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# The Edge-Tangent Sphere of a Circumscriptible Tetrahedron

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**Abstract**. A tetrahedron is circumscriptible if there is a sphere tangent to each of its six edges. We prove that the radius $\ell$ of the edge-tangent sphere is at least $\sqrt{3}$ times the radius of its inscribed sphere. This settles affirmatively a problem posed by Z. C. Lin and H. F. Zhu. We also briefly examine the generalization into higher dimension, and pose an analogous problem for an $n-$dimensional simplex admitting a sphere tangent to each of its edges.

## 1. Introduction

Every tetrahedron has a circumscribed sphere passing through its four vertices and an inscribed sphere tangent to each of its four faces. A tetrahedron is said to be circumscriptible if there is a sphere tangent to each of its six edges (see [1, §§786–794]). We call this the edge-tangent sphere of the tetrahedron.

Let $\mathscr{P}$ denote a tetrahedron $P_0P_1P_2P_3$ with edge lengths $P_iP_j = a_{ij}$ for $0 \leq i < j \leq 3$. The following necessary and sufficient condition for a tetrahedron to admit an edge-tangent sphere can be found in [1, §§787, 790, 792]. See also [4, 6].

**Theorem 1.** *The following statement for a tetrahedron $\mathscr{P}$ are equivalent.*
(1) *$\mathscr{P}$ has an edge-tangent sphere.*
(2) $a_{01} + a_{23} = a_{02} + a_{13} = a_{03} + a_{12}$*;*
(3) *There exist $x_i > 0$, $i = 0, 1, 2, 3$, such that $a_{ij} = x_i + x_j$ for $0 \leq i < j \leq 3$.*

For $i = 0, 1, 2, 3$, $x_i$ is the length of a tangent from $P_i$ to the edge-tangent sphere of $\mathscr{P}$. Let $\ell$ denote the radius of this sphere.

**Theorem 2.** [1, §793] *The radius of the edge-tangent sphere of a circumscriptible tetrahedron of volume $V$ is given by*

$$\ell = \frac{2x_0x_1x_2x_3}{3V}. \tag{1}$$

Lin and Zhu [4] have given the formula (1) in the form

$$\ell^2 = \frac{(2x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3 \sum_{0\leq i<j\leq 3} x_ix_j - (x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2)}. \tag{2}$$

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The fact that this latter denominator is $(3V)^2$ follows from the formula for the volume of a tetrahedron in terms of its edges:

$$V^2 = \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (x_0+x_1)^2 & (x_0+x_2)^2 & (x_0+x_3)^2 \\ 1 & (x_0+x_1)^2 & 0 & (x_1+x_2)^2 & (x_1+x_3)^2 \\ 1 & (x_0+x_2)^2 & (x_1+x_2)^2 & 0 & (x_2+x_3)^2 \\ 1 & (x_0+x_3)^2 & (x_1+x_3)^2 & (x_2+x_3)^2 & 0 \end{vmatrix}.$$

Lin and Zhu *op. cit.* obtained several inequalities for the edge-tangent sphere of $\mathscr{P}$. They also posed the problem of proving or disproving $\ell^2 \geq 3r^2$ for a circumscriptible tetrahedron. See also [2]. The main purpose of this paper is to settle this problem affirmatively.

**Theorem 3.** *For a circumscriptible tetrahedron with inradius $r$ and edge-tangent sphere of radius $\ell$, $\ell \geq \sqrt{3}r$.*

## 2. Two inequalities

**Lemma 4.** *If $x_i > 0$ for $0 \leq i \leq 3$, then*

$$\left(\frac{x_1+x_2+x_3}{x_1x_2x_3} + \frac{x_2+x_3+x_0}{x_2x_3x_0} + \frac{x_3+x_0+x_1}{x_3x_0x_1} + \frac{x_0+x_1+x_2}{x_0x_1x_2}\right) \cdot \frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3 \sum\limits_{0\leq i<j\leq 3} x_ix_j - (x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2)} \geq 6. \tag{3}$$

*Proof.* From

$$x_0^2x_1^2(x_2-x_3)^2 + x_0^2x_2^2(x_1-x_3)^2 + x_0^2x_3^2(x_1-x_2)^2 + x_1^2x_2^2(x_0-x_3)^2 + x_1^2x_3^2(x_0-x_2)^2 + x_2^2x_3^2(x_0-x_1)^2 \geq 0,$$

we have

$$x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2 \geq \frac{2}{3}x_0x_1x_2x_3 \sum_{0\leq i<j\leq 3} x_ix_j,$$

and

$$2x_0x_1x_2x_3 \sum_{0\leq i<j\leq 3} x_ix_j - (x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2) \leq \frac{4}{3}x_0x_1x_2x_3 \sum_{0\leq i<j\leq 3} x_ix_j,$$

or

$$\frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3 \sum\limits_{0\leq i<j\leq 3} x_ix_j - (x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2)} \geq \frac{4(x_0x_1x_2x_3)^2}{\frac{4}{3}x_0x_1x_2x_3 \sum\limits_{0\leq i<j\leq 3} x_ix_j} = \frac{3x_0x_1x_2x_3}{\sum\limits_{0\leq i<j\leq 3} x_ix_j}. \tag{4}$$

On the other hand, it is easy to see that

$$\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{x_2+x_3+x_0}{x_2x_3x_0}+\frac{x_3+x_0+x_1}{x_3x_0x_1}+\frac{x_0+x_1+x_2}{x_0x_1x_2}=\frac{2\sum\limits_{0\le i<j\le 3}x_ix_j}{x_0x_1x_2x_3}. \tag{5}$$

Inequality (3) follows immediately from (4) and (5). □

**Corollary 5.** *For a circumscriptible tetrahedron $\mathscr{P}$ with an edge-tangent sphere of radius $\ell$, and faces with inradii $r_0$, $r_1$, $r_2$, $r_3$,*

$$\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)\ell^2\ge 6.$$

*Equality holds if and only if $\mathscr{P}$ is a regular tetrahedron.*

*Proof.* From the famous Heron formula, the inradius of a triangle $ABC$ of side-lengths $a=y+z$, $b=z+x$ and $c=x+y$ is given by

$$r^2=\frac{xyz}{x+y+z}.$$

Applying this to the four faces of $\mathscr{P}$, we see that the first factor on the left hand side of (3) is $\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)$. Now the result follows from (2). □

**Proposition 6.** *Let $\mathscr{P}$ be a circumscriptible tetrahedron of volume $V$. If, for $i=0,1,2,3$, the opposite face of vertex $P_i$ has area $\triangle_i$ and inradius $r_i$, then*

$$(\triangle_0+\triangle_1+\triangle_2+\triangle_3)^2\ge\frac{9V^2}{2}\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right). \tag{6}$$

*Equality holds if and only if $\mathscr{P}$ is a regular tetrahedron.*

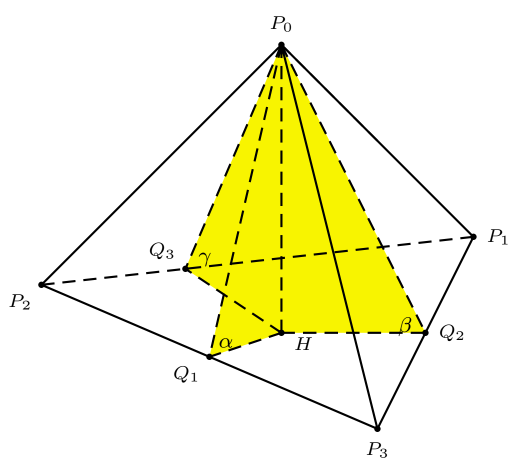


Figure 1.

*Proof.* Let $\alpha$ be the angle between the planes $P_0P_2P_3$ and $P_1P_2P_3$. If the perpendiculars from $P_0$ to the line $P_2P_3$ and to the plane $P_1P_2P_3$ intersect these at $Q_1$ and $H$ respectively, then $\angle P_0QH = \alpha$. See Figure 1. Similarly, we have the angles $\beta$ between the planes $P_0P_3P_1$ and $P_1P_2P_3$, and $\gamma$ between $P_0P_1P_2$ and $P_1P_2P_3$. Note that

$$P_0H = P_0Q_1 \cdot \sin\alpha = P_0Q_2 \cdot \sin\beta = P_0Q_3 \cdot \sin\gamma.$$

Hence,

$$P_0H \cdot P_2P_3 = 2\triangle_1 \sin\alpha = 2\sqrt{(\triangle_1 + \triangle_1\cos\alpha)(\triangle_1 - \triangle_1\cos\alpha)}, \tag{7}$$

$$P_0H \cdot P_3P_1 = 2\triangle_2 \sin\beta = 2\sqrt{(\triangle_2 + \triangle_2\cos\beta)(\triangle_2 - \triangle_2\cos\beta)}, \tag{8}$$

$$P_0H \cdot P_1P_2 = 2\triangle_3 \sin\gamma = 2\sqrt{(\triangle_3 + \triangle_3\cos\gamma)(\triangle_3 - \triangle_3\cos\gamma)}. \tag{9}$$

From (7–9), together with $P_0H = \frac{3V}{\triangle_0}$ and $\frac{\triangle_0}{r_0} = \frac{1}{2}(P_1P_2 + P_2P_3 + P_3P_1)$, we have

$$\begin{aligned}\frac{3V}{r_0} =& \sqrt{(\triangle_1 + \triangle_1\cos\alpha)(\triangle_1 - \triangle_1\cos\alpha)} \\ &+ \sqrt{(\triangle_2 + \triangle_2\cos\beta)(\triangle_2 - \triangle_2\cos\beta)} \\ &+ \sqrt{(\triangle_3 + \triangle_3\cos\gamma)(\triangle_3 - \triangle_3\cos\gamma)}.\end{aligned} \tag{10}$$

Applying Cauchy's inequality and noting that

$$\triangle_0 = \triangle_1\cos\alpha + \triangle_2\cos\beta + \triangle_3\cos\gamma,$$

we have

$$\begin{aligned}\left(\frac{3V}{r_0}\right)^2 \leq& (\triangle_1 + \triangle_1\cos\alpha + \triangle_2 + \triangle_2\cos\beta + \triangle_3 + \triangle_3\cos\gamma) \\ &\cdot(\triangle_1 - \triangle_1\cos\alpha + \triangle_2 - \triangle_2\cos\beta + \triangle_3 - \triangle_3\cos\gamma) \\ =& (\triangle_1 + \triangle_2 + \triangle_3 + \triangle_0)(\triangle_1 + \triangle_2 + \triangle_3 - \triangle_0) \\ =& (\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2,\end{aligned} \tag{11}$$

or

$$(\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2 \geq \left(\frac{3V}{r_0}\right)^2. \tag{12}$$

It is easy to see that equality in (12) holds if and only if

$$\frac{\triangle_1 + \triangle_1\cos\alpha}{\triangle_1 - \triangle_1\cos\alpha} = \frac{\triangle_2 + \triangle_2\cos\beta}{\triangle_2 - \triangle_2\cos\beta} = \frac{\triangle_3 + \triangle_3\cos\gamma}{\triangle_3 - \triangle_3\cos\gamma}.$$

Equivalently, $\cos\alpha = \cos\beta = \cos\gamma$, or $\alpha = \beta = \gamma$. Similarly, we have

$$(\triangle_2 + \triangle_3 + \triangle_0)^2 - \triangle_1^2 \geq \left(\frac{3\,V}{r_1}\right)^2, \tag{13}$$

$$(\triangle_3 + \triangle_0 + \triangle_1)^2 - \triangle_2^2 \geq \left(\frac{3\,V}{r_2}\right)^2, \tag{14}$$

$$(\triangle_0 + \triangle_1 + \triangle_2)^2 - \triangle_3^2 \geq \left(\frac{3\,V}{r_3}\right)^2. \tag{15}$$

Summing (12) to (15), we obtain the inequality (6), with equality precisely when all dihedral angles are equal, *i.e.*, when $\mathscr{P}$ is a regular tetrahedron. □

*Remark.* Inequality (6) is obtained by X. Z. Yang in [5].

## 3. Proof of Theorem 3

Since $r = \frac{3V}{\triangle_0+\triangle_1+\triangle_2+\triangle_3}$, it follows from Proposition 6 and Corollary 5 that

$$\ell^2 \geq \frac{6}{\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}} \geq \frac{27V^2}{(\triangle_0+\triangle_1+\triangle_2+\triangle_3)^2} = 3r^2.$$

This completes the proof of Theorem 3.

## 4. A generalization with an open problem

As a generalization of the tetrahedron, we say that an $n-$dimensional simplex is circumscriptible if there is a sphere tangent to each of its edges. The following basic properties of a circumscriptible simplex can be found in [3].

**Theorem 7.** *Suppose the edge lengths of an $n$-simplex $\mathscr{P} = P_0P_1\cdots P_n$ are $P_iP_j = a_{ij}$ for $0 \leq i < j \leq n$. The $n$-simplex has an edge-tangent sphere if and only if there exist $x_i$, $i = 0, 1, \ldots, n$, satisfying $a_{ij} = x_i + x_j$ for $0 \leq i \neq j \leq n$. In this case, the radius of the edge-tangent sphere is given by*

$$\ell^2 = -\frac{D_1}{2D_2}, \tag{16}$$

*where*

$$D_1 = \begin{vmatrix} -2x_0^2 & 2x_0x_1 & \cdots & 2x_0x_{n-1} \\ 2x_0x_1 & -2x_1^2 & \cdots & 2x_1x_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 2x_0x_{n-1} & 2x_1x_{n-1} & \cdots & -2x_{n-1}^2 \end{vmatrix},$$

*and*

$$D_2 = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \cdot & \cdots & \cdot \\ \vdots & \vdots & D_1 & \vdots \\ 1 & \cdot & \cdots & \cdot \end{vmatrix}.$$

We conclude this paper with an open problem: for a circumscriptible $n$-simplex with a circumscribed sphere of radius $R$, an inscribed sphere of radius $r$ and an edge-tangent sphere of radius $\ell$, prove or disprove that

$$R \geq \sqrt{\frac{2n}{n-1}}l \geq nr.$$

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