

## A Stronger Triangle Inequality for Neutral Geometry

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**Abstract.** Bailey and Bannister [*College Math. Journal*, 28 (1997) 182–186] proved that a stronger triangle inequality holds in the Euclidean plane for all triangles having largest angle less than  $\arctan(\frac{24}{7}) \approx 74^\circ$ . We use hyperbolic trigonometry to show that a stronger triangle inequality holds in the hyperbolic plane for all triangles having largest angle less than or equal to  $65.87^\circ$ .

### 1. Introduction

One of the most fundamental results of neutral geometry is the triangle inequality. How can this cherished inequality be strengthened? Under certain restrictions, the sum of the lengths of two sides of a triangle is greater than the length of the remaining side plus the length of the altitude to this side.

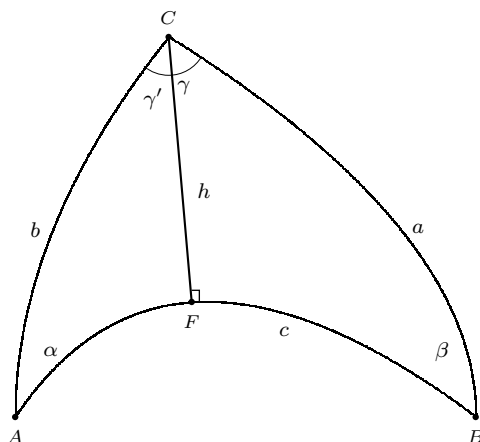


Figure 1. Strong triangle inequality  $a + b > c + h$

Let  $ABC$  be a triangle belonging to neutral geometry (see Figure 1). Let  $a$ ,  $b$  and  $c$  be the lengths of sides  $BC$ ,  $AC$  and  $AB$ , respectively. Also, let  $\alpha$ ,  $\beta$  and  $\gamma$  denote the angles at  $A$ ,  $B$  and  $C$  respectively. If we let  $F$  be the foot of the perpendicular from  $C$  onto side  $AB$  and if  $h$  is the length of the segment  $CF$ , when is it true that  $a + b > c + h$ ? Since  $a > h$  and  $b > h$ , this question is of interest only if  $c$  is the length of the longest side of  $ABC$ , or, equivalently, if  $\gamma$  is the the largest angle of  $ABC$ . With this notation, if the inequality  $a + b > c + h$  holds where  $\gamma$  is the largest angle of the triangle  $ABC$ , we say that  $ABC$  satisfies the *strong triangle inequality*.

The following result, due to Bailey and Bannister [1], explains what happens if the triangle  $ABC$  belongs to Euclidean geometry.

**Theorem 1.** *If  $ABC$  is a Euclidean triangle having largest angle  $\gamma < \arctan(\frac{24}{7}) \approx 74^\circ$ , then  $ABC$  satisfies the strong triangle inequality.*

An elegant trigonometric proof of Theorem 1 can be found in [3]. It should be noted that the bound of  $\arctan(\frac{24}{7})$  is the best possible since any isosceles Euclidean triangle with  $\gamma = \arctan(\frac{24}{7})$  violates the strong triangle inequality.

The goal of this note is to extend the Bailey and Bannister result to neutral geometry. To get the appropriate bound for the extended result we need the function

$$f(\gamma) := -1 - \cos \gamma + \sin \gamma + \sin \frac{\gamma}{2} \sin \gamma. \quad (1)$$

Observe that  $f'(\gamma) = \sin \gamma + \cos \gamma + \sin \frac{\gamma}{2} \cos \gamma + \frac{1}{2} \cos \frac{\gamma}{2} \sin \gamma > 0$  on the interval  $[0, \frac{\pi}{2}]$ . Therefore,  $f(\gamma)$  is strictly monotone increasing on the interval  $(0, \frac{\pi}{2})$ . Since  $f(0) = -2$ ,  $f(\frac{\pi}{2}) = \frac{\sqrt{2}}{2}$ , and  $f$  is continuous it follows that  $f$  has a unique root  $r$  in the interval  $(0, \frac{\pi}{2})$ . In fact,  $r$  is approximately 1.15 (radians) or  $65.87^\circ$ . See Figure 2.

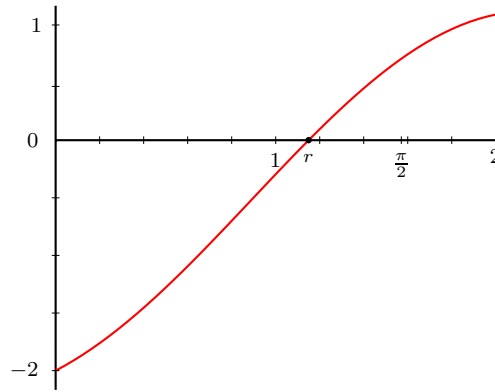


Figure 2. Graph of  $f(\gamma)$

**Theorem 2.** *In neutral geometry a triangle  $ABC$  having largest angle  $\gamma$  satisfies the strong triangle inequality if  $\gamma \leq r \approx 1.15$  radians or  $65.87^\circ$ .*

The proof of Theorem 2 is based on the fact that a model of neutral geometry is isomorphic to either the Euclidean plane or a hyperbolic plane. Given Theorem 1, it is enough to establish our result for the case of hyperbolic geometry. Moreover, since the strong triangle inequality holds if and only if  $ka + kb > kc + kh$  for any positive constant  $k$ , it is enough to assume that the distance scale in hyperbolic geometry is 1. An explanation about the distance scale  $k$  and how it is used in hyperbolic geometry can be found in [4].

## 2. Hyperbolic trigonometry

Recall that the hyperbolic sine and hyperbolic cosine functions are given by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

The formulas needed to prove the main result are given below. First, there are the standard identities

$$\cosh^2 x - \sinh^2 x = 1 \tag{2}$$

and

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y. \tag{3}$$

If  $ABC$  is a hyperbolic triangle with a right angle at  $C$ , i.e.,  $\gamma = \frac{\pi}{2}$ , then

$$\sinh a = \sinh c \sin \alpha \tag{4}$$

and

$$\cosh a \sin \beta = \cos \alpha. \tag{5}$$

For any hyperbolic triangle  $ABC$ ,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma, \tag{6}$$

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}, \tag{7}$$

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}. \tag{8}$$

See [2, Chapter 10] or [5, Chapter 8] for more details regarding (4 – 8).

## 3. Proof of Theorem 2

The strong triangle inequality  $a + b > c + h$  holds if and only if  $\cosh(a + b) > \cosh(c + h)$ . Expanding both sides by the identity given in (3) we have

$$\cosh a \cosh b + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h,$$

$$\cosh c + \sinh a \sinh b \cos \gamma + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h, \text{ by (6)}$$

$$\cosh c (1 - \cosh h) + \sinh a \sinh b (\cos \gamma + 1) - \sinh c \sinh h > 0.$$

Since  $ACF$  is a right triangle with the length of  $CF$  equal to  $h$ , it follows from (4) that  $\sinh h = \sinh b \sin \alpha$ . Applying (7), we have

$$\cosh c (1 - \cosh h) + \sinh a \sinh b (\cos \gamma + 1) - \frac{\sinh a}{\sin \alpha} \cdot \sin \gamma \sinh b \sin \alpha > 0,$$

$$\cosh c (1 - \cosh h) + \sinh a \sinh b (\cos \gamma + 1 - \sin \gamma) > 0,$$

$$\cosh c (1 - \cosh^2 h) + \sinh a \sinh b (1 + \cosh h) (\cos \gamma + 1 - \sin \gamma) > 0,$$

$$\cosh c (-\sinh^2 h) + \sinh a \sinh b (1 + \cosh h) (\cos \gamma + 1 - \sin \gamma) > 0, \quad \text{by (2)}$$

$$\cosh c (-\sinh^2 b \sin^2 \alpha) + \sinh a \sinh b (1 + \cosh h) (\cos \gamma + 1 - \sin \gamma) > 0.$$

Dividing both sides of the inequality by  $\sinh b > 0$ , we have

$$-\cosh c \sinh b \sin^2 \alpha + \sinh a(1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0.$$

By (7) and (8), we have

$$-\left(\frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}\right) \frac{\sinh a \sin \beta}{\sin \alpha} \sin^2 \alpha + \sinh a(1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0.$$

Simplifying and dividing by  $\sinh a > 0$ , we have

$$\begin{aligned} & -(\cos \alpha \cos \beta + \cos \gamma) \sinh a + \sinh a(1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0, \\ & -(\cos \alpha \cos \beta + \cos \gamma) + (1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0, \end{aligned} \quad (9)$$

We have manipulated the original inequality into one involving the original angles,  $\alpha$ ,  $\beta$ , and  $\gamma$ , and the length of the altitude on  $AB$ . In the right triangle  $ACF$ , let  $\gamma'$  be the angle at  $C$ . We may assume  $\gamma' \leq \frac{\gamma}{2}$  (otherwise we can work with the right triangle  $BCF$ ). Applying (5) to triangle  $ACF$  gives  $\cosh h = \frac{\cos \alpha}{\sin \gamma'}$ . Now continuing with the inequality (9) we get

$$-(\cos \alpha \cos \beta + \cos \gamma) + \left(1 + \frac{\cos \alpha}{\sin \gamma'}\right) (1 + \cos \gamma - \sin \gamma) > 0$$

Multiplying both sides by  $-\sin \gamma' < 0$ , we have

$$\sin \gamma' (\cos \alpha \cos \beta + \cos \gamma) - (\sin \gamma' + \cos \alpha) (1 + \cos \gamma - \sin \gamma) < 0,$$

Simplifying this and rearranging terms, we have

$$\cos \alpha (\sin \gamma' \cos \beta - 1 - \cos \gamma + \sin \gamma) + \sin \gamma' (\sin \gamma - 1) < 0. \quad (10)$$

If  $\sin \gamma' \cos \beta - 1 - \cos \alpha + \sin \alpha > 0$ , then

$$\begin{aligned} & \cos \alpha (\sin \gamma' \cos \beta - 1 - \cos \gamma + \sin \gamma) + \sin \gamma' (\sin \gamma - 1) \\ & < \sin \gamma' - 1 - \cos \gamma + \sin \gamma + \sin \gamma' (\sin \gamma - 1) \\ & = -1 - \cos \gamma + \sin \gamma + \sin \gamma' \sin \gamma \\ & \leq -1 - \cos \gamma + \sin \gamma + \sin \frac{\gamma}{2} \sin \gamma. \end{aligned}$$

Note that this last expression is  $f(\gamma)$  defined in (1). We have shown that

$$\cos \alpha (\sin \gamma' \cos \beta - 1 - \cos \gamma + \sin \gamma) + \sin \gamma' (\sin \gamma - 1) < \max\{0, f(\gamma)\}.$$

For  $\gamma \leq r$ , we have  $f(\gamma) \leq 0$  and the strong triangle inequality holds.

This completes the proof of Theorem 2.

If  $r < \gamma < \frac{\pi}{2}$ , then  $f(\gamma) > 0$ . In this case, we can find an angle  $\alpha$  such that  $0 < \alpha < \frac{\pi}{2} - \frac{\gamma}{2}$  and

$$\cos \alpha \left( \sin \frac{\gamma}{2} \cos \alpha - 1 - \cos \alpha + \sin \alpha \right) + \sin \frac{\gamma}{2} (\sin \gamma - 1) > 0.$$

Since  $\gamma + 2\alpha < \pi$  it follows from [5, Theorem 6.7] that there exists a hyperbolic triangle  $ABC$  with angles  $\alpha$ ,  $\alpha$ , and  $\gamma$ . Our previous work shows that the

triangle  $ABC$  satisfies the strong triangle inequality if and only if (10) holds. Consequently,  $a + b > c + h$  provided  $f(\gamma) \leq 0$ . Therefore, the bound  $r$  given in Theorem 2 is the best possible.

## References

- [1] H. R. Bailey and R. Bannister, A stronger triangle inequality, *College Math. J.*, 28 (1997) 182–186.
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