Forum Geometricorum
Volume 7 (2007) 53–65.

FORUM GEOM
ISSN 1534-1178

Midcircles and the Arbelos

Eric Danneels and Floor van Lamoen

Abstract. We begin with a study of inversions mapping one given circle into another. The results are applied to the famous configuration of an arbelos. In particular, we show how to construct three infinite Pappus chains associated with the arbelos.

1. Inversions swapping two circles

Given two circles $O_i(r_i)$, $i = 1, 2$, in the plane, we seek the inversions which transform one of them into the other. Set up a cartesian coordinate system such that for $i = 1,\ 2$, O_i is the point $(a_i,\ 0)$. The endpoints of the diameters of the circles on the x-axis are $(a_i \pm r_i,\ 0)$. Let $(a,\ 0)$ and Φ be the center and the power of inversion. This means, for an appropriate choice of $\varepsilon = \pm 1$,

$$(a_1 + \varepsilon \cdot r_1 - a)(a_2 + r_2 - a) = (a_1 - \varepsilon \cdot r_1 - a)(a_2 - r_2 - a) = \Phi.$$

Solving these equations we obtain

$$a = \frac{r_2 a_1 + \varepsilon \cdot r_1 a_2}{r_2 + \varepsilon \cdot r_1}, \tag{1}$$

$$\Phi = \frac{\varepsilon \cdot r_1 r_2((r_2 + \varepsilon \cdot r_1)^2 - (a_1 - a_2)^2)}{(r_2 + \varepsilon \cdot r_1)^2}. \tag{2}$$

From (1) it is clear that the center of inversion is a center of similitude of the two circles, internal or external according as $\varepsilon = +1$ or -1. The two circles of inversion, real or imaginary, are given by $(x - a)^2 + y^2 = \Phi$, or more explicitly,

$$r_2((x - a_1)^2 + y^2 - r_1^2) + \varepsilon \cdot r_1((x - a_2)^2 + y^2 - r_2^2) = 0. \tag{3}$$

They are members of the pencil of circles generated by the two given circles. Following Dixon [1, pp.86–88], we call these the *midcircles* $\mathcal{M}_\varepsilon$, $\varepsilon = \pm 1$, of the two given circles $O_i(r_i)$, $i = 1, 2$. From (2) we conclude that
(i) the *internal* midcircle $\mathcal{M}_+$ is real if and only if $r_1 + r_2 > d$, the distance between the two centers, and
(ii) the *external* midcircle $\mathcal{M}_-$ is real if and only if $|r_1 - r_2| < d$.
In particular, if the two given circles intersect, then there are two real circles of inversion through their common points, with centers at the centers of similitudes. See Figure 1.

Lemma 1. *The image of the circle with center B, radius r, under inversion at a point A with power Φ is the circle of radius $\left|\frac{\Phi}{d^2 - r^2}\right| r$, and center dividing AB at the ratio $AP : PB = \Phi : d^2 - r^2 - \Phi$, where d is the distance between A and B.*

Publication Date: February 19, 2007. Communicating Editor: Paul Yiu.

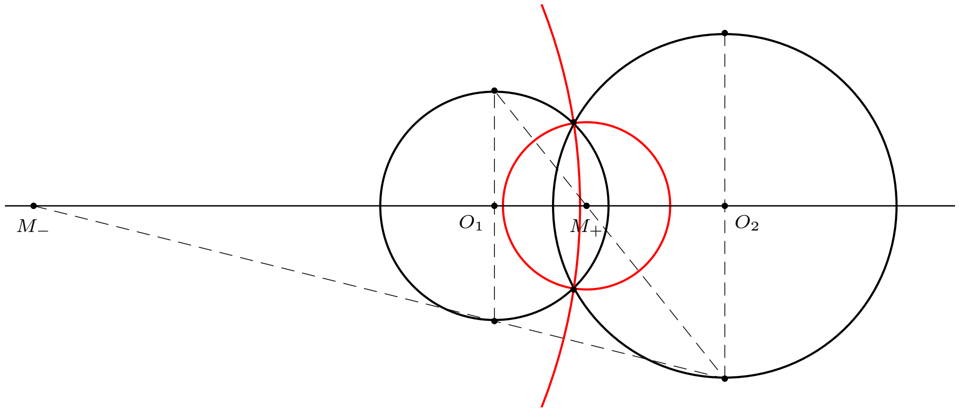

Figure 1.

2. A locus property of the midcircles

Proposition 2. *The locus of the center of inversion mapping two given circles $O_i(a_i)$, $i = 1, 2$, into two congruent circles is the union of their midcircles $\mathcal{M}_+$ and $\mathcal{M}_-$.*

Proof. Let $d(P, Q)$ denote the distance between two points P and Q. Suppose inversion in P with power Φ transforms the given circles into congruent circles. By Lemma 1,

$$\frac{d(P, O_1)^2 - r_1^2}{d(P, O_2)^2 - r_2^2} = \varepsilon \cdot \frac{r_1}{r_2} \tag{4}$$

for $\varepsilon = \pm 1$. If we set up a coordinate system so that $O_i = (a_i, 0)$ for $i = 1, 2$, $P = (x, y)$, then (4) reduces to (3), showing that the locus of P is the union of the midcircles $\mathcal{M}_+$ and $\mathcal{M}_-$. $\square$

Corollary 3. *Given three circles, the common points of their midcircles taken by pairs are the centers of inversion that map the three given circles into three congruent circles.*

For $i, j = 1, 2, 3$, let $\mathcal{M}_{ij}$ be a midcircle of the circles $\mathcal{C}_i = O_i(R_i)$ and $\mathcal{C}_j = O_j(R_j)$. By Proposition 2 we have $\mathcal{M}_{ij} = R_j \cdot \mathcal{C}_i + \varepsilon_{ij} \cdot R_i \cdot \mathcal{C}_j$ with $\varepsilon_{ij} = \pm 1$. If we choose ε_{ij} to satisfy $\varepsilon_{12} \cdot \varepsilon_{23} \cdot \varepsilon_{31} = -1$, then the centers of $\mathcal{M}_{12}$, $\mathcal{M}_{23}$ and $\mathcal{M}_{31}$ are collinear. Since the radical center P of the triad $\mathcal{C}_i$, $i = 1, 2, 3$, has the same power with respect to these circles, they form a pencil and their common points X and Y are the poles of inversion mapping the circles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ into congruent circles.

The number of common points that are the poles of inversion mapping the circles $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ into a triple of congruent circles depends on the configuration of these circles.

(1) The maximal number is 8 and occurs when each pair of circles $\mathcal{C}_i$ and $\mathcal{C}_j$ have two distinct intersections. Of these 8 points, two correspond to

the three external midcircles while each pair of the remaining six points correspond to a combination of one external and two internal midcircles.
(2) The minimal number is 0. This occurs for instance when the circles belong to a pencil of circles without common points.

Corollary 4. *The locus of the centers of the circles that intersect three given circles at equal angles are* 0, 1, 2, 3 *or* 4 *lines through their radical center* P *perpendicular to a line joining three of their centers of similitude.*

Proof. Let $\mathcal{C}_1 = A(R_1)$, $\mathcal{C}_2 = B(R_2)$, and $\mathcal{C}_3 = C(R_3)$ be the given circles. Consider three midcircles with collinear centers. If X is an intersection of these midcircles, reflection in the center line gives another common point Y. Consider an inversion τ with pole X that maps circle $\mathcal{C}_3$ to itself. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ become $\mathcal{C}_1' = A'(R_3)$ and $\mathcal{C}_2 = B'(R_3)$. If P' is the radical center of the circles $\mathcal{C}_1$, $\mathcal{C}_2'$ and $\mathcal{C}_3'$, then every circle $\mathcal{C} = P'(R)$ will intersect these 3 circles at equal angles. When we apply the inversion τ once again to the circles $\mathcal{C}_1'$, $\mathcal{C}_2'$, $\mathcal{C}_3$ and $\mathcal{C}$ we get the 3 original circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and a circle $\mathcal{C}'$ and since an inversion preserves angles circle $\mathcal{C}'$ will also intersect these original circles at equal angles.

The circles orthogonal to all circles $\mathcal{C}'$ are mapped by τ to lines through P'. This means that the circles orthogonal to $\mathcal{C}'$ all pass through the inversion pole X. By symmetry they also pass through Y, and thus form the pencil generated by the triple of midcircles we started with. The circles $\mathcal{C}'$ form therefore a pencil as well, and their centers lie on XY as X and Y are the limit-points of this pencil. □

Remark. Not every point on the line leads to a real circle, and not every real circle leads to real intersections and real angles.

As an example we consider the A-, B- and C-Soddy circles of a triangle ABC. Recall that the A-Soddy circle of a triangle is the circle with center A and radius $s - a$, where s is the semiperimeter of triangle ABC. The area enclosed in the interior of ABC by the A-, B- and C-Soddy circles form a skewed arbelos, as defined in [5]. The circles $\mathcal{F}_\phi$ making equal angles to the A-, B- and C-Soddy circles form a pencil, their centers lie on the Soddy line of ABC, while the only real line of three centers of midcircles is the tripolar of the Gergonne point X_7. [1]

The points X and Y in the proof of Corollary 4 are the limit points of the pencil generated by $\mathcal{F}_\phi$. In barycentric coordinates, these points are, for $\varepsilon = \pm 1$,

$$(4R + r) \cdot X_7 + \varepsilon \cdot \sqrt{3}s \cdot I = \left(2r_a + \varepsilon \cdot \sqrt{3}a : 2r_b + \varepsilon \cdot \sqrt{3}b : 2r_c + \varepsilon \cdot \sqrt{3}c\right),$$

where R, r, r_a, r_b, r_c are the circumradius, inradius, and inradii. The midpoint of XY is the Fletcher-point X_{1323}. See Figure 2.

3. The Arbelos

Now consider an arbelos, consisting of two interior semicircles $O_1(r_1)$ [2] and $O_2(r_2)$ and an exterior semicircle $O(r) = O_0(r)$, $r = r_1 + r_2$. Their points of

[1] The numbering of triangle centers following numbering in [2, 3].

[2] We adopt notations as used in [4]: By (PQ) we denote the circle with diameter PQ, by $P(r)$ the circle with center P and radius r, while $P(Q)$ is the circle with center P through Q and (PQR)

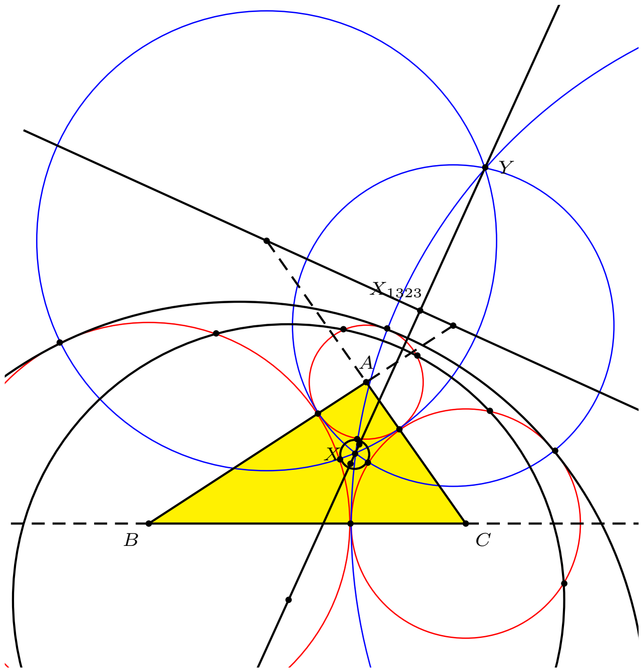

Figure 2.

tangency are A, B and C as indicated in Figure 3. The arbelos has an incircle (O'). For simple constructions of (O'), see [7, 8].

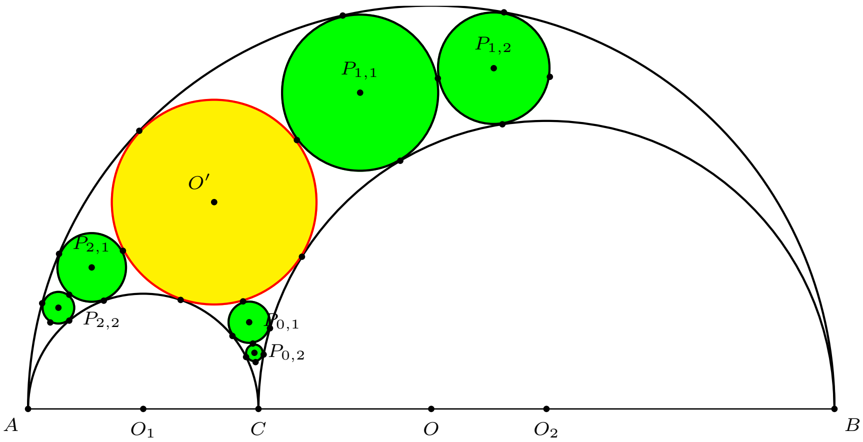

Figure 3.

We consider three Pappus chains $(\mathcal{P}_{i,n})$, $i = 0, 1, 2$. If (i, j, k) is a permutation of $(0, 1, 2)$, the Pappus chain $(\mathcal{P}_{i,n})$ is the sequence of circles tangent to both (O_j)

---

is the circle through P, Q and R. The circle (P) is the circle with center P, and radius clear from context.

and (O_k) defined recursively by
(i) $\mathcal{P}_{i,0} = (O')$, the incircle of the arbelos,
(ii) for $n \geq 1$, $\mathcal{P}_{i,n}$ is tangent to $\mathcal{P}_{i,n-1}$, (O_j) and (O_k),
(iii) for $n \geq 2$, $\mathcal{P}_{i,n}$ and $\mathcal{P}_{i,n-2}$ are distinct circles.

These Pappus chains are related to the centers of similitude of the circles of the arbelos. We denote by M_0 the external center of similitude of (O_1) and (O_2), and, for $i, j = 1, 2$, by M_i the internal center of similitude of (O) and (O_j). The midcircles are $M_0(C)$, $M_1(B)$ and $M_2(A)$. Each of the three midcircles leaves (O') and its reflection in AB invariant, so does each of the circles centered at A, B and C respectively and orthogonal to (O'). These six circles are thus members of a pencil, and O' lies on the radical axis of this pencil. Each of the latter three circles inverts two of the circles forming the arbelos to the tangents to (O') perpendicular to AB, and the third circle into one tangent to (O'). See Figure 4.

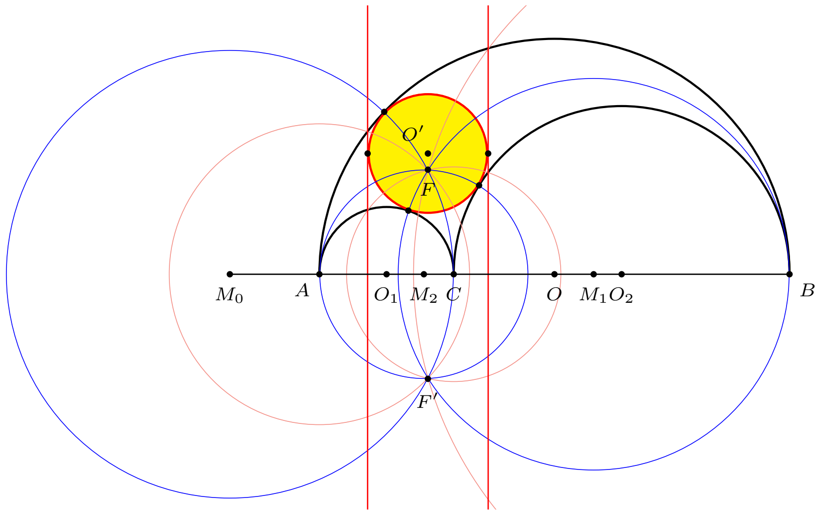

Figure 4.

We make a number of interesting observations pertaining to the construction of the Pappus chains. Denote by $P_{i,n}$ the center of the circle $\mathcal{P}_{i,n}$.

3.1. For $i = 0, 1, 2$, inversion in the midcircle (M_i) leaves $(P_{i,n})$ invariant. Consequently,

(1) the point of tangency of $(P_{i,n})$ and $(P_{i,n+1})$ lies on (M_i) and their common tangent passes through M_i;
(2) for every permuation (i, j, k) of $(0, 1, 2)$, the points of tangency of $(P_{i,n})$ with (O_j) and (O_k) are collinear with M_i. See Figure 5.

3.2. For every permutation (i, j, k) of $(0, 1, 2)$, inversion in (M_i) swaps $(P_{j,n})$ and $(P_{k,n})$. Hence,

(1) M_i, $P_{j,n}$ and $P_{k,n}$ are collinear;

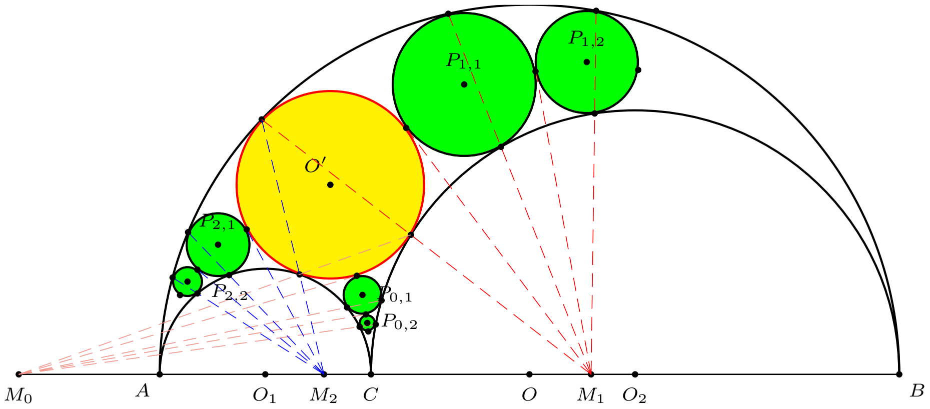

Figure 5.

(2) the points of tangency of $(P_{j,n})$ and $(P_{k,n})$ with (O_i) are collinear with M_i;
(3) the points of tangency of $(P_{j,n})$ with $(P_{j,n+1})$, and of $(P_{k,n})$ with $(P_{k,n+1})$ are collinear with M_i;
(4) the points of tangency of $(P_{j,n})$ with (O_k), and of $(P_{k,n})$ with (O_j) are collinear with M_i. See Figure 6.

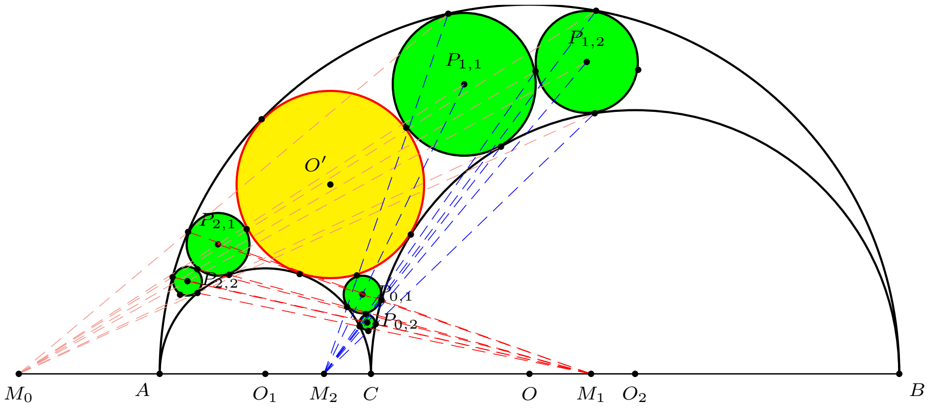

Figure 6.

3.3. Let (i, j, k) be a permutation of $(0, 1, 2)$. There is a circle $\mathcal{I}_i$ which inverts (O_j) and (O_k) respectively into the two tangents ℓ_1 and ℓ_2 of (O') perpendicular to AB. The Pappus chain $(P_{i,n})$ is inverted to a chain of congruent circles (Q_n) tangent to ℓ_1 and ℓ_2 as well, with $(Q_0) = (O')$. See Figure 7. The lines joining A to

(i) the point of tangency of (Q_n) with ℓ_1 (respectively ℓ_2) intersect $\mathcal{C}_0$ (respectively $\mathcal{C}_1$) at the points of tangency with $\mathcal{P}_{2,n}$,

(ii) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_2$ at the point of tangency of $\mathcal{P}_{2,n}$ and $\mathcal{P}_{2,n-1}$.

From these points of tangency the circle $(P_{2,n})$ can be constructed.

Similarly, the lines joining B to

(iii) the point of tangency of (Q_n) with ℓ_1 (respectively ℓ_2) intersect $\mathcal{C}_2$ (respectively $\mathcal{C}_0$) at the points of tangency with $\mathcal{P}_{1,n}$,

(iv) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_1$ at the point of tangency of $(P_{1,n})$ and $(P_{1,n-1})$.

From these points of tangency the circle $(P_{1,n})$ can be constructed.

Finally, the lines joining C to

(v) the point of tangency of (Q_n) with ℓ_i, $i = 1, 2$, intersect $\mathcal{C}_i$ at the points of tangency with $\mathcal{P}_{0,n}$,

(vi) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_0$ at the point of tangency of $(P_{0,n})$ and $(P_{0,n-1})$.

From these points of tangency the circle $(P_{0,n})$ can be constructed.

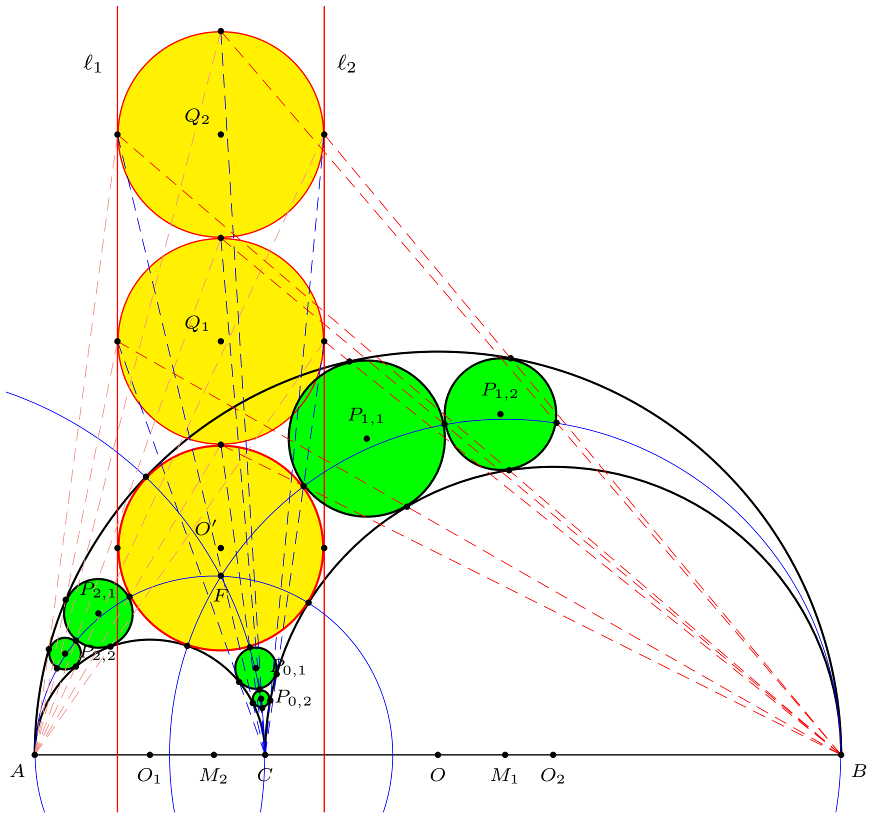

Figure 7.

3.4. Now consider the circle $\mathcal{K}_n$ through the points of tangency of $(P_{i,n})$ with (O_j) and (O_k) and orthogonal to $\mathcal{I}_i$. Then by inversion in $\mathcal{I}_i$ we see that $\mathcal{K}_n$ also passes through the points of tangency of (Q_n) with ℓ_1 and ℓ_2. Consequently the center K_n of $\mathcal{K}_n$ lies on the line through O' parallel to ℓ_1 and ℓ_2, which is the radical axis of the pencil of $\mathcal{I}_i$ and (M_i). By symmetry $\mathcal{K}_n$ passes through the points of tangency $(P_{i',n})$ with $(O_{j'})$ and $(O_{k'})$ for other permutations (i', j', k') of $(0, 1, 2)$ as well. The circle $\mathcal{K}_n$ thus passes through eight points of tangency, and all $\mathcal{K}_n$ are members of the same pencil.

With a similar reasoning the circle $\mathcal{L}_n = (L_n)$ tangent to $P_{i,n}$ and $P_{i,n+1}$ at their point of tangency as well as to (Q_n) and (Q_{n+1}) at their point of tangency, belongs to the same pencil as $\mathcal{K}_n$. See Figure 8.

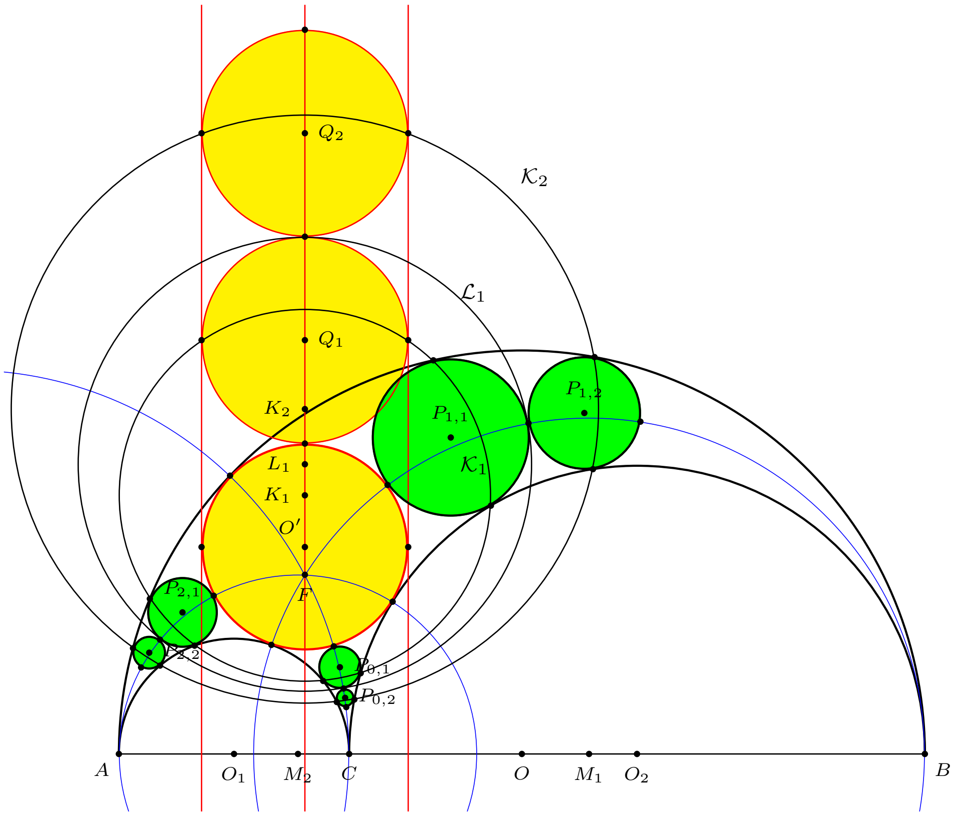

Figure 8.

The circles $\mathcal{K}_n$ and $\mathcal{L}_n$ make equal angles to the three arbelos semicircles (O), (O_1) and (O_2). In §5 we dive more deeply into circles making equal angles to three given circles.

4. λ-Archimedean circles

Recall that in the arbelos the twin circles of Archimedes have radius $r_A = \frac{r_1 r_2}{r}$. Circles congruent to these twin circles with relevant additional properties in the arbelos are called Archimedean.

Now let the homothety $\mathsf{h}(A, \mu)$ map O and O_1 to O' and O'_1. In [4] we have seen that the circle tangent to O' and O'_1 and to the line through C perpendicular to AB is Archimedean for any μ within obvious limitations. On the other hand from this we can conclude that when we apply the homothety $\mathsf{h}(A, \lambda)$ to the line through C perpendicular to AB, to find the line ℓ, then the circle tangent to ℓ, O and O' has radius λr_A. These circles are described in a different way in [6]. We call circles with radius λr_A and with additional relevant properties λ-Archimedean.

We can find a family of λ-Archimedean circles in a way similar to Bankoff's triplet circle. A proof showing that Bankoff's triplet circle is Archimedean uses the inversion in $A(B)$, that maps O and O_1 to two parallel lines perpendicular to AB, and (O_2) and the Pappus chain $(P_{2,n})$ to a chain of tangent circles enclosed by these two lines. The use of a homothety through A mapping Bankoff's triplet circle (W_3) to its inversive image shows that it is Archimedean. We can use this homothety as (W_3) circle is tangent to AB. This we know because (W_3) is invariant under inversion in (M_0), and thus intersects (M_0) orthogonally at C. In the same way we find λ-Archimedean circles.

Proposition 5. *For $i, j = 1, 2$, let $V_{i,n}$ be the point of tangency of (O_i) and $(P_{j,n})$. The circle $(CV_{1,n}V_{2,n})$ is $(n+1)$-Archimedean.*

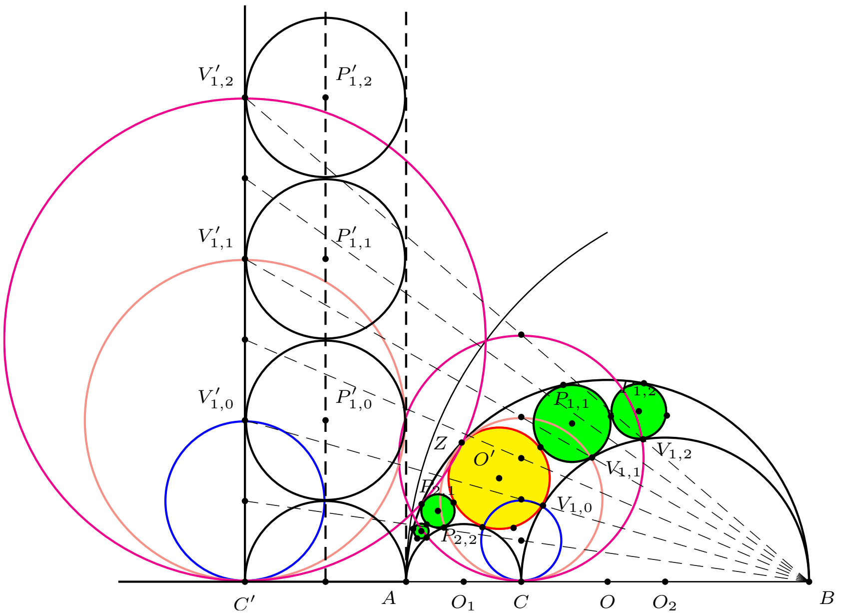

Figure 9.

A special circle of this family is $(L) = (CV_{1,1}V_{2,1})$, which tangent to (O) and (O') at their point of tangency Z, as can be easily seen from the figure after inversion. See Figure 9. We will meet again this circle in the final section.

Let $W_{1,n}$ be the point of tangency of $(P_{0,n})$ and (O_1). Similarly let $W_{2,n}$ be the point of tangency of $(P_{0,n})$ and (O_2). The circles $(CW_{1,n}W_{2,n})$ are invariant under inversion through (M_0), hence are tangent to AB. We may consider AB itself as preceding element of these circles, as we may consider (O) as $(P_{0,-1})$. Inversion through C maps $(P_{0,n})$ to a chain of tangent congruent circles tangent to two lines perpendicular to AB, and maps the circles $(CW_{1,n}W_{2,n})$ to equidistant lines parallel to AB and including AB. The diameters through C of $(CW_{1,n}W_{2,n})$ are thus, by inversion back of these equidistant lines, proportional to the harmonic sequence. See Figure 10.

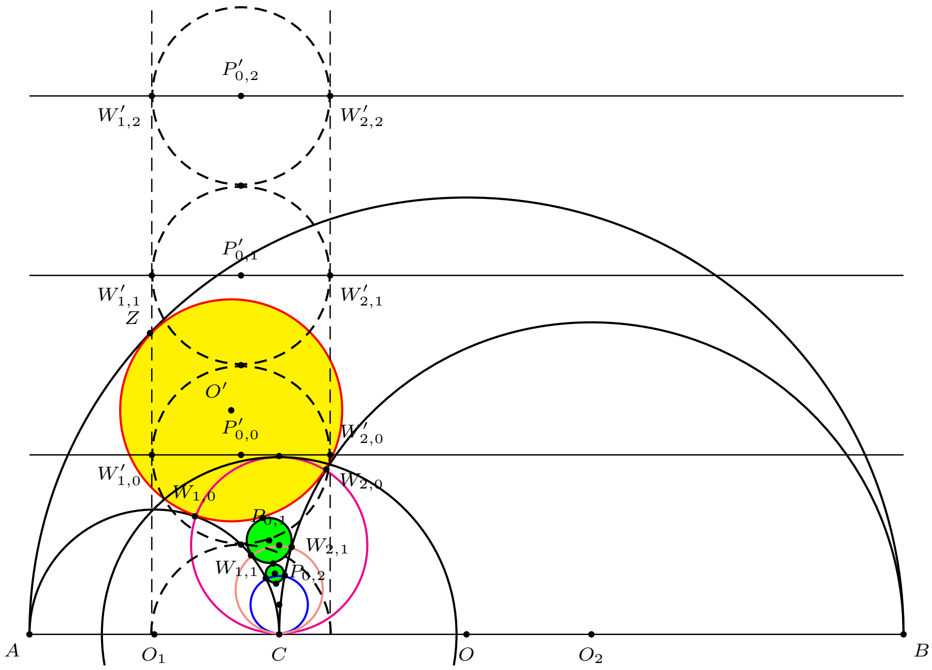

Figure 10.

Proposition 6. *The circle* $(CW_{1,n}W_{2,n})$ *is* $\frac{1}{n+1}$*-Archimedean.*

5. Inverting the arbelos to congruent circles

Let F_1 and F_2 be the intersection points of the midcircles (M_0), (M_1) and (M_2) of the arbelos. Inversion through F_i maps the circles (O), (O_1) and (O_2) to three congruent and pairwise tangent circles $(E_{i,0})$, $(E_{i,1})$ and $(E_{i,2})$. Triangle $E_{i,0}E_{i,1}E_{i,2}$ of course is equilateral, and stays homothetic independent of the power of inversion.

The inversion through F_i maps (M_0) to a straight line which we may consider as the midcircle of the two congruent circles $(E_{i,1})$ and $(E_{i,2})$. The center M'_0 of this degenerate midcircle we may consider at infinity. It follows that the line $F_iM_0 = F_iM'_0$ is parallel to the central $E_{i,1}E_{i,2}$ of these circles. Hence the lines through F_i parallel to the sides of $E_{i,1}E_{i,2}E_{i,3}$ pass through the points M_0, M_1 and M_2.

Now note that A, B, and C are mapped to the midpoints of triangle $E_{i,0}E_{i,1}E_{i,2}$, and the line AB thus to the incircle of $E_{i,0}E_{i,1}E_{i,2}$. The point F_i is thus on this circle, and from inscribed angles in this incircle we see that the directed angles (F_iA, F_iB), (F_iB, F_iC), (F_iC, F_iA) are congruent modulo π.

Proposition 7. *The points F_1 and F_2 are the Fermat-Torricelli points of degenerate triangles ABC and $M_0M_1M_2$.*

Let the diameter of (O') parallel AB meet (O') in G_1 and G_2 and Let G_1' and G_2' be their feet of the perpendicular altitudes on AB. From Pappus' theorem we know that $G_1G_2G_1'G_2'$ is a square. Construction 4 in [7] tells us that O' and its reflection through AB can be found as the Kiepert centers of base angles $\pm \arctan 2$. Multiplying all distances to AB by $\frac{\sqrt{3}}{2}$ implies that the points F_i form equilateral triangles with G_1' and G_2'. See Figure 11.

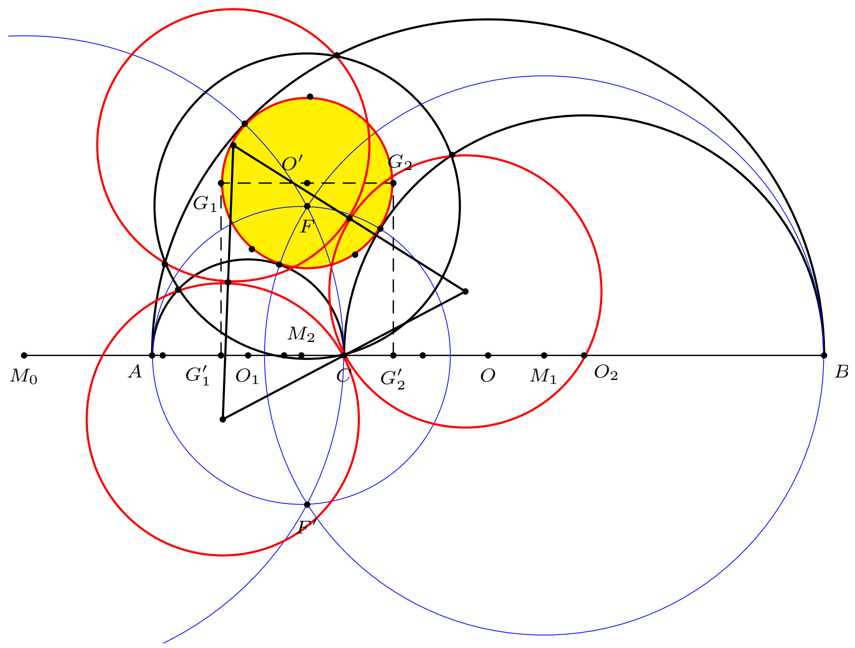

Figure 11.

A remarkable corollary of this and Proposition 7 is that the arbelos erected on $M_0M_1M_2$ shares its incircle with the original arbelos. See Figure 12.

Let F_1 be at the same side of ABC as the Arbelos semicircles. The inversion in $F_1(C)$ maps (O), (O_1) and (O_2) to three 2-Archimedean circles (E_0), (E_1) and (E_2), which can be shown with calculations, that we omit here. The 2-Archimedean circle (L) we met earlier meets (E_1) and (E_2) in their "highest" points H_1 and H_2 respectively. This leads to new Archimedean circles (E_1H_1) and (E_2H_2), which are tangent to Bankoff's triplet circle. Note that the points E_1, E_2, L, the point of tangency of (E_0) and (E_1) and the point of tangency of (E_0)

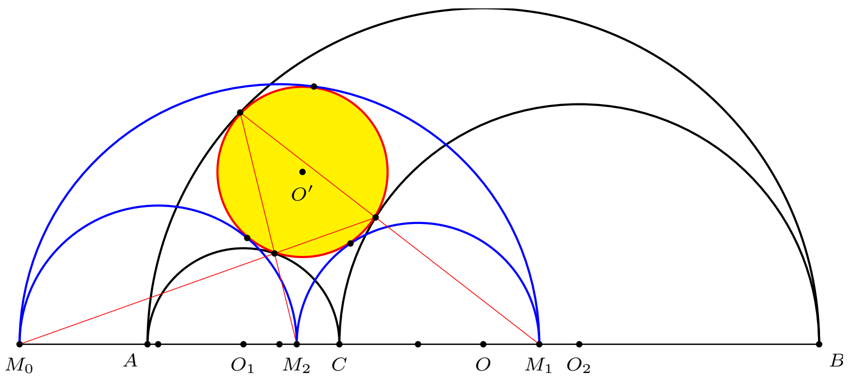

Figure 12.

and (E_2) lie on the 2-Archimedean circle with center C tangent to the common tangent of (O_1) and (O_2). See Figure 13.

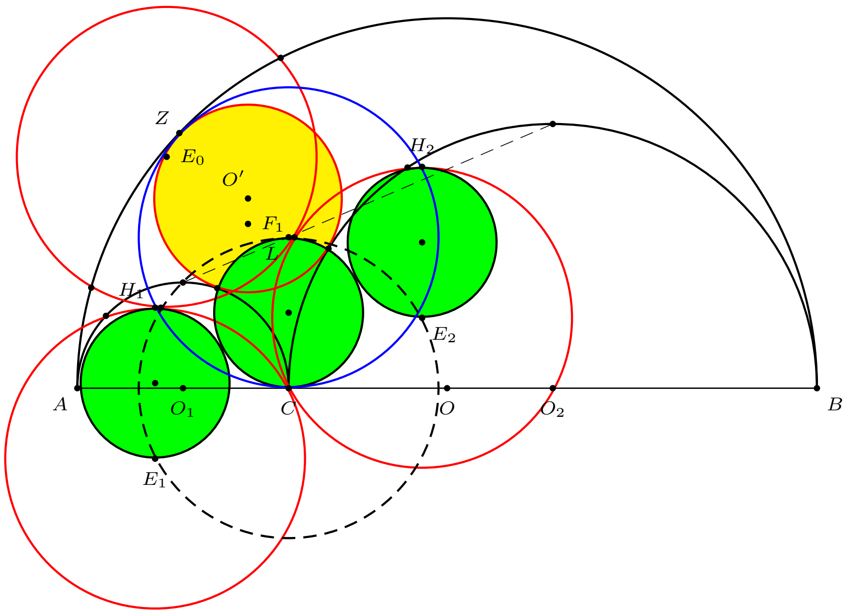

Figure 13.

References

[1] R. D. Dixon, *Mathographics*, Dover, 1991.
[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[4] F. M. van Lamoen, Archimedean adventures, *Forum Geom.*, 6 (2006) 79–96.
[5] H. Okumara and M. Watanabe, The twin circles of Archimedes in a skewed arbelos, *Forum Geom.*, 4 (2004) 229–251.
[6] H. Okumura and M. Watanabe, A generalization of Power's Archimedean circle, *Forum Geom.*, 6 (2006) 103–105.
[7] P. Y. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
[8] P. Yiu, Elegant geometric constructions, *Forum Geom.*, 5 (2005) 75–96.

Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: `eric.danneels@pandora.be`

Floor van Lamoen: St. Willibrordcollege, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: `fvanlamoen@planet.nl`