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The Method of Punctured Containers

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Abstract. We introduce the method of punctured containers, which geometrically relates volumes and centroids of complicated solids to those of simpler punctured prismatic solids. This method goes to the heart of some of the basic properties of the sphere, and extends them in natural and significant ways to solids assembled from cylindrical wedges (Archimedean domes) and to more general solids, especially those with nonuniform densities.

1. Introduction

Archimedes (287-212 B.C.) is regarded as the greatest mathematician of ancient times because of his masterful and innovative treatment of a remarkable range of topics in both pure and applied mathematics. One landmark discovery is that the volume of a solid sphere is two-thirds the volume of its circumscribing cylinder, and that the surface area of the sphere is also two-thirds the total surface area of the same cylinder. Archimedes was so proud of this revelation that he wanted the sphere and circumscribing cylinder engraved on his tombstone. He discovered the volume ratio by balancing slices of the sphere against slices of a *larger* cylinder and cone, using centroids and the law of the lever, which he had also discovered.

Today we know that the volume ratio for the sphere and cylinder can be derived more simply by an elementary geometric method that Archimedes overlooked. It is illustrated in Figure 1. By symmetry it suffices to consider a hemisphere, as in Figure 1a, and its circumscribing cylindrical container. Figure 1b shows the cylinder with a solid cone removed. The punctured cylindrical container has exactly the same volume as the hemisphere, because every horizontal plane cuts the hemisphere and the punctured cylinder in cross sections of equal area. The cone's volume is one-third that of the cylinder, hence the hemisphere's volume is two-thirds that of the cylinder, which gives the Archimedes volume ratio for the sphere and its circumscribing cylinder.

This geometric method extends to more general solids we call Archimedean domes. They and their punctured prismatic containers are described below in Section 2. Any plane parallel to the equatorial base cuts such a dome and its punctured container in cross sections of equal area. This implies that two planes parallel to the base cut the dome and the punctured container in slices of equal volumes, equality of volumes being a consequence of the following:

Slicing principle. *Two solids have equal volumes if their horizontal cross sections taken at any height have equal areas.*

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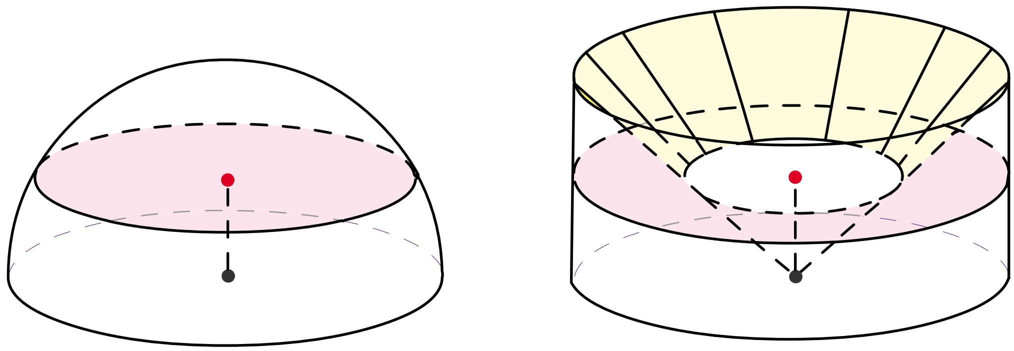


(a) (b)

Figure 1. (a) A hemisphere and (b) a punctured cylindrical container of equal volume.

This statement is often called Cavalieri's principle in honor of Bonaventura Cavalieri (1598-1647), who attempted to prove it for general solids. Archimedes used it sixteen centuries earlier for special solids, and he credits Eudoxus and Democritus for using it even earlier in their discovery of the volume of a cone. Cavalieri employed it to find volumes of many solids, and tried to establish the principle for general solids by applying Archimedes' method of exhaustion, but it was not demonstrated rigorously until integral calculus was developed in the 17th century. We prefer using the neutral and more descriptive term *slicing principle*.

To describe the slicing principle in the language of calculus, cut two solids by horizontal planes that produce cross sections of equal area $A(x)$ at an arbitrary height x above a fixed base. The integral $\int_{x_1}^{x_2} A(x)\,dx$ gives the volume of the portion of each solid cut by all horizontal planes as x varies over some interval $[x_1,\, x_2]$. Because the integrand $A(x)$ is the same for both solids, the corresponding volumes are also equal. We could just as well integrate any function $f(x, A(x))$, and the integral over the interval $[x_1,\, x_2]$ would be the same for both solids. For example, $\int_{x_1}^{x_2} xA(x)\,dx$ is the first moment of the area function over the interval $[x_1,\, x_2]$, and this integral divided by the volume gives the altitude of the *centroid* of the slice between the planes $x = x_1$ and $x = x_2$. Thus, not only are the volumes of these slices equal, but also the altitudes of their centroids are equal. Moreover, all moments $\int_{x_1}^{x_2} x^k A(x)\,dx$ with respect to the plane of the base are equal for both slices.

In [1; Theorem 6a] we showed that the lateral surface area of any slice of an Archimedean dome between two parallel planes is equal to the lateral surface area of the corresponding slice of the circumscribing (unpunctured) prism. This was deduced from the fact that Archimedean domes circumscribe hemispheres. It implies that the total surface area of a sphere is equal to the lateral surface area of its circumscribing cylinder which, in turn, is two-thirds the total surface area of the cylinder. The surface area ratio was discovered by Archimedes by a completely different method.

This paper extends our geometric method further, from Archimedean domes to more general solids. First we dilate an Archimedean dome in a vertical direction to produce a dome with elliptic profiles, then we replace its base by an arbitrary polygon, not necessarily convex. This leads naturally to domes with arbitrary curved bases. Such domes and their punctured prismatic containers have equal volumes and equal moments relative to the plane of the base because of the slicing principle, but if these domes do not circumscribe hemispheres the corresponding lateral surface areas will not be equal. This paper relaxes the requirement of equal surface areas and concentrates on solids having the same volume and moments as their punctured prismatic containers. We call such solids *reducible* and describe them in Section 3. Section 4 treats reducible domes and shells with polygonal bases, then Section 5 extends the results to domes with curved bases, and formulates reducibility in terms of mappings that preserve volumes and moments.

The full power of our method, which we call *the method of punctured containers*, is revealed by the treatment of nonuniform mass distributions in Section 6. Problems of calculating masses and centroids of nonuniform wedges, shells, and their slices with elliptic profiles, including those with cavities, are reduced to those of *simpler punctured prismatic containers*. Section 7 gives explicit formulas for volumes and centroids, and Section 8 reveals the surprising fact that uniform domes are reducible to their punctured containers if and only if they have elliptic profiles.

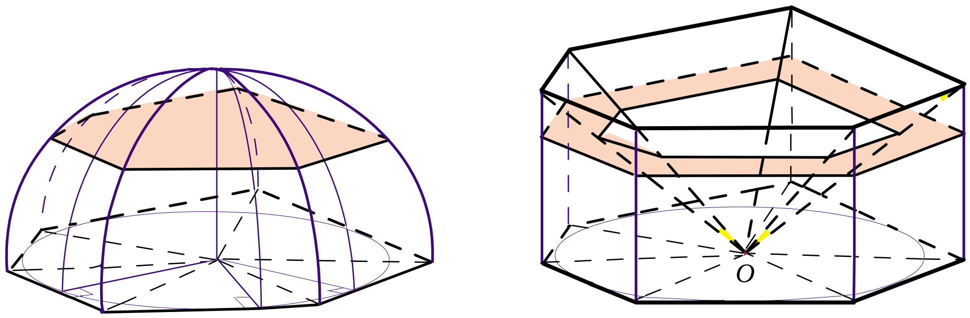
2. Archimedean domes

Archimedean domes are solids of the type shown in Figure 2a, formed by assembling portions of circular cylindrical wedges. Each dome circumscribes a hemisphere, and its horizontal base is a polygon, not necessarily regular, circumscribing the equator of the hemisphere. Cross sections cut by planes parallel to the base are similar polygons circumscribing the cross sections of the hemisphere. Figure 2b shows the dome's punctured prismatic container, a circumscribing prism, from which a pyramid with congruent polygonal base has been removed as indicated. The shaded regions in Figure 2 illustrate the fundamental relation between any Archimedean dome and its punctured prismatic container:

Each horizontal plane cuts both solids in cross sections of equal area.

Hence, by the slicing principle, any two horizontal planes cut both solids in slices of equal volume. Because the removed pyramid has volume one-third that of the unpunctured prism, we see that the volume of any Archimedean dome is two-thirds that of its punctured prismatic container.

We used the name "Archimedean dome" because of a special case considered by Archimedes. In his preface to The Method [3; Supplement, p. 12] Archimedes announced (without proof) that the volume of intersection of two congruent orthogonal circular cylinders is two-thirds the volume of the circumscribing cube. In [3; pp. 48-50], Zeuthen verifies this with the method of centroids and levers employed by Archimedes in treating the sphere. However, if we observe that half the solid of intersection is an Archimedean dome with a square base, and compare its volume with that of its punctured prismatic container, we immediately obtain the required two-thirds volume ratio.



(a) Archimedean dome (b) Punctured prismatic container

Figure 2. Each horizontal plane cuts the dome and its punctured prismatic container in cross sections of equal area.

As a limiting case, when the polygonal cross sections of an Archimedean dome become circles, and the punctured container becomes a circumscribing cylinder punctured by a cone, we obtain a purely geometric derivation of the Archimedes volume ratio for a sphere and cylinder.

When an Archimedean dome and its punctured container are *uniform* solids, made of material of the same constant density (mass per unit volume), the corresponding horizontal slices also have equal masses, and the center of mass of each slice lies at the same height above the base [1; Section 9].

3. Reducible solids

This paper extends the method of punctured containers by applying it first to general dome-like structures far removed from Archimedean domes, and then to domes with *nonuniform* mass distributions. The generality of the structures is demonstrated by the following examples.

Cut any Archimedean dome and its punctured container into horizontal slices and assign to each pair of slices the same constant density, which can differ from pair to pair. Because the masses are equal slice by slice, the total mass of the dome is equal to that of its punctured container, and the centers of mass are at the same height. Or, cut the dome and its punctured container into wedges by vertical half planes through the polar axis, and assign to each pair of wedges the same constant density, which can differ from pair to pair. Again, the masses are equal wedge by wedge, so the total mass of the dome is equal to that of its punctured container, and the centers of mass are at the same height. Or, imagine an Archimedean dome divided into thin concentric shell-like layers, like those of an onion, each assigned its own constant density, which can differ from layer to layer. The punctured container is correspondingly divided into coaxial prismatic layers, each assigned the same constant density as the corresponding shell layer. In this case the masses are equal shell by shell, so the total mass of the dome is equal to that of its punctured container, and again the centers of mass are at the same height. We are interested in a class of solids, with pyramidically punctured prismatic containers, that share the following property with Archimedean domes:

Definition. (Reducible solid) A solid is called reducible if an arbitrary horizontal slice of the solid and its punctured container have equal volumes, equal masses, and hence centers of mass at the same height above the base.

Every uniform Archimedean done is reducible, and in Section 5 we exhibit some nonuniform Archimedean domes that are reducible as well.

The method of punctured containers enables us to reduce both volume and mass calculations of domes to those of simpler prismatic solids, thus generalizing the profound volume relation between the sphere and cylinder discovered by Archimedes. Another famous result of Archimedes [3; Method, Proposition 6] states that the centroid of a uniform solid hemisphere divides its altitude in the ratio 5:3. Using the method of punctured containers we show that the same ratio holds for uniform Archimedean domes and other more general domes (Theorem 7), and we also extend this result to the center of mass of a more general class of nonuniform reducible domes (Theorem 8).

4. Polygonal elliptic domes and shells

To easily construct a more general class of reducible solids, start with any Archimedean dome, and dilate it and its punctured container in a vertical direction by the same scaling factor $\lambda > 0$. The circular cylindrical wedges in Figure 2a become elliptic cylindrical wedges, as typified by the example in Figure 3a. A circular arc of radius a is dilated into an elliptic arc with horizontal semi axis a and vertical semi axis λa. Dilation changes the altitude of the prismatic wedge from a to λa (Figure 3b). The punctured container is again a prism punctured by a pyramid.

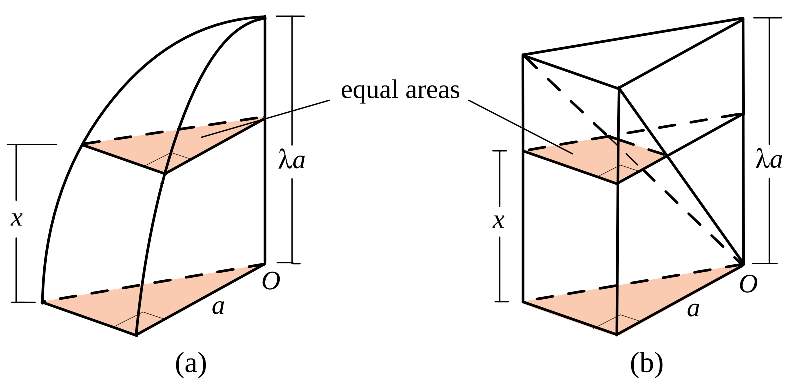


Figure 3. (a) Vertical dilation of a cylindrical wedge by a factor λ. (b) Its punctured prismatic container.

Each horizontal plane at a given height above the base cuts both the elliptic wedge and the corresponding punctured prismatic wedge in cross sections of equal area. Consequently, any two horizontal planes cut both solids in slices of equal volume.

If the elliptic and prismatic wedges have the same constant density, then they also have the same mass, and their centers of mass are at the same height above the base. In other words, we have:

Theorem 1. *Every uniform elliptic cylindrical wedge is reducible.*

Now assemble a finite collection of nonoverlapping elliptic cylindrical wedges with their horizontal semi axes, possibly of different lengths, in the same horizontal plane, but having a *common vertical semi axis*, which meets the base at a point O called the *center*. We assume the density of each component wedge is constant, although this constant may differ from component to component. For each wedge, the punctured circumscribing prismatic container with the same density is called its *prismatic counterpart*. The punctured containers assembled in the same manner produce the counterpart of the wedge assemblage. We call an assemblage *nonuniform* if some of its components can have different constant densities. This includes the special case of a *uniform* assemblage where all components have the same constant density. Because each wedge is reducible we obtain:

Corollary 1. *Any nonuniform assemblage of elliptic cylindrical wedges is reducible.*

Polygonal elliptic domes. Because the base of a finite assemblage is a polygon (a union of triangles with a common vertex O) we call the assemblage a *polygonal elliptic dome*. The polygonal base need not circumscribe a circle and it need not be convex. Corollary 1 gives us:

Corollary 2. *The volume of any polygonal elliptic dome is equal to the volume of its circumscribing punctured prismatic container, that is, two-thirds the volume of the unpunctured prismatic container, which, in turn, is the area of the base times the height.*

In the special limiting case when the equatorial polygonal base of the dome turns into an ellipse with center at O, the dome becomes half an ellipsoid, and the circumscribing prism becomes an elliptic cylinder. In this limiting case, Corollary 2 reduces to:

Corollary 3. *The volume of any ellipsoid is two-thirds that of its circumscribing elliptic cylinder.*

In particular, we have Archimedes' result for "spheroids" [3; Method, Proposition 3]:

Corollary 4. (Archimedes) *The volume of an ellipsoid of revolution is two-thirds that of its circumscribing circular cylinder.*

Polygonal elliptic shells. A *polygonal elliptic shell* is the solid between two concentric similar polygonal elliptic domes. From Theorem 1 we also obtain:

Theorem 2. *The following solids are reducible:*
(a) *Any uniform polygonal elliptic shell.*
(b) *Any wedge of a uniform polygonal elliptic shell.*
(c) *Any horizontal slice of a wedge of type* (b).
(d) *Any nonuniform assemblage of shells of type* (a).
(e) *Any nonuniform assemblage of wedges of type* (b).
(f) *Any nonuniform assemblage of slices of type* (c).

By using as building blocks horizontal slices of wedges cut from a polygonal elliptic shell, we can see intuitively how one might construct, from such building blocks, very general polygonal elliptic domes that are reducible and have more or less arbitrary mass distribution. By considering limiting cases of polygonal bases with many edges, and building blocks with very small side lengths, we can imagine elliptic shells and domes whose bases are more or less arbitrary plane regions, for example, elliptic, parabolic or hyperbolic segments.

The next section describes an explicit construction of general reducible domes with curvilinear bases.

5. General elliptic domes

Replace the polygonal base by any plane region bounded by a curve whose polar coordinates (r, θ) relative to a "center" O satisfy an equation $r = \rho(\theta)$, where ρ is a given piecewise continuous function, and θ varies over an interval of length 2π. Above this base we build an elliptic dome as follows. First, the altitude of the dome is a segment of fixed height $h > 0$ along the polar axis perpendicular to the base at O. We assume that each vertical half plane through the polar axis at angle θ cuts the surface of the dome along a quarter of an ellipse with horizontal semi axis $\rho(\theta)$ and the same vertical semi axis h, as in Figure 4a. The ellipse will be degenerate at points where $\rho(\theta) = 0$. Thus, an elliptic wedge is a special case of an elliptic dome.

When $\rho(\theta) > 0$, the cylindrical coordinates (r, θ, z) of points on the surface of the dome satisfy the equation of an ellipse:

$$\left(\frac{r}{\rho(\theta)}\right)^2 + \left(\frac{z}{h}\right)^2 = 1. \tag{1}$$

The dome is circumscribed by a cylindrical solid of altitude h whose base is congruent to that of the dome (Figure 4b). Incidentally, we use the term "cylindrical solid" with the understanding that the solid is a prism when the base is polygonal.

Each point (r', θ', z') on the lateral surface of the cylinder in Figure 4b is related to the corresponding point (r, θ, z) on the surface of the dome by the equations

$$\theta' = \theta, \quad z' = z, \quad r' = \rho(\theta).$$

From this cylindrical solid we remove a conical solid whose surface points have cylindrical coordinates (r'', θ, z), where $z/h = r''/\rho(\theta)$, or

$$r'' = z\rho(\theta)/h.$$

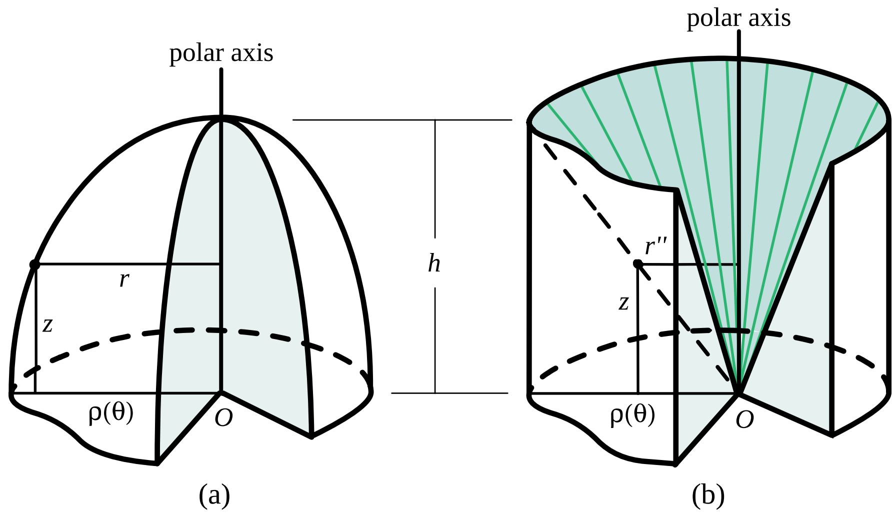


Figure 4. An elliptic dome (a), and its circumscribing punctured prismatic container (b).

When $z = h$, this becomes $r'' = \rho(\theta)$, so the base of the cone is congruent to the base of the elliptic dome. When the base is polygonal, the conical solid is a pyramid.

More reducible domes. The polar axis of an elliptic dome depends on the location of center O. For a given curvilinear base, we can move O to any point inside the base, or even to the boundary. Moving O will change the function $\rho(\theta)$ describing the boundary of the base, with a corresponding change in the shape of the ellipse determined by (1). Thus, this construction generates not one, but *infinitely many elliptic domes* with a given base. For any such dome, we can generate another family as follows: Imagine the dome and its prismatic counterpart made up of very thin horizontal layers, like two stacks of cards. Deform each solid by a horizontal translation and rotation of each horizontal layer. The shapes of the solids will change, but their cross-sectional areas will not change. In general, such a deformation may alter the shape of each ellipse on the surface to some other curve, and the deformed dome will no longer be elliptic. The same deformation applied to the prismatic counterpart will change the punctured container to a nonprismatic punctured counterpart. Nevertheless, all the results of this paper (with the exception of Theorem 11) will hold for such deformed solids and their counterparts.

However, if the deformation is a linear shearing that leaves the base fixed but translates each layer by a distance proportional to its distance from the base, then straight lines are mapped onto straight lines and the punctured prismatic solid is deformed into another prism punctured by a pyramid with the same base. The correspondingly sheared dome will be *elliptic* because each elliptic curve on the surface of the dome is deformed into an elliptic curve. To visualize a physical model of such a shearing, imagine a general elliptic dome and its counterpart sliced horizontally to form stacks of cards. Pierce each stack by a long pin along the polar axis, and let O be the point where the tip of the pin touches the base. Tilting the pin away from the vertical polar axis, keeping O fixed, results in horizontal linear shearing of the stacks and produces infinitely many elliptic domes, all with the same polygonal base. The prismatic containers are correspondingly tilted, and the domes are reducible.

Reducibility mapping. For a given general elliptic dome, we call the corresponding circumscribing punctured cylindrical solid *its punctured container*. Our goal is to show that *every uniform general elliptic dome is reducible*. This will be deduced from a more profound property, stated below in Theorem 3. It concerns a mapping that relates elliptic domes and their punctured containers.

To determine this mapping, regard the dome as a collection of layers of similar elliptic domes, like layers of an onion. Choose O as the center of similarity, and for each scaling factor $\mu \leq 1$, imagine a surface $E(\mu)$ such that a vertical half plane through the polar axis at angle θ intersects $E(\mu)$ along a quarter of an ellipse with semiaxes $\mu\rho(\theta)$ and μh. When $\rho(\theta) > 0$, the coordinates r and z of points on this similar ellipse satisfy

$$\left(\frac{r}{\mu\rho(\theta)}\right)^2 + \left(\frac{z}{\mu h}\right)^2 = 1. \tag{2}$$

Regard the punctured container as a collection of coaxial layers of similar punctured cylindrical surfaces $C(\mu)$.

It is easy to relate the cylindrical coordinates $(r',\, \theta',\, z')$ of each point on $C(\mu)$ to the coordinates $(r,\, \theta,\, z)$ of the corresponding point on $E(\mu)$. First, we have

$$\theta' = \theta, \quad z' = z, \quad r' = \mu\rho(\theta). \tag{3}$$

From (2) we find $r^2 + z^2\rho(\theta)^2/h^2 = \mu^2\rho(\theta)^2$, hence (3) becomes

$$\theta' = \theta, \quad z' = z, \quad r' = \sqrt{r^2 + z^2\rho(\theta)^2/h^2}. \tag{4}$$

The three equations in (4), which are independent of μ, describe a *mapping* from each point $(r,\, \theta,\, z)$, not on the polar axis, of the solid elliptic dome to the corresponding point $(r',\, \theta',\, z')$ on its punctured container. On the polar axis, $r = 0$ and θ is undefined.

Using (2) in (4) we obtain (3), hence points on the ellipse described by (2) are mapped onto the vertical segment of length μh through the base point $(\mu\rho(\theta),\, \theta)$. It is helpful to think of the solid elliptic dome as made up of *elliptic fibers* emanating from the points on the base. Mapping (4) converts each elliptic fiber into a vertical fiber through the corresponding point on the base of the punctured container.

Preservation of volumes. Now we show that mapping (4) preserves volumes. The volume element in the $(r,\, \theta,\, z)$ system is given by $r\,dr\,d\theta\,dz$, while that in the $(r',\, \theta', z')$ system is $r'dr'd\theta'dz'$. From (4) we have

$$(r')^2 = r^2 + z^2\rho(\theta)^2/h^2$$

which, for fixed z and θ, gives $r'dr' = r dr$. From (4) we also have $d\theta' = d\theta$ and $dz' = dz$, so the volume elements are equal: $r\,dr\,d\theta\,dz = r'dr'd\theta'dz'$. This proves:

Theorem 3. *Mapping* (4)*, from a general elliptic dome to its punctured prismatic container, preserves volumes. In particular, every general uniform elliptic dome is reducible.*

As an immediate consequence of Theorem 3 we obtain:

Corollary 5. *The volume of a general elliptic dome is equal to the volume of its circumscribing punctured cylindrical container, that is, two-thirds the volume of the circumscribing unpunctured cylindrical container which, in turn, is simply the area of the base times the height.*

The same formulas show that for a fixed altitude z, we have $r\,dr\,d\theta = r' dr' d\theta'$. In other words, the mapping also preserves areas of horizontal cross sections cut from the elliptic dome and its punctured container. This also implies Corollary 5 because of the slicing principle.

Lambert's classical mapping as a special case. Our mapping (4) generalizes Lambert's classical mapping [2], which is effected by wrapping a tangent cylinder about the equator, and then projecting the surface of the sphere onto this cylinder by rays through the axis which are parallel to the equatorial plane. Lambert's mapping takes points on the spherical surface (not at the north or south pole) and maps them onto points on the lateral cylindrical surface in a way that preserves areas. For a solid sphere, our mapping (4) takes each point not on the polar axis and maps it onto a point of the punctured solid cylinder in a way that preserves volumes. Moreover, analysis of a thin shell (similar to that in [1; Section 6]) shows that (4) also preserves areas when the surface of an Archimedean dome is mapped onto the lateral surface of its prismatic container. Consequently, we have:

Theorem 4. *Mapping* (4)*, from the surface of an Archimedean dome onto the lateral surface of its prismatic container, preserves areas.*

In the limiting case when the Archimedean dome becomes a hemisphere we get:

Corollary 6. (Lambert) *Mapping* (4)*, from the surface of a sphere to the lateral surface of its tangent cylinder, preserves areas.*

If the hemisphere in this limiting case has radius a, it is easily verified that (4) reduces to Lambert's mapping: $\theta' = \theta, \;\; z' = z, \;\; r' = a.$

6. Nonuniform elliptic domes

Mapping (4) takes each point P of an elliptic dome and carries it onto a point P' of its punctured container. Imagine an arbitrary mass density assigned to P, and assign the same mass density to its image P'. If a set of points P fills out a portion of the dome of volume v and total mass m, say, then the image points P' fill out a solid, which we call the *counterpart*, having the same volume v and the same total mass m. This can be stated as an extension of Theorem 3:

Theorem 5. *Any portion of a general nonuniform elliptic dome is reducible.*

By analogy with Theorem 3, we can say that mapping (4) “with weights” also preserves masses.

Fiber-elliptic and shell-elliptic domes. Next we describe a special way of assigning variable mass density to the points of a general elliptic dome and its punctured container so that corresponding portions of the dome and its counterpart have the same mass. The structure of the dome as a collection of similar domes plays an essential role in this description.

First assign mass density $f(r, \theta)$ to each point (r, θ) on the base of the dome and of its cylindrical container. Consider the elliptic fiber that emanates from any point $(\mu\rho(\theta),\, \theta)$ on the base, and assign the same mass density $f(\mu\rho(\theta),\, \theta)$ to each point of this fiber. In other words, the mass density along the elliptic fiber has a constant value inherited from the point at which the fiber meets the base. Of course, the constant may differ from point to point on the base. The elliptic fiber maps into a vertical fiber in the punctured container (of length μh, where h is the altitude of the dome), and we assign the same mass density $f(\mu\rho(\theta),\, \theta)$ to each point on this vertical fiber. In this way we produce a nonuniform elliptic dome and its punctured container, each with variable mass density inherited from the base. We call such a dome *fiber-elliptic*. The punctured container with density assigned in this manner is called the *counterpart* of the dome. The volume element multiplied by mass density is the same for both the dome and its counterpart.

An important special case occurs when the assigned density is also constant along the base curve $r = \rho(\theta)$ and along each curve $r = \mu\rho(\theta)$ similar to the base curve, where the constant density depends only on μ. Then each elliptic surface $E(\mu)$ will have its own constant density. We call domes with this assignment of mass density *shell-elliptic*. For fiber-elliptic and shell-elliptic domes, horizontal slices cut from any portion of the dome and its counterpart have equal masses, and their centers of mass are at the same height above the base. Thus, as a consequence of Theorem 3 we have:

Corollary 7. (a) *Any portion of a fiber-elliptic dome is reducible.*

(b) *In particular, any portion of a shell-elliptic dome is reducible.*

(c) *In particular, a sphere with spherically symmetric mass distribution is reducible.*

The reducibility properties of an elliptic dome also hold for the more general case in which we multiply the mass density $f(\mu\rho(\theta),\, \theta)$ by any function of z. Such change of density could be imposed, for example, by an external field (such as atmospheric density in a gravitational field that depends only on the height z). Consequently, not only are the volume and mass of any portion of this type of nonuniform elliptical dome equal to those of its counterpart, but the same is true for all moments with respect to the horizontal base.

Elliptic shells and cavities. Consider a general elliptic dome of altitude h, and denote its elliptic surface by $E(1)$. Scale $E(1)$ by a factor μ, where $0 < \mu < 1$, to produce a similar elliptic surface $E(\mu)$. The region between the two surfaces $E(\mu)$ and $E(1)$ is called an *elliptic shell*. It can be regarded as an elliptic dome with a cavity, or, equivalently, as a shell-elliptic dome with density 0 assigned to each point between $E(\mu)$ and the center.

Figure 5a shows an elliptic shell element, and Figure 5b shows its counterpart. Each base in the equatorial plane is bounded by portions of two curves with polar equations $r = \rho(\theta)$ and $r = \mu\rho(\theta)$, and two segments with $\theta = \theta_1$ and $\theta = \theta_2$. The shell element has two vertical plane faces, each consisting of a region between two similar ellipses. If μ is close to 1 and if θ_1 and θ_2 are nearly equal, the elliptic shell element can be thought of as a thin elliptic fiber, as was done earlier.

Consider a horizontal slice between two horizontal planes that cut both the inner and outer elliptic boundaries of the shell element. In other words, both planes are pierced by the cavity. The prismatic counterpart of this slice has horizontal cross sections congruent to the base, so its centroid lies *midway* between the two cutting planes. The same is true for the slice of the shell and for the center of mass of a slice cut from an assemblage of uniform elliptic shell elements, each with its own constant density.

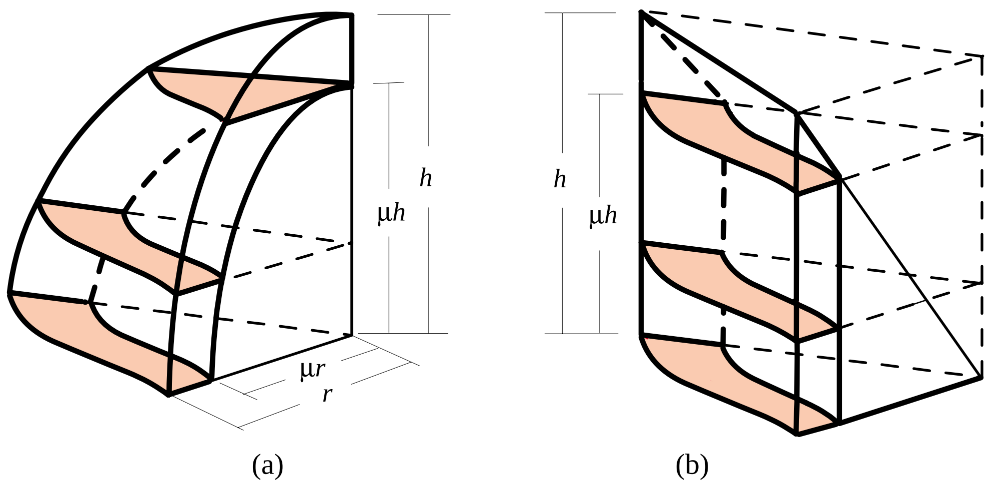


Figure 5. An elliptic shell element (a) and its counterpart (b).

In the same way, if we build a nonuniform shell-elliptic solid with a finite number of similar elliptic shells, each with its density inherited from the base, then any horizontal slice pierced by the cavity has its center of mass midway between the two horizontal cutting planes. Moreover, the following theorem holds for every such shell-elliptic wedge.

Theorem 6. *Any horizontal slice pierced by the cavity of a nonuniform shell-elliptic wedge has volume and mass equal, respectively, to those of its prismatic counterpart. Each volume and mass is independent of the height above the base and each is proportional to the thickness of the slice. Consequently, the center of mass of such a slice lies midway between the two cutting planes.*

Corollary 8. (Sphere with cavity) *Consider a spherically symmetric distribution of mass inside a solid sphere with a concentric cavity. Any slice between parallel* *planes pierced by the cavity has volume and mass proportional to the thickness of the slice, and is independent of the location of the slice.*

Corollary 8 implies that the one-dimensional vertical projection of the density is constant along the cavity. This simple result has profound consequences in tomography, which deals with the inverse problem of reconstructing spatial density distributions from a knowledge of their lower dimensional projections. Details of this application will appear elsewhere.

7. Formulas for volume and centroid

This section uses reducibility to give specific formulas for volumes and centroids of various building blocks of elliptic domes with an arbitrary curvilinear base.

Volume of a shell element. We begin with the simplest case. Cut a wedge from an elliptic dome of altitude h by two vertical half planes $\theta = \theta_1$ and $\theta = \theta_2$ through the polar axis, and then remove a similar wedge scaled by a factor μ, where $0 < \mu < 1$, as shown in Figure 5a. Assume the unpunctured cylindrical container in Figure 5b has volume V. By Corollary 5 the outer wedge has volume $2V/3$, and the similar inner wedge has volume $2\mu^3 V/3$, so the volume v of the shell element and its prismatic counterpart is the difference

$$v = \frac{2}{3}V(1-\mu^3). \tag{5}$$

Now $V = Ah$, where A is the area of the base of both the elliptic wedge and its container. The base of the elliptic shell element and its unpunctured container have area $B = A - \mu^2 A$, so $A = B/(1-\mu^2)$, $V = Bh/(1-\mu^2)$, and (5) can be written as

$$v = \frac{2}{3}Bh\frac{1-\mu^3}{1-\mu^2}. \tag{6}$$

Formula (6) also holds for the total volume of any assemblage of elliptic shell elements with a given h and μ, with B representing the total base area. The product Bh is the volume of the corresponding unpunctured cylindrical container of altitude h, so (6) gives us the formula

$$v_\mu(h) = \frac{2}{3}v_{cyl}\frac{1-\mu^3}{1-\mu^2}\,, \tag{7}$$

where $v_\mu(h)$ is the volume of the assemblage of elliptic shell elements and of the counterpart, and v_{cyl} is the volume of its *unpunctured* cylindrical container. When $\mu = 0$ in (7), the assemblage of elliptic wedges has volume $v_0(h) = 2v_{cyl}/3$, so we can write (7) in the form

$$v_\mu(h) = v_0(h)\frac{1-\mu^3}{1-\mu^2}, \tag{8}$$

where $v_0(h)$ is the volume of the outer dome of the assemblage and its counterpart. If μ approaches 1 the shell becomes very thin, the quotient $(1-\mu^3)/\left(1-\mu^2\right)$ approaches 3/2, and (7) shows that $v_\mu(h)$ approaches v_{cyl}. In other words, a very thin elliptic shell element has volume very nearly equal to that of its very thin unpunctured cylindrical container. An Archimedean shell has constant thickness equal to that of the prismatic container, so the lateral surface area of any assemblage of Archimedean wedges is equal to the lateral surface area of its prismatic container, a result derived in [1]. Note that this argument cannot be used to find the surface area of an nonspherical elliptic shell because it does not have constant thickness.

Next we derive a formula for the height of the centroid of any uniform elliptic wedge above the plane of its base.

Theorem 7. *Any uniform elliptic wedge or dome of altitude h has volume two-thirds that of its unpunctured prismatic container. Its centroid is located at height c above the plane of the base, where*

$$c = \frac{3}{8}h. \tag{9}$$

Proof. It suffices to prove (9) for the prismatic counterpart. For any prism of altitude h, the centroid is at a distance $h/2$ above the plane of the base. For a cone or pyramid with the same base and altitude it is known that the centroid is at a distance $3h/4$ from the vertex. To determine the height c of the centroid of a punctured prismatic container above the plane of the base, assume the unpunctured prismatic container has volume V and equate moments to get

$$c\left(\frac{2}{3}V\right) + \frac{3h}{4}\left(\frac{1}{3}V\right) = \frac{h}{2}V,$$

from which we find (9). By Theorem 5, the centroid of the inscribed elliptic wedge is also at height $3h/8$ above the base. The result is also true for any uniform elliptic dome formed as an assemblage of wedges. □

Equation (9) is equivalent to saying, in the style of Archimedes, that the centroid divides the altitude in the ratio 3:5.

Corollary 9. (a) *The centroid of a uniform Archimedean dome divides its altitude in the ratio* 3:5.

(b) (Archimedes) *The centroid of a uniform hemisphere divides its altitude in the ratio* 3:5.

Formula (9) is obviously true for the center of mass of any nonuniform assemblage of elliptic wedges of altitude h, each with its own constant density.

Centroid of a shell element. Now we can find, for any elliptic shell element, the height $c_\mu(h)$ of its centroid above the plane of its base. The volume and centroid results are summarized as follows:

Theorem 8. *Any nonuniform assemblage of elliptic shell elements with common altitude h and scaling factor μ has volume $v_\mu(h)$ given by* (8). *The height $c_\mu(h)$ of the centroid above the plane of its base is given by*

$$c_\mu(h) = \frac{3}{8}h\frac{1-\mu^4}{1-\mu^3}. \tag{10}$$

Proof. Consider first a single uniform elliptic shell element. Again it suffices to do the calculation for the prismatic counterpart. The inner wedge has altitude μh, so by (9) its centroid is at height $3\mu h/8$. The centroid of the outer wedge is at height $3h/8$. If the outer wedge has volume V_{outer}, the inner wedge has volume $\mu^3 V_{outer}$, and the shell element between them has volume $(1-\mu^3)V_{outer}$. Equating moments and canceling the common factor V_{outer} we find

$$\left(\frac{3}{8}\mu h\right)\mu^3 \;+\; c_\mu(h)(1-\mu^3) \;=\frac{3}{8}h,$$

from which we obtain (10). Formula (10) also holds for any nonuniform assemblage of elliptic shell elements with the same h and μ, each of constant density, although the density can differ from element to element. □

When $\mu = 0$, (10) gives $c_0(h) = 3h/8$.

When $\mu \to 1$, the shell becomes very thin and the limiting value of $c_\mu(h)$ in (10) is $h/2$. This also follows from Theorem 6 when the shell is very thin and the slice includes the entire dome. It is also consistent with Corollary 15 of [1], which states that the centroid of the surface area of an Archimedean dome is at the midpoint of its altitude.

Centroid of a slice of a wedge. More generally, we can determine the centroid of any slice of altitude z of a uniform elliptic wedge. By reducing this calculation to that of the prismatic counterpart, shown in Figure 6, the analysis becomes very simple. For clarity, the base in Figure 6 is shown as a triangle, but the same argument applies to a more general base like that in Figure 5. The slice in question is obtained from a prism of altitude z and volume $V(z) = \lambda V$, where V is the volume of the unpunctured prismatic container of altitude h, and $\lambda = z/h$. The centroid of the slice is at an altitude $z/2$ above the base. We remove from this slice a pyramidal portion of altitude z and volume $v(z) = \lambda^3 V/3$, whose centroid is at an altitude $3z/4$ above the base. The portion that remains has volume

$$V(z) - v(z) \;=\left(\lambda - \frac{1}{3}\lambda^3\right)V \tag{11}$$

and centroid at altitude $c(z)$ above the base. To determine $c(z)$, equate moments to obtain

$$\frac{3z}{4}v(z) + c(z)(V(z) - v(z)) = \frac{z}{2}V(z),$$

which gives

$$c(z) = \frac{\frac{z}{2}\,V(z) - \frac{3z}{4}v(z)}{V(z) - v(z)}.$$

Because $V(z) = \lambda V$, and $v(z) = \lambda^3 V/3$, we obtain the following theorem.

Theorem 9. *Any slice of altitude z cut from a uniform elliptic wedge of altitude h has volume given by* (11)*, where $\lambda = z/h$ and V is the volume of the unpunctured prismatic container. The height $c(z)$ of the centroid is given by*

$$c(z) = \frac{3}{4}z\frac{2-\lambda^2}{3-\lambda^2}. \tag{12}$$

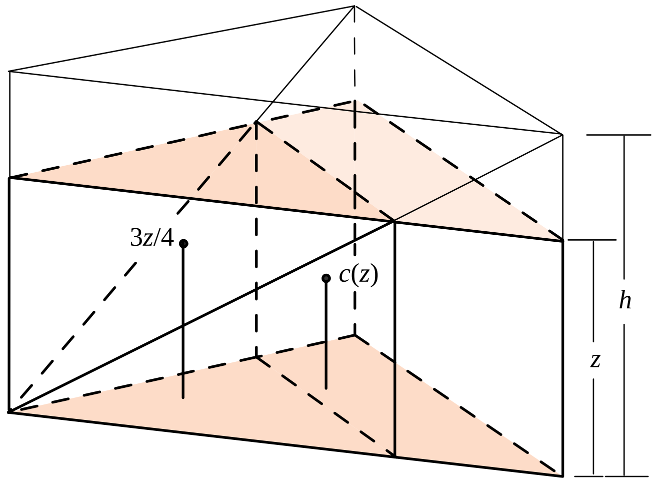


Figure 6. Calculating the centroid of a slice of altitude z cut from a wedge of altitude h.

When $z = h$ then $\lambda = 1$ and this reduces to (9). For small z the right member of (12) is asymptotic to $z/2$. This is reasonable because for small z the walls of the dome are nearly perpendicular to the plane of the equatorial base, so the dome is almost cylindrical near the base.

Centroid of a slice of a wedge shell element. There is a common generalization of (10) and (12). Cut a slice of altitude z from a shell element having altitude h and scaling factor μ, and let $c_\mu(z)$ denote the height of its centroid above the base. Again, we simplify the calculation of $c_\mu(z)$ by reducing it to that of its prismatic counterpart. The slice in question is obtained from an unpunctured prism of altitude z, whose centroid has altitude $z/2$ above the base. As in Theorem 9, let $\lambda = z/h$. If $\lambda \le \mu$, the slice lies within the cavity, and the prismatic counterpart is the same unpunctured prism of altitude z, in which case we know from Theorem 6 that

$$c_\mu(z) = \frac{z}{2}, \quad (\lambda \le \mu). \tag{13}$$

But if $\lambda \ge \mu$, the slice cuts the outer elliptic dome as shown in Figure 7a. In this case the counterpart slice has a slant face due to a piece removed by the puncturing pyramid, as indicated in Figure 7b.

Let V denote the volume of the unpunctured prismatic container of the outer dome. Then λV is the volume of the unpunctured prism of altitude z. Remove from this prism the puncturing pyramid of volume $\lambda^3 V/3$, leaving a solid whose volume is

$$V(z) = \lambda V - \frac{1}{3}\lambda^3 V, \quad (\lambda \ge \mu) \tag{14}$$

and whose centroid is at altitude $c(z)$ given by (12). This solid, in turn, is the union of the counterpart slice in question, and an adjacent pyramid with vertex O,

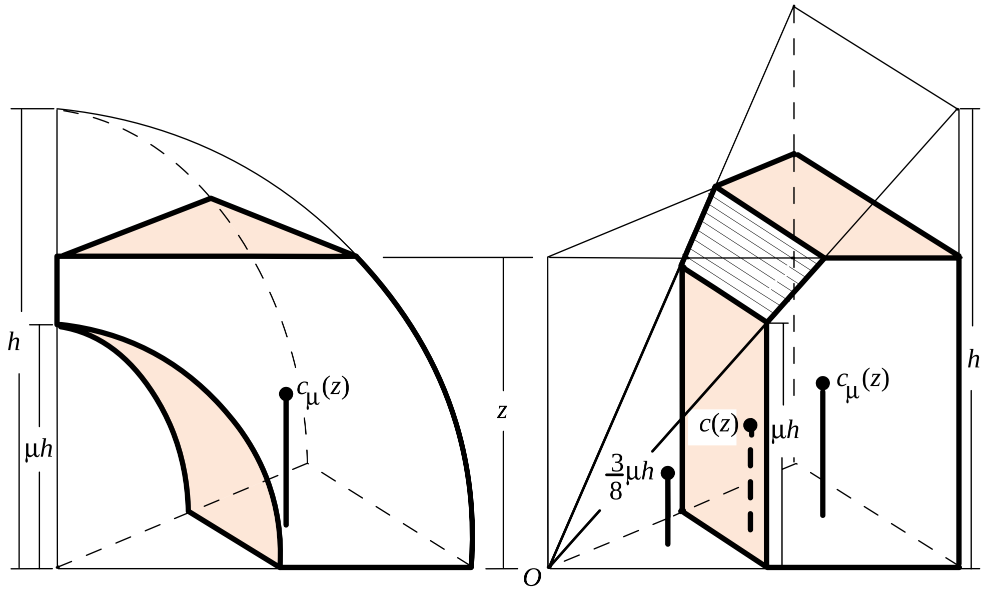


Figure 7. Determining the centroid of a slice of altitude $z \geq \mu h$ cut from an elliptic shell element.

altitude $\mu h,$ volume

$$v_\mu = \frac{2}{3}\mu^3 V, \tag{15}$$

and centroid at altitude $3\mu h/8$. The counterpart slice in question has volume

$$V(z) - v_\mu = \left(\lambda - \frac{1}{3}\lambda^3 - \frac{2}{3}\mu^3\right) V. \tag{16}$$

To find the altitude $c_\mu(z)$ of its centroid we equate moments and obtain

$$\left(\frac{3}{8}\mu h\right) v_\mu + c_\mu(z)(V(z) - v_\mu) = c(z)V(z),$$

from which we find

$$c_\mu(z) = \frac{c(z)V(z) - \left(\frac{3}{8}\mu h\right) v_\mu}{V(z) - v_\mu}.$$

Now we use (12), (14), (15)and (16). After some simplification we find the result

$$c_\mu(z) = \frac{3}{4}h\,\frac{\lambda^2(2-\lambda^2) - \mu^4}{\lambda(3-\lambda^2) - 2\mu^3} \quad (\lambda \geq \mu). \tag{17}$$

When $\lambda = \mu$, (17) reduces to (13); when $\lambda = 1$ then $z = h$ and (17) reduces to (10); and when $\mu = 0$, (17) reduces to (12). The results are summarized by the following theorem.

Theorem 10. *Any horizontal slice of altitude $z \geq \mu h$ cut from a wedge shell element of altitude h and scaling factor μ has volume given by* (16)*, where $\lambda = z/h$. The altitude of its centroid above the base is given by* (17)*. In particular these formulas hold for any slice of a shell of an Archimedean, elliptic, or spherical dome.*

Note: Theorem 6 covers the case $z \leq \mu h$.

In deriving the formulas in this section we made no essential use of the fact that the shell elements are elliptic. The important fact is that each shell element is the region between two similar objects.

8. The necessity of elliptic profiles

We know that every horizontal plane cuts an elliptic dome and its punctured cylindrical container in cross sections of equal area. This section reveals the surprising fact that the elliptical shape of the dome is actually a consequence of this property.

Consider a dome of altitude h, and its punctured prismatic counterpart having a congruent base bounded by a curve satisfying a polar equation $r = \rho(\theta)$. Each vertical half plane through the polar axis at angle θ cuts the dome along a curve we call a *profile*, illustrated by the example in Figure 8a. This is like the elliptic dome in Figure 5a, except that we do not assume that the profiles are elliptic. Each profile passes through a point $(\rho(\theta),\, \theta)$ on the outer edge of the base. At altitude z above the base a point on the profile is at distance r from the polar axis, where r is a function of z that determines the shape of the profiles. We define a general profile dome to be one in which each horizontal cross section is similar to the base. Figure 8a shows a portion of a dome in which $\rho(\theta) > 0$. This portion is a wedge with two vertical plane faces that can be thought of as "walls" forming part of the boundary of the wedge.

Suppose that a horizontal plane at distance z above the base cuts a region of area $A(z)$ from the wedge and a region of area $B(z)$ from the punctured prism. We know that $A(0) = B(0)$. Now we assume that $A(z) = B(z)$ for some $z > 0$ and deduce that the point on the profile with polar coordinates (r, θ, z) satisfies the equation

$$\left(\frac{r}{\rho(\theta)}\right)^2 + \left(\frac{z}{h}\right)^2 = 1 \tag{18}$$

if $\rho(\theta) > 0$. In other words, the point on the profile at a height where the areas are equal lies on an ellipse with vertical semi axis of length h, and horizontal semi axis of length $\rho(\theta)$. Consequently, if $A(z) = B(z)$ for every z from 0 to h, the profile will fill out a quarter of an ellipse and the dome will necessarily be elliptic. Note that (18) implies that $r \to 0$ as $z \to h$.

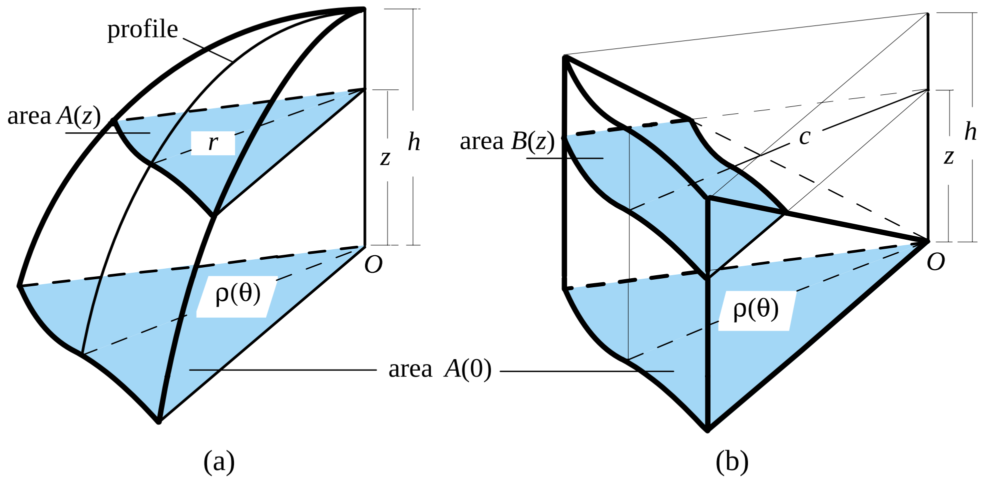


Figure 8. Determining the elliptic shape of the profiles as a consequence of the relation $A(z) = B(z)$.

To deduce (18), note that the horizontal cross section of area $A(z)$ in Figure 8a is similar to the base with similarity ratio $r/\rho(\theta)$, where $\rho(\theta)$ denotes the radial distance to the point where the profile intersects the base, and r is the length of the radial segment at height z. By similarity, $A(z) = (r/\rho(\theta))^2 A(0)$. In Figure 8b, area $B(z)$ is equal to $A(0)$ minus the area of a smaller similar region with similarity ratio $c/\rho(\theta)$, where c is the length of the parallel radial segment of the smaller similar region at height z. By similarity, $c/\rho(\theta) = z/h$, hence $B(z) = (1 - (z/h)^2)A(0)$. Equating this to $A(z)$ we find $(1 - (z/h)^2)A(0) = (r/\rho(\theta))^2 A(0)$, which gives (18). And, of course, we already know that (18) implies $A(z) = B(z)$ for every z. Thus we have proved:

Theorem 11. *Corresponding horizontal cross sections of a general profile uniform dome and its punctured prismatic counterpart have equal areas if, and only if, each profile is elliptic.*

As already remarked in Section 5, an elliptic dome can be deformed in such a way that areas of horizontal cross sections are preserved but the deformed dome no longer has elliptic profiles. At first glance, this may seem to contradict Theorem 11. However, such a deformation will distort the vertical walls; the dome will not satisfy the requirements of Theorem 11, and also the punctured counterpart will no longer be prismatic.

An immediate consequence of Theorem 11 is that any reducible general profile dome necessarily has elliptic profiles, because if all horizontal slices of such a dome and its counterpart have equal volumes then the cross sections must have equal areas. We have also verified that Theorem 11 can be extended to nonuniform general profile domes built from a finite number of general profile similar shells, each with its own constant density, under the condition that corresponding horizontal slices of the dome and its counterpart have equal masses, with no requirements on volumes or reducibility.

Concluding remarks. The original motivation for this research was to extend to more general solids classical properties which seemed to be unique to spheres and hemispheres. Initially an extension was given for Archimedean domes and a further extension was made by simply dilating these domes in a vertical direction. These extensions could also have been analyzed by using properties of inscribed spheroids.

A significant extension was made when we introduced polygonal elliptic domes whose bases could be arbitrary polygons, not necessarily circumscribing the circle. In this case there are no inscribed spheroids to aid in the analysis, but the method of punctured containers was applicable. This led naturally to general elliptic domes with arbitrary base, and the method of punctured containers was formulated in terms of mappings that preserve volumes.

But the real power of the method is revealed by the treatment of nonuniform mass distributions. Problems of determining volumes and centroids of elliptic wedges, shells, and their slices, including those with cavities, were reduced to those of simpler prismatic containers. Finally, we showed that domes with elliptic profiles are essentially the only ones that are reducible.

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