triangle inscribed in Σ with center O, radius R and orthocenter H, then the ellipse touches the sides of TUV. The porism is established.
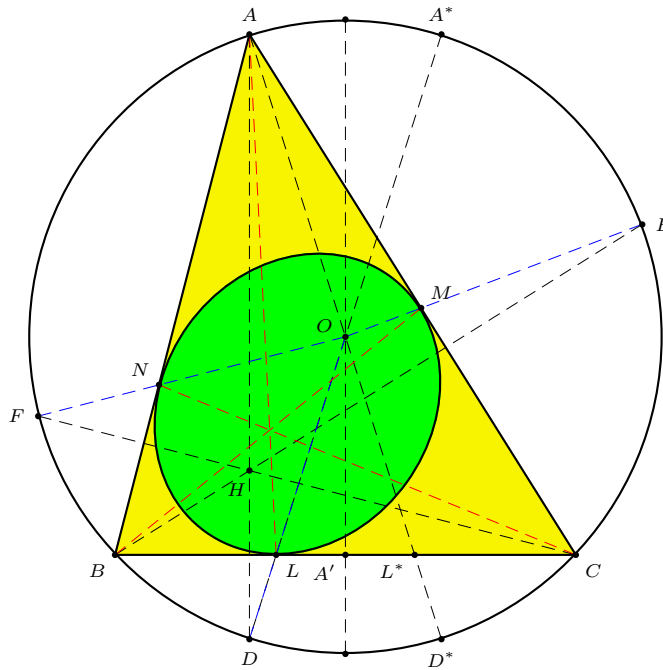
Figure 3. The inconic of the Euler line porism

Identification of the Brianchon point. This is the point of concurrence of AL, BM, CN. Since O and H are isogonal conjugates, it follows that D^* and A^* are reflections of D and A in the line which is the perpendicular bisector of BC. The same applies to B^*, C^*, E^* and F^* with respect to other perpendicular bisectors. Thus A^*D and AD^* are reflections of each other in the perpendicular bisector. Thus L^* is the reflection of L and thus $A'L = A'L^*$. Thus since AL^*, BM^*, CN^* are concurrent at O, the lines AL, BM and CN are concurrent at O_t, the isotomic conjugate of O.

The obtuse triangle case. Refer to Figure 4. Using the same notation as before, now consider the hyperbola defined as the locus of points P such that $|HP - OP| = R$. We now have $HL - OL = LD - OL = R$ so that L lies on the hyperbola.

Also $\angle A^*LB = \angle CLD = \angle HLC$, so the hyperbola touches BC at L, and the argument proceeds as before.

It is a routine matter to obtain the Cartesian equation of this inconic. Scaling so that $R = 1$ we may assume that O is at $(0, 0)$ and H at $(c, 0)$ where $0 \leq c < 3$ but $c \neq 1$.
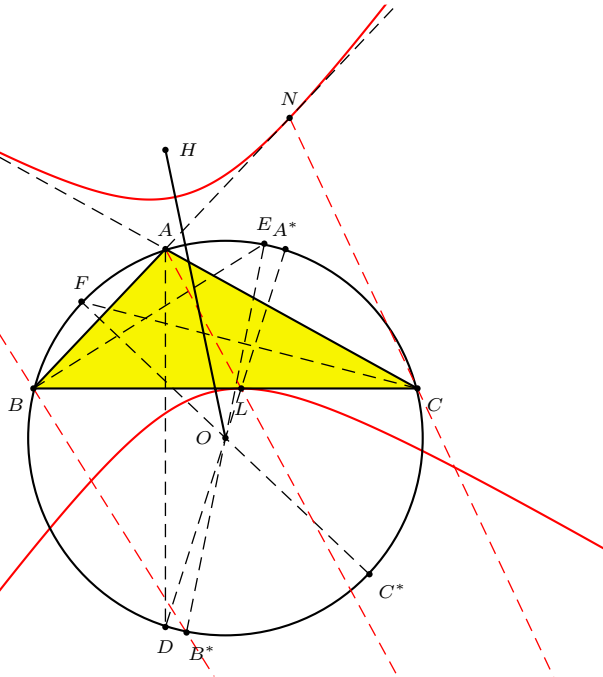
Figure 4. The Euler inconic can be a hyperbola

The inconic then has equation

$$4y^2 + (1 - c^2)(2x - c)^2 = (1 - c^2). \tag{1}$$

When $c < 1$, so H is internal to Σ, this represents an ellipse, but when $c > 1$ it represents a hyperbola. In all cases the center is at $(\frac{c}{2}, 0)$, which is the nine-point center.

One of the axes of the ellipse is the Euler line itself, whose equation is $y = 0$. We see from Equation (1) that the eccentricity of the inconic is $c = \frac{OH}{R}$ and of course its foci are at O and H. Not every tangent line to the inconic arises as a side of a triangle in the porism if H is outside Σ.

Areal analysis. One can also perform the geometric analysis of the envelope using areal co-ordinates, and we briefly report relevant equations for the reader interested in further areal work. Take ABC as triangle of reference and define $u = \cot B \cot C$, $v = \cot C \cot A$, $w = \cot A \cot B$ so that $H(u, v, w)$ and $O(v + w, w + u, u + v)$. This means that the isotomic conjugate O_t of O has co-ordinates

$$O_t \left(\frac{1}{v + w}, \frac{1}{w + u}, \frac{1}{u + v} \right).$$

The altitudes are AH, BH, CH with equations $wy = vz$, $uz = wx$, $vx = uy$ respectively.

The equation of the inconic is

$$(v + w)^2 x^2 + (w + u)^2 y^2 + (u + v)^2 z^2 - 2(w + u)(u + v)yz$$
$$-2(u + v)(v + w)zx - 2(v + w)(w + u)xy = 0. \qquad (2)$$

This curve can be parameterized by the formulas:

$$x = \frac{(1 + q)^2}{v + w}, y = \frac{1}{w + u}, z = \frac{q^2}{u + v}, \qquad (3)$$

where q has any real value (including infinity). The perpendicular lines l and l through H may be taken to pass through the points at infinity with co-ordinates $((1 + t), -t, -1)$ and $((1 + s), -s, -1)$ and then the Droz-Farny line has equation

$$-(sw + tw - 2v)(2stw - sv - tv)x - (sw + tw + 2(u + w))(2stw - sv - tv)y$$
$$+(sw + tw - 2v)(2st(u + v) + sv + tv)z = 0. \qquad (4)$$

In Equation (4) for the midpoints X, Y, Z to be collinear we must take

$$s = -\frac{v(tw + u + w)}{w(t(u + v) + v)}. \qquad (5)$$

If we now substitute Equation (3) into Equation (4) and use Equation (5), a discriminant test on the resulting quadratic equation with the help of DERIVE confirms the tangency for all values of t.

Incidentally, nowhere in this areal analysis do we use the precise values of u, v, w in terms of the angles A, B, C. Therefore we have a bonus theorem: if H is replaced by another point K, then given a line through K, there is always a second line through K (but not generally at right angles to it) so that XYZ is a straight line. As the line l rotates, l' also rotates (but not at the same rate). However the rotations of these lines is such that the variable points X, Y, Z remain collinear and the line XYZ also envelops a conic. This affine generalization of the Droz-Farny theorem was discovered independently by Charles Thas [5] in a paper published after the original submission of this article. We happily cede priority.

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